

**Spectral representations of graphons  
in very large network systems control**

S. Gao,  
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# Spectral representations of graphons in very large network systems control

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**Abstract:** Graphon-based control has recently been proposed and developed to solve control problems for dynamical systems on networks which are very large or growing without bound (see Gao and Caines, CDC 2017, CDC 2018). In this paper, spectral representations, eigenfunctions and approximations of graphons, and their applications to graphon-based control are studied. First, spectral properties of graphons are presented and then approximations based on Fourier approximated eigenfunctions are analyzed. Within this framework, two classes of graphons with simple spectral representations are given. Applications to graphon-based control analysis are next presented; in particular, the controllability of systems distributed over very large networks is expressed in terms of the properties of the corresponding graphon dynamical systems. Moreover, spectral analysis based upon real-world network data is presented, which demonstrates that low-dimensional spectral approximations of networks are possible. Finally, an initial, exploratory investigation of the utility of the spectral analysis methodology in graphon systems control to study the control of epidemic spread is presented.

**Keywords:** Large-scale networks, complex networks, graphon control, spectral decompositions, controllability, epidemic spread

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## 1 Introduction

Graphon theory has been developed to study large networks and graph limits [1, 2, 3]. Recently it has been applied to study dynamical systems such as the heat equation [4], coupled oscillators [5] and power network synchronization [6], to analyze network centrality [7], to investigate static and dynamic games [8, 9], and to control large networks of dynamical systems [10, 11, 12, 13, 9]. Graphon theory provides a theoretical tool for the study of arbitrarily large and, in the limit, infinite network systems, and thus enables low-complexity approximate solutions to control problems on such systems [10, 11, 12, 13, 9].

Among these applications, graphon spectral properties are very significant [14]. In fact, the spectral analysis of large-scale systems has been studied since the late 1960s [15] and it plays a key role to the low-complexity control synthesis of such systems [16, 17, 15]. This leads us to the study of spectral representations and approximations of graphons in this paper.

As operators, graphons are Hilbert-Schmidt integral operators and hence are compact. Moreover, the symmetry property of graphons ensures that the Spectral Theorem [18] applies to graphon operators. Topics on Hilbert-Schmidt integral operators and self-adjoint compact operators have been extensively studied in the literature (see e.g. [19, 20, 18]), and the spectral properties of graphons, for instance, are investigated in [21].

This paper studies the control and analysis of graphon systems and their associated networks via the exploitation of their spectral properties. The contributions of this paper include 1) the presentation of the spectral representations of two types of graphons, 2) an analysis of the exact controllability of a class of graphon dynamical systems based on spectral decompositions, 3) the study of network spectral properties based on real-world network data which demonstrates that low-dimensional spectral approximations of networks are possible, and 4) the initial, exploratory investigation of the utility of the spectral analysis methodology in graphon systems control to study the controlled epidemic spread process.

### Notation

$\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{Z}_+$  represent respectively the sets of all real numbers, all positive real numbers and all positive integers.  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  denote respectively inner product and norm. Bold face letters (e.g.  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{u}$ ) are used to represent graphons and functions. Blackboard bold letters (e.g.  $\mathbb{I}$ ,  $\mathbb{A}$ ,  $\mathbb{B}$ ,  $\mathbb{W}$ ) are used to denote linear operators which are not necessarily compact; in particular,  $\mathbb{I}$  denotes the identity operator.

## 2 Graphs and graphons

Network structures can be modeled as graphs. A graph  $G = (V, E)$  is specified by an node set  $V$  and an edge set  $E \subset V \times V$ . It has a representation by the corresponding adjacency matrix  $A = [a_{ij}]$  where the element  $a_{ij}$  is one when there is an edge from node  $i$  to node  $j$ , and zero otherwise. For a weighted graph, the elements of its adjacency matrix represent the corresponding edge weights. Furthermore, if one embeds the adjacency matrix on  $[0, 1]^2$  as a pixel picture where each pixel has a side length  $\frac{1}{|V|}$  with  $|V|$  representing the cardinality of  $V$ , then it gives a function  $\mathbf{A} : [0, 1]^2 \rightarrow [0, 1]$ .

Formally, graphons are defined as bounded symmetric Lebesgue measurable functions  $\mathbf{A} : [0, 1]^2 \rightarrow [0, 1]$ , which can be interpreted as weighted graphs on the node set  $[0, 1]$ . A meaningful convergence with respect to the *cut metric* (see e.g. [3]) is defined for sequences of dense and finite graphs. Graphons are then the limit objects of converging graph sequences. Moreover, graphons can be used as generative models for exchangeable random graphs [22]. These properties make graphons suitable to model extremely large-scale networks. We note that in some papers, for instance [23], the word “graphon” refers to symmetric, integrable functions from  $[0, 1]^2$  to  $\mathbb{R}$ . In this paper, unless stated otherwise, the term “graphon” refers to a bounded symmetric Lebesgue measurable function  $\mathbf{A}_1 : [0, 1]^2 \rightarrow [-1, 1]$ ,

so as to include networks with possibly negative weights, and  $\mathcal{W}_1$  denotes the set of all graphons. Let  $\mathcal{W}_0$  represent the set of all graphons satisfying  $\mathbf{A}_0 : [0, 1]^2 \rightarrow [0, 1]$ .  $\mathcal{W}_0$  and  $\mathcal{W}_1$  are both compact under the cut metric (after identifying points of zero distance) [3].

### 3 Graphon operators

#### 3.1 Graphon operators

Let  $L^2(\Omega)$  be the standard Lebesgue space defined on  $\Omega$  endowed with the 2-norm  $\|x\|_2 = (\int_{\Omega} x(\alpha)^2 d\alpha)^{\frac{1}{2}}$ .  $L^2[0, 1]$  and  $L^2[0, 1]^2$  are defined by specializing  $\Omega$  to be  $[0, 1]$  and  $[0, 1]^2$ , respectively.

**Definition 1** A linear map  $\mathbb{T} : L^2[0, 1] \rightarrow L^2[0, 1]$  is said to be compact if  $\mathbb{T}$  maps the open unit ball in  $L^2[0, 1]$  to a set in  $L^2[0, 1]$  that has compact closure.

**Definition 2 ([3])** A graphon operator  $\mathbb{T}_{\mathbf{A}} : L^2[0, 1] \rightarrow L^2[0, 1]$  is defined by a graphon  $\mathbf{A} \in \mathcal{W}_1$  as follows:

$$[\mathbb{T}_{\mathbf{A}}\mathbf{f}](x) = \int_0^1 \mathbf{A}(x, y)\mathbf{f}(y)dy, \quad \mathbf{f}(\cdot) \in L^2[0, 1]. \quad (1)$$

Clearly, the operator  $\mathbb{T}_{\mathbf{A}}$  is Hermitian (or self-adjoint), since for any  $x, y$  in  $L^2[0, 1]$ ,  $\langle x, \mathbb{T}_{\mathbf{A}}y \rangle = \langle \mathbb{T}_{\mathbf{A}}x, y \rangle$ . Moreover, the graphon operator  $\mathbb{T}_{\mathbf{A}}$  is a linear operator that is bounded, (hence) continuous, and compact [24].

For simplicity of notation we henceforth use the bold face letter (e.g.  $\mathbf{A}$ ,  $\mathbf{B}$ ) to represent both a graphon and its corresponding graphon operator. Let  $\mathbf{A}\mathbf{f}$  denote the function defined by (1).

The graphon operator product is then defined by

$$[\mathbf{A}\mathbf{B}](x, y) = \int_0^1 \mathbf{A}(x, z)\mathbf{B}(z, y)dz, \quad (2)$$

where  $\mathbf{A}, \mathbf{B} \in \mathcal{W}_1$ . Let  $\mathbf{A}\mathbf{B}$  denote the graphon given by the convolution in (2). Consequently, the power  $\mathbf{A}^n$  of an operator  $\mathbf{A} \in \mathcal{W}_1$  is given by

$$\mathbf{A}^n(x, y) = \int_{[0, 1]^n} \mathbf{A}(x, \alpha_1) \cdots \mathbf{A}(\alpha_{n-1}, y) d\alpha_1 \cdots d\alpha_{n-1}$$

with  $\mathbf{A}^n \in \mathcal{W}_1, n \in \mathbb{Z}_+$ .  $\mathbf{A}^0$  is formally defined as the identity operator  $\mathbb{I}$  on functions in  $L^2[0, 1]$ , and hence  $\mathbf{A}^0$  is neither a graphon nor a compact operator. Furthermore,  $e^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k$  of a graphon operator  $\mathbf{A}$  is a bounded linear operator from  $L^2[0, 1]$  to  $L^2[0, 1]$ .

#### 3.2 Spectral representations

Denote the operator norm for a linear operator  $\mathbb{T}$  on  $L^2[0, 1]$  as

$$\|\mathbb{T}\|_{\text{op}} = \sup_{\mathbf{f} \in L^2[0, 1], \|\mathbf{f}\|_2=1} \|\mathbb{T}\mathbf{f}\|_2. \quad (3)$$

Following [19], we define *kernel space*, *spectrum*, *eigenvalue* and *eigenfunction* below.

**Definition 3** Define the kernel space (or null space) for a linear operator  $\mathbb{T}$  on  $L^2[0, 1]$  as :

$$\text{Ker}(\mathbb{T}) := \{x \in L^2[0, 1] : \mathbb{T}x = 0\}.$$

**Definition 4** The spectrum  $\sigma(\mathbb{T})$  of a linear bounded operator  $\mathbb{T}$  on  $L^2[0, 1]$  is the set of all (complex or real) scalars  $\lambda$  such that  $\mathbb{T} - \lambda\mathbb{I}$  is not invertible, where  $\mathbb{I}$  is the identity operator. Thus  $\lambda \in \sigma(\mathbb{T})$  if and only if at least one of the following two statements is true:

(i) The range of  $\mathbb{T} - \lambda\mathbb{I}$  is not all of  $L^2[0, 1]$ , i.e.,  $\mathbb{T} - \lambda\mathbb{I}$  is not onto.

(ii)  $\mathbb{T} - \lambda\mathbb{I}$  is not one-to-one.

If (ii) holds,  $\lambda$  is said to be an eigenvalue of  $\mathbb{T}$ ; the corresponding eigenspace is  $\text{Ker}(\mathbb{T} - \lambda\mathbb{I})$ ; each  $x \in \text{Ker}(\mathbb{T} - \lambda\mathbb{I})$  (except  $x = 0$ ) is an eigenfunction of  $\mathbb{T}$ ; it satisfies the equation  $\mathbb{T}x = \lambda x$ .

**Lemma 1 ([12])** For any graphon  $\mathbf{A}$  or any function  $\mathbf{A}$  in  $L^2[0, 1]^2$ ,

$$\|\mathbf{A}\|_{\text{op}} \leq \|\mathbf{A}\|_2,$$

where  $\|\mathbf{A}\|_2 = \left( \int_0^1 \int_0^1 (\mathbf{A}(x, y))^2 dx dy \right)^{\frac{1}{2}}$ .

Hence a graphon sequence convergences under  $\|\cdot\|_2$  implies it convergences under  $\|\cdot\|_{\text{op}}$ . Furthermore, the following inequalities hold [25, 8]:

$$\|\mathbf{A}\|_{\square} \leq \|\mathbf{A}\|_{\text{op}} \leq \sqrt{8\|\mathbf{A}\|_{\square}}, \quad (4)$$

where the cut norm of a graphon  $\mathbf{A} \in \mathcal{W}_1$  is defined as

$$\|\mathbf{A}\|_{\square} = \sup_{M, T \subset [0, 1]} \left| \int_{M \times T} \mathbf{A}(x, y) dx dy \right|. \quad (5)$$

The *cut metric* between two graphons  $\mathbf{V}$  and  $\mathbf{W}$  is then given by

$$\delta_{\square}(\mathbf{W}, \mathbf{V}) = \inf_{\phi \in S_{[0, 1]}} \|\mathbf{W}^{\phi} - \mathbf{V}\|_{\square}, \quad (6)$$

where  $\mathbf{W}^{\phi}(x, y) = \mathbf{W}(\phi(x), \phi(y))$  and  $S_{[0, 1]}$  denotes the set of measure preserving bijections from  $[0, 1]$  to  $[0, 1]$ . See [3, 25] for more details on different norms.

**Proposition 1** Consider the graphon operator  $\mathbf{A}$  corresponding to a graphon  $\mathbf{A} \in \mathcal{W}_1$ . Then there is a set  $\{\mathbf{f}_{\ell}\}$  consisting of a countable number of orthonormal elements in  $L^2[0, 1]$  such that the elements  $\mathbf{f}_{\ell}$  are eigenfunctions to the eigenvalues  $\lambda_{\ell} \in \mathbb{R}$  ordered as follows:

$$\|\mathbf{A}\|_{\text{op}} = |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq 0,$$

If the set  $\{\mathbf{f}_{\ell}\}$  is infinite, we have the asymptotic behavior  $\lim_{\ell \rightarrow \infty} \lambda_{\ell} = 0$ . Furthermore, for any  $\mathbf{u} \in L^2[0, 1]$ ,  $\mathbf{A}\mathbf{u}$  has representations as:

$$\mathbf{A}\mathbf{u} = \sum_{\ell=1}^{\infty} \lambda_{\ell} \langle \mathbf{f}_{\ell}, \mathbf{u} \rangle \mathbf{f}_{\ell} \quad \text{and} \quad \langle \mathbf{u}, \mathbf{A}\mathbf{u} \rangle = \sum_{\ell=1}^{\infty} \lambda_{\ell} |\langle \mathbf{f}_{\ell}, \mathbf{u} \rangle|^2.$$

This result is a special case of [18, Theorem 7.3], where the compact Hermitian operators there are specialized to graphon operators.

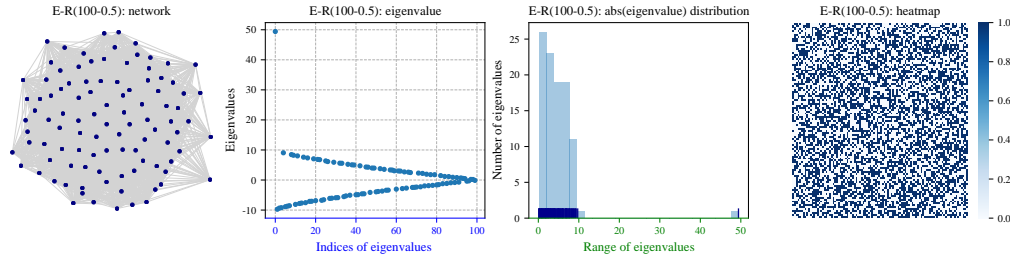
### 3.3 Convergence of eigenvalues

For a graphon  $\mathbf{A}$ , the eigenvalues form two sequences  $\mu_1(\mathbf{A}) \geq \mu_2(\mathbf{A}) \geq \dots \geq 0$  and  $\mu'_1(\mathbf{A}) \leq \mu'_2(\mathbf{A}) \leq \dots \leq 0$  converging to zero, where  $\mu_i(\mathbf{A})$  and  $\mu'_i(\mathbf{A})$  denote respectively the  $i^{\text{th}}$  non-negative eigenvalue and the  $i^{\text{th}}$  non-positive eigenvalue.

**Theorem 1 ([2])** Let  $\{\mathbf{A}_i\}_{i=1}^{\infty}$  be a sequence of uniformly bounded graphons, converging in the cut metric to a graphon  $\mathbf{A}$ . Then for every  $i \geq 1$ ,

$$\mu_i(\mathbf{A}_n) \rightarrow \mu_i(\mathbf{A}) \quad \text{and} \quad \mu'_i(\mathbf{A}_n) \rightarrow \mu'_i(\mathbf{A}) \quad \text{as } n \rightarrow \infty.$$

This implies that if a sequence of graphs converges in the cut metric [3] to a graphon limit with a simple spectral characterization by a few non-zero eigenvalues, then the sequence of graphs admits simple low-dimensional spectral approximations. Furthermore, if the graphs in the sequence are increasing in size, then the low-dimensional approximations perform better as the networks increase in size. This can be illustrated by the sequence of random graphs generated by the Erdős-Rényi model in Figure 1. For general random graphs generated by dense low-rank models, reasonable low-rank approximations exist [26].



**Figure 1: The eigenvalue distribution of a graph with 100 nodes in a convergent sequence of random graphs to the graphon limit  $W(x, y) = 0.5$ . Notice that the eigenvalues accumulate at 0 and  $0.5 \times 100$  which correspond to the eigenvalues of the graphon  $W(x, y) = 0.5$ . This feature makes it possible to approximate certain large random networks by low dimensional spectral characterizations.**

## 4 Approximation of graphons

If the eigenvalues and the corresponding eigenfunctions of a graphon are known, one can approximate the graphon by a finite spectral sum. Consider the approximation of a graphon  $\mathbf{A}$  by  $\mathbf{A}_m(x, y) = \sum_{\ell=1}^m \lambda_{\ell} \mathbf{f}_{\ell}(x) \mathbf{f}_{\ell}(y)$ . Then the mean square error is given as follows:

$$\|\mathbf{A} - \mathbf{A}_m\|_2 = \sqrt{\|\mathbf{A}\|_2^2 - \sum_{\ell=1}^m \lambda_{\ell}^2}. \quad (7)$$

Denote  $p_{\ell}(\cdot)$  as a polynomial function and  $p_{\ell}(e^{2\pi i \cdot})$  is used to approximate the  $\ell^{\text{th}}$  eigenfunction  $\mathbf{f}_{\ell}(\cdot)$  of  $\mathbf{A}$  with  $\mathbf{A}(x, y) = \sum_{\ell=1}^{\infty} \lambda_{\ell} \mathbf{f}_{\ell}(x) \mathbf{f}_{\ell}(y)$ . Note that if the polynomial  $p_{\ell}(\cdot)$  permits terms up to infinite order,  $p_{\ell}(e^{2\pi i \cdot})$  is simply the Fourier representation of  $\mathbf{f}_{\ell}(\cdot)$ . Denote the spectral sum with Fourier approximated eigenfunctions as

$$\mathbf{A}_{pm}(\vartheta, \psi) = \sum_{\ell=1}^m \lambda_{\ell} p_{\ell}(e^{2\pi i \vartheta}) p_{\ell}(e^{2\pi i \psi}) \quad (8)$$

There are two levels of approximations: (a) Fourier approximation of eigenfunctions; (b) spectral decomposition approximation of the graphon operator. Hence the approximation error is bounded as follows:

$$\|\mathbf{A} - \mathbf{A}_{pm}\|_2 \leq \|\mathbf{A} - \mathbf{A}_m\|_2 + \|\mathbf{A}_m - \mathbf{A}_{pm}\|_2.$$

Moreover, the approximation error for functions of operators is given in the following.

**Proposition 2 ([13])** *If there exists  $c > 0$  such that  $\|\mathbf{A}\|_2 \leq c$  and  $\|\mathbf{A}_{pm}\|_2 \leq c$ , then*

$$\|\mathbf{A}^n - (\mathbf{A}_{pm})^n\|_2 \leq nc^n \|\mathbf{A} - \mathbf{A}_{pm}\|_2, \quad (9)$$

$$\|e^{\mathbf{A}} - e^{\mathbf{A}_{pm}}\|_{\text{op}} \leq ce^c \|\mathbf{A} - \mathbf{A}_{pm}\|_2. \quad (10)$$

Consider a polynomial  $p_\ell(e^{2\pi i \cdot})$  with highest order  $2n$ , then there exists a matrix  $T_\ell \in \mathbb{R}^{n \times n}$  such that

$$p_\ell(e^{2\pi i \cdot}) = (e_1, e_2, \dots, e_n) T_\ell (e_1, e_2, \dots, e_n)^T$$

with  $e_k = e^{2\pi k i}$ . Since Fourier basis forms a complete basis for  $L^2[0, 1]$  (see e.g. [27]), any function  $\mathbf{f} \in L^2[0, 1]$  can be approximated by finite Fourier series and hence by polynomial functions of  $e^{2\pi i \cdot}$ .

## 5 Graphons with simple spectral representations

### 5.1 Sinusoidal graphons

A *sinusoidal graphon* is defined as any graphon that can be represented by

$$\mathbf{A}(\varphi, \vartheta) = a_0 + \sum_{k=1}^{\infty} b_k \cos(2\pi k(\varphi - \vartheta)), \quad (\varphi, \vartheta) \in [0, 1]^2. \quad (11)$$

Clearly sinusoidal graphons are symmetric and diagonally constant and hence they are suitable to fit Toeplitz matrices [28].

Sinusoidal graphons have simple spectral characterizations. The eigenfunctions of the graphon in (11) are  $L^2[0, 1]$  functions as follows:

$$1, \{\sqrt{2} \cos 2\pi k(\cdot) : k \in \mathbb{Z}_+\}, \{\sqrt{2} \sin 2\pi k(\cdot) : k \in \mathbb{Z}_+\}.$$

The corresponding eigenvalues are:  $a_0, \{\frac{b_k}{2} : k \in \mathbb{Z}_+\}, \{\frac{b_k}{2} : k \in \mathbb{Z}_+\}$ . Moreover, the eigenfunctions form a complete orthonormal basis for  $L^2[0, 1]$  (see [27]).

#### 5.1.1 Operation on $L^2[0, 1]$ functions

Consider a function  $\mathbf{x} \in L^2[0, 1]$  represented by Fourier series as

$$\mathbf{x}(\vartheta) = K_0 + \sum_{k=1}^{\infty} (\alpha_k \sin(\vartheta) + \beta_k \cos(\vartheta)), \quad \vartheta \in [0, 1].$$

Then operating a sinusoidal graphon  $\mathbf{A}$  on  $\mathbf{x}$  yields

$$[\mathbf{A}\mathbf{x}](\vartheta) = a_0 K_0 + \sum_{k=1}^{\infty} \frac{b_k}{2} (\alpha_k \sin(\vartheta) + \beta_k \cos(\vartheta)), \quad \vartheta \in [0, 1].$$

Similarly, for  $m \in \mathbb{Z}_+$  and  $\vartheta \in [0, 1]$ ,

$$\begin{aligned} [\mathbf{A}^m \mathbf{x}](\vartheta) &= a_0^m K_0 + \sum_{k=1}^{\infty} \left(\frac{b_k}{2}\right)^m (\alpha_k \sin(\vartheta) + \beta_k \cos(\vartheta)), \\ [e^{\mathbf{A}} \mathbf{x}](\vartheta) &= e^{a_0} K_0 + \sum_{k=1}^{\infty} e^{\frac{b_k}{2}} (\alpha_k \sin(\vartheta) + \beta_k \cos(\vartheta)). \end{aligned}$$

#### 5.1.2 Functions of sinusoidal graphons

Power functions of a sinusoidal graphon  $\mathbf{A}$  are given by

$$\mathbf{A}^m(\varphi, \vartheta) = a_0^m + \sum_{k=1}^{\infty} \left(\frac{b_k}{2}\right)^{m-1} \cdot b_k \cos(2\pi k(\varphi - \vartheta)),$$

where  $(\varphi, \vartheta) \in [0, 1]^2, m \in \mathbb{Z}_+$ . Moreover, the time-varying exponential function of a sinusoidal graphon is given by  $e^{\mathbf{A}t} = \mathbb{I} + \mathbf{U}_t$ ,  $t \in \mathbb{R}$ , where  $\mathbf{U}_t$  is defined as follows: for all  $(\varphi, \vartheta) \in [0, 1]^2$

$$\mathbf{U}_t(\varphi, \vartheta) = (e^{a_0 t} - 1) + 2 \sum_{k=1}^{\infty} \left(e^{\frac{b_k}{2} t} - 1\right) \cos 2\pi k(\varphi - \vartheta).$$

We note that, for any  $t \in \mathbb{R}$ ,  $e^{\mathbf{A}t}$  is an element of the graphon unitary operator algebra [11].

## 5.2 Step function graphons

A graphon  $\mathbf{A} \in \mathcal{W}_1$  is a *step function* if there is a partition  $Q = \{Q_1, \dots, Q_N\}$  of  $[0, 1]$  into measurable sets such that  $\mathbf{A}$  is constant on every product set  $Q_i \times Q_j$ . A *uniform partition*  $P^N = \{P_1, P_2, \dots, P_N\}$  of  $[0, 1]$  is given by setting  $P_k = [\frac{k-1}{N}, \frac{k}{N})$ ,  $k \in \{1, N-1\}$  and  $P_N = [\frac{N-1}{N}, 1]$ . In this paper, we focus on step functions with uniform partitions. For general step functions, see [3].

The *piece-wise constant function*  $\mathbb{S}_u$  in  $L^2[0, 1]$  corresponding to  $u \in \mathbb{R}^N$  is defined as

$$\mathbb{S}_u(x) = \sum_{i=1}^N \mathbb{1}_{P_i}(x)u_i, \quad \forall x \in [0, 1] \quad (12)$$

where  $\mathbb{1}_{P_i}(\cdot)$  denotes the indicator function, that is,  $\mathbb{1}_{P_i}(x) = 1$  if  $x \in P_i$  and  $\mathbb{1}_{P_i}(x) = 0$  if  $x \notin P_i$ . Let  $\mathbb{S}_u \cdot \mathbb{S}_v^T$  be given as follows:

$$\mathbb{S}_u \cdot \mathbb{S}_v^T t(x, y) := \sum_{i=1}^N \mathbb{1}_{P_i}(x)u_i \sum_{j=1}^N \mathbb{1}_{P_j}(y)v_j \quad (13)$$

where  $\mathbb{S}_u$  and  $\mathbb{S}_v^T$  share the same uniform partition  $P^N = \{P_1, \dots, P_N\}$ .

**Proposition 3** *If the matrix  $A = [a_{ij}]$  has a spectral decomposition  $A = V\Lambda_d V^T$ , where  $\Lambda_d = \text{diag}(\lambda_1, \dots, \lambda_d)$  and  $V = (v_1, \dots, v_d)$  with  $v_\ell$  representing the normalized eigenvector of  $\lambda_\ell$ , then the step function graphon  $\mathbf{A}$  defined by*

$$\mathbf{A}(x, y) = \sum_{i=1}^N \sum_{j=1}^N \mathbb{1}_{P_i}(x)\mathbb{1}_{P_j}(y)a_{ij}, \quad (x, y) \in [0, 1]^2 \quad (14)$$

has a spectral representation given by

$$\mathbf{A}(x, y) = \sum_{\ell=1}^d \lambda_\ell [\mathbb{S}_{v_\ell} \cdot \mathbb{S}_{v_\ell}^T](x, y), \quad (x, y) \in [0, 1]^2, \quad (15)$$

where the underlying partition  $\{P_1, \dots, P_N\}$  of  $[0, 1]$  is uniform.

**Proof.** For all  $(x, y) \in [0, 1]^2$ ,

$$\begin{aligned} \mathbf{A}(x, y) &= \sum_{i=1}^N \sum_{j=1}^N \mathbb{1}_{P_i}(x)\mathbb{1}_{P_j}(y) \sum_{\ell=1}^d \lambda_\ell v_\ell(i)v_\ell(j) \\ &= \sum_{\ell=1}^d \lambda_\ell \sum_{i=1}^N \mathbb{1}_{P_i}(x)v_\ell(i) \sum_{j=1}^N \mathbb{1}_{P_j}(y)v_\ell(j) \\ &= \sum_{\ell=1}^d \lambda_\ell [\mathbb{S}_{v_\ell} \cdot \mathbb{S}_{v_\ell}^T](x, y), \end{aligned}$$

where  $v_\ell(i)$  denotes the  $i^{\text{th}}$  element of  $v_\ell$ . □

We note that  $\langle \mathbb{S}_{v_\ell}, \mathbb{S}_{v_k} \rangle = 0$ , if  $\ell \neq k$ ;  $\langle \mathbb{S}_{v_\ell}, \mathbb{S}_{v_k} \rangle = \frac{1}{N}$ , if  $\ell = k$  and hence the corresponding eigenvalues for  $\mathbf{A}$  are given by  $\{\frac{1}{N}\lambda_\ell\}_{\ell=1}^d$ .

Piece-wise constant functions in  $L^2[0, 1]$  form eigenfunctions of step function graphons. Since piece-wise constant functions in  $L^2[0, 1]$  form a dense subset of  $L^2[0, 1]$  space, they can also be used to approximate eigenfunctions of general graphons.

## 6 Graphon dynamical systems

Consider a group of linear dynamical subsystems  $\{S_i^N; 1 \leq i \leq N\}$  coupled over an undirected graph  $G_N$ . The subsystem  $S_i^N$  at the node  $i$  of  $G_N$  has interactions with  $S_j^N, 1 \leq j \leq N$ , specified as below:

$$\begin{aligned} \dot{x}_t^i &= \alpha_0 x_t^i + \frac{1}{N} \sum_{j=1}^N a_{ij} x_t^j + \beta_0 u_t^i + \frac{1}{N} \sum_{j=1}^N b_{ij} u_t^j, \\ t &\in [0, T], \quad \alpha_0, \beta_0 \in \mathbb{R}, \quad x_t^i, u_t^i \in \mathbb{R}, \end{aligned} \quad (16)$$

with  $A_N = [a_{ij}]$  and  $B_N = [b_{ij}] \in \mathbb{R}^{N \times N}$  as the symmetric adjacency matrices of  $G_N$  and of the input graph. Let  $x_t = [x_1, \dots, x_N]^T$  and  $u_t = [u_1, \dots, u_N]^T$ .

Define the graphon step functions  $\mathbf{A}^{[N]}$  and  $\mathbf{B}^{[N]}$  that correspond to  $A_N$  and  $B_N$  according to (14), respectively. Next, define the piece-wise constant functions  $\mathbf{x}_t^{[N]}$  and  $\mathbf{u}_t^{[N]}$  that correspond respectively to  $x_t$  and  $u_t$  according to (12). Then the corresponding graphon dynamical system is given by

$$\begin{aligned} \dot{\mathbf{x}}_t^{[N]} &= (\alpha_0 \mathbb{I} + \mathbf{A}^{[N]}) \mathbf{x}_t^{[N]} + (\beta_0 \mathbb{I} + \mathbf{B}^{[N]}) \mathbf{u}_t^{[N]}, \quad t \in [0, T], \\ \alpha_0, \beta_0 &\in \mathbb{R}, \quad \mathbf{x}_t^{[N]}, \mathbf{u}_t^{[N]} \in L_{pwc}^2[0, 1], \quad \mathbf{A}^{[N]}, \mathbf{B}^{[N]} \in \mathcal{W}_1 \end{aligned} \quad (17)$$

where  $L_{pwc}^2[0, 1]$  represents the set of all piece-wise constant functions in  $L^2[0, 1]$ . The trajectories of the system in (16) correspond one-to-one to the trajectories of the system (17).

We formulate the infinite dimensional graphon linear system as follows:

$$\dot{\mathbf{x}}_t = (\alpha_0 \mathbb{I} + \mathbf{A}) \mathbf{x}_t + (\beta_0 \mathbb{I} + \mathbf{B}) \mathbf{u}_t, \quad t \in [0, T], \quad (18)$$

where  $\alpha_0, \beta_0 \in \mathbb{R}$ ,  $\mathbf{A}, \mathbf{B} \in \mathcal{W}_1$ , and  $\mathbf{x}_t, \mathbf{u}_t \in L^2[0, 1]$ .  $\mathbf{x}_t$  and  $\mathbf{u}_t$  represent respectively the system state and the control input at time  $t$ . The space of admissible control is taken to be  $L^2([0, T]; L^2[0, 1])$ , that is, the Banach space of equivalence classes of strongly measurable mappings  $\mathbf{x} : [0, T] \rightarrow L^2[0, 1]$  that are integrable with the norm  $\|\mathbf{x}\|_{L^2([0, T]; L^2[0, 1])} = (\int_0^T \int_0^1 \mathbf{x}_\tau(\alpha)^2 d\alpha d\tau)^{\frac{1}{2}}$ .

Evidently,  $\alpha_0 \mathbb{I} + \mathbf{A}$  and  $\beta_0 \mathbb{I} + \mathbf{B}$  are bounded linear operators on  $L^2[0, 1]$ . Therefore the system model is well defined and has a unique mild solution following [29]. For simplicity, let  $\mathbb{A} = (\alpha_0 \mathbb{I} + \mathbf{A})$  and  $\mathbb{B} = (\beta_0 \mathbb{I} + \mathbf{B})$ . Hence  $\mathbb{A}$  and  $\mathbb{B}$  lie in the graphon unitary operator algebra [12]. Denote the graphon dynamical system in (18) by  $(\mathbb{A}; \mathbb{B})$ .

The system in (18) can represent the limit system for (17) when the underlying step function graphons convergence in the operator sense or the  $L^2[0, 1]^2$  sense [12].

## 7 Controllability analysis based on spectral representations

A graphon  $\mathbf{A}$  is a compact operator, and hence it has a discrete spectrum. Its spectral decomposition is given as follows  $\mathbf{A}(x, y) = \sum_{\ell \in I_\lambda} \lambda_\ell \mathbf{f}_\ell(x) \mathbf{f}_\ell(y)$  for all  $(x, y) \in [0, 1]^2$ . where  $\mathbf{f}_\ell$  is the normalized eigenfunction corresponding to the non-zero eigenvalues  $\lambda_\ell$  and  $I_\lambda$  is the index set for non-zero eigenvalues of  $\mathbf{A}$ , which contains a countable number of elements [3].

For infinite dimensional systems there are two notions of controllability: approximate controllability and exact controllability [30]. We only discuss exact controllability in this paper.

**Definition 5** *A graphon dynamical system  $(\mathbb{A}; \mathbb{B})$  in (18) is exactly controllable in  $L^2[0, 1]$  over the time horizon  $[0, T]$  if the system state can be driven to the origin at time  $T$  from any initial state  $\mathbf{x}_0 \in L^2[0, 1]$ .*

We define the *controllability Gramian operator* as  $\mathbb{W}_T = \int_0^T e^{\mathbb{A}\tau} \mathbb{B} \mathbb{B}^\top e^{\mathbb{A}^\top \tau} d\tau$ . If there exists  $c > 0$  such that, for every  $\lambda \in \sigma(\mathbb{W}_T)$ ,  $|\lambda| \geq c$  holds, then the system is exactly controllable [12] and the minimum control energy  $J = \int_0^T \|\mathbf{u}_t\|_2^2 dt$  (see. e.g. [10]) required to drive the system from state  $\mathbf{x}_0 \in L^2[0, 1]$  to the origin at time  $T$  is given by  $J(\mathbf{x}_0) = \langle e^{\mathbb{A}T} \mathbf{x}_0, \mathbb{W}_T^{-1} e^{\mathbb{A}^\top T} \mathbf{x}_0 \rangle$ .

For any graphon system  $(\mathbb{A}; \mathbb{B})$  with a compact operator  $\mathbb{B}$ , exact controllability cannot be achieved over a finite horizon [31]. If  $\mathbb{B}$  lies in the graphon unitary operator algebra [12], then exact controllability of  $(\mathbb{A}; \mathbb{B})$  in (18) over  $[0, T]$  implies  $\beta_0 \neq 0$ .

In general, it is not obvious how to find the explicit forms of the controllability Gramian operator. However, when  $\mathbf{A}$  and  $\mathbf{B}$  share the same structure, explicit forms are possible.

**Proposition 4** *Assume  $\mathbf{A} \in \mathcal{W}_1$  and  $\mathbf{B} = \sum_{k=1}^d \beta_k \mathbf{A}^k$ . Denote  $\eta_\ell = \sum_{k=0}^d \beta_k \lambda_\ell^k$ . Then the controllability Gramian operator for the system  $(\mathbb{A}, \mathbb{B})$  in (18) is explicitly given by*

$$\begin{aligned} \mathbb{W}_T &= \int_0^T e^{\alpha_0 t} dt \beta_0^2 \mathbb{I} \\ &+ \sum_{\ell \in I_\lambda} \left( (\eta_\ell)^2 \int_0^T e^{2(\alpha_0 + \lambda_\ell)t} dt - \int_0^T e^{\alpha_0 t} dt \beta_0^2 \right) \mathbf{f}_\ell \mathbf{f}_\ell^\top; \end{aligned} \quad (19)$$

furthermore, if  $\beta_0 \neq 0$ , then the inverse of the controllability Gramian operator for  $(\mathbb{A}; \mathbb{B})$  in (18) is explicitly given by

$$\begin{aligned} \mathbb{W}_T^{-1} &= \frac{1}{\int_0^T e^{\alpha_0 t} dt \beta_0^2} \mathbb{I} \\ &- \frac{1}{\int_0^T e^{\alpha_0 t} dt \beta_0^2} \sum_{\ell \in I_\lambda} \frac{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt - T \beta_0^2}{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt} \mathbf{f}_\ell \mathbf{f}_\ell^\top. \end{aligned} \quad (20)$$

**Proof.** Consider any  $\mathbf{z} \in L^2[0, 1]$ . Let  $\check{\mathbf{z}} = \mathbf{z} - \sum_{\ell \in I_\lambda} \langle \mathbf{z}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell$ . Then

$$\begin{aligned} \mathbb{W}_T \mathbf{z} &= \int_0^T e^{\mathbf{A}t} \mathbb{B} \mathbb{B}^\top e^{\mathbf{A}^\top t} dt \left( \check{\mathbf{z}} + \sum_{\ell \in I_\lambda} \langle \mathbf{z}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell \right) \\ &= \int_0^T e^{\mathbf{A}t} \mathbb{B} \mathbb{B}^\top e^{\mathbf{A}^\top t} dt \check{\mathbf{z}} + \sum_{\ell \in I_\lambda} \int_0^T e^{\mathbf{A}t} \mathbb{B} \mathbb{B}^\top e^{\mathbf{A}^\top t} dt \langle \mathbf{z}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell \\ &= \int_0^T e^{2\alpha_0 t} dt \beta_0^2 \check{\mathbf{z}} + \sum_{\ell \in I_\lambda} \int_0^T e^{2(\alpha_0 + \lambda_\ell)t} \eta_\ell^2 dt \langle \mathbf{z}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell \\ &= \int_0^T e^{2\alpha_0 t} dt \beta_0^2 \mathbf{z} \\ &+ \sum_{\ell \in I_\lambda} \left( \eta_\ell^2 \int_0^T e^{2(\alpha_0 + \lambda_\ell)t} dt - \int_0^T e^{2\alpha_0 t} dt \beta_0^2 \right) \langle \mathbf{z}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell. \end{aligned} \quad (21)$$

This yields the equivalent representation in (19).

If  $\beta_0 \neq 0$ ,  $\mathbb{W}_T$  is invertible [12]. Suppose  $\mathbf{u} = \mathbb{W}_T \mathbf{z}$ . To find  $\mathbb{W}_T^{-1}$ , we need to find the operator that maps  $\mathbf{u}$  back to  $\mathbf{z}$ . Taking the inner product with  $\mathbf{f}_\ell$  on both sides of (21) yields:

$$\langle \mathbf{u}, \mathbf{f}_\ell \rangle = \sum_{\ell \in I_\lambda} \left( \eta_\ell^2 \int_0^T e^{2(\alpha_0 + \lambda_\ell)t} dt \right) \langle \mathbf{z}, \mathbf{f}_\ell \rangle. \quad (22)$$

Then by replacing  $\langle \mathbf{z}, \mathbf{f}_\ell \rangle$  in (21) based on (22), we obtain

$$\begin{aligned} \mathbf{z} &= \frac{1}{\int_0^T e^{\alpha_0 t} dt \beta_0^2} \mathbf{u} \\ &- \frac{1}{\int_0^T e^{\alpha_0 t} dt \beta_0^2} \sum_{\ell \in I_\lambda} \frac{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt - T \beta_0^2}{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt} \langle \mathbf{u}, \mathbf{f}_\ell \rangle \mathbf{f}_\ell^\top, \end{aligned} \quad (23)$$

and hence (20) holds.  $\square$

If  $\alpha_0 = 0$ , then the controllability Gramian operator for the system  $(\mathbb{A}; \mathbb{B})$  in (18) is given by

$$\mathbb{W}_T = T\beta_0^2 \mathbb{I} + \sum_{\ell \in I_\lambda} \left( (\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt - T\beta_0^2 \right) \mathbf{f}_\ell \mathbf{f}_\ell^\top; \quad (24)$$

furthermore, if  $\beta_0 \neq 0$ , then the inverse of the controllability Gramian operator for  $(\mathbb{A}; \mathbb{B})$  in (18) is explicitly given by

$$\mathbb{W}_T^{-1} = \frac{1}{T\beta_0^2} \mathbb{I} - \frac{1}{T\beta_0^2} \sum_{\ell \in I_\lambda} \frac{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt - T\beta_0^2}{(\eta_\ell)^2 \int_0^T e^{2\lambda_\ell t} dt} \mathbf{f}_\ell \mathbf{f}_\ell^\top.$$

These then recover the result on exact controllability in [12].

## 8 Spectral analysis of network data

Real world networks are finite in size and can be represented by step function graphons. The corresponding eigenfunctions for non-zero eigenvalues are necessarily  $L^2_{pwc}[0, 1]$  functions. By analyzing these finite networks, we infer possible properties of the limits if such limits exist. In this section, numerical properties of the low-rank approximations to finite networks are analyzed.

The spectral properties of real-world network structures based on an open data set available online [35] are shown in Figure reffig:spectral-approximation. It is observed that most of the eigenvalues of many symmetric networks are distributed around zero and a few eigenvalues are large. This implies that low dimensional approximations of these networks are possible. Notice that for these networks the data set only characterizes interactions among different nodes and hence there are no self-loops in the network structures. This means that the diagonal elements of the corresponding adjacency matrices are zeros and hence the trace (i.e. sum of all eigenvalues) of each is zero. Note that only connection structures are captured in the data sets and the underlying dynamical systems need to be investigated in the future.

The spectral approximation by 10% of the most significant eigendirections is given for each network data. As shown in Figure 2, this approximation preserves the patterns of the corresponding graphon. In practice, the threshold for selecting the eigenvalues in this approximation depends on the tolerance of the approximation error. The spectral approximations of these sparsely connected graphs typically give rise to graphs with dense structures since the eigenvectors corresponding to the removed eigenvalues contain many non-zero elements.

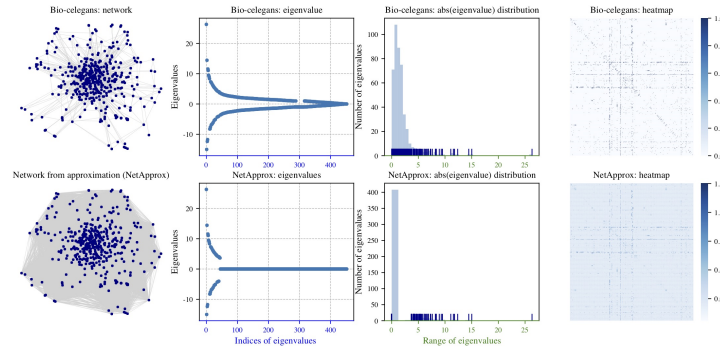
## 9 Controlling epidemic processes on networks based on spectral decomposition

Consider the process of an infectious disease spreading over contact networks where controls via vaccinations and medications are possible. Each node on the network has a state representing the infection level and each node can have a control action to receive medications or vaccinations to reduce the level of the infection. The nodes affect each other through the underlying contact network and their actions influence each other. The objective is to optimally reduce the level of the infection in the whole network with some cost constraints.

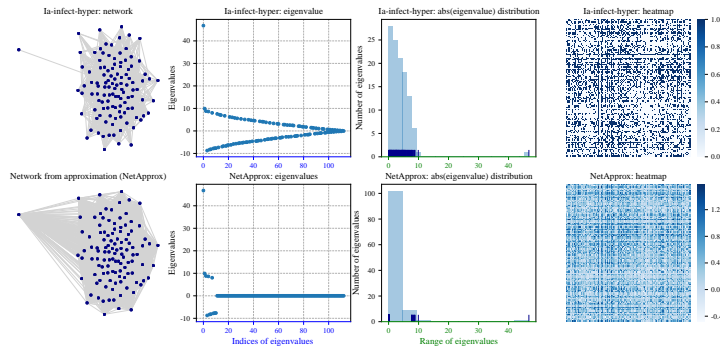
Based on the meta-population model for the epidemic process [37], the dynamics of the spread over a network is described by:

$$\dot{p}_t^i = -\alpha p_t^i + \eta \sum_{j=1}^N a_{ij} p_t^j (1 - p_t^i), \quad t \in \mathbb{R}_+, \quad (25)$$

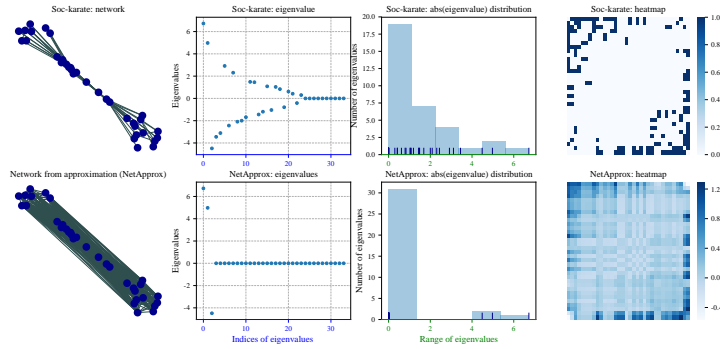
where  $p_t^i \in [0, 1]$  is interpreted as the fraction of the  $i^{\text{th}}$  subpopulation that is infected,  $\alpha$  is the recovering rate,  $\eta$  is the infection strength and  $N$  is the number of subpopulations (i.e. communities,



(a) C-elegans metabolic network where edges represent metabolic reactions between substrates [32].



(b) Infectious contact network [33].



(c) Zachary karate club network [34].

**Figure 2: Structures, eigenvalues, distributions of the absolute values of eigenvalues, and the corresponding step function graphons for networks and their spectral approximations by 10% of the most significant eigendirections.**

cities). The origin is a global asymptotic stable equilibrium if and only if  $\alpha \geq \eta \lambda_{\max}(A)$  [37]. If the underlying networks grows (i.e. more nodes are connected to the networks), the limit spectral properties of the networks would be useful to estimate the  $\lambda_{\max}(A)$ . It is also important to recognize the network eigendirections with significant eigenvalues and act in those directions.

Notice  $(1 - p_t^i) \leq 1$  is close to 1 when  $p_t^i$  is close to zero. Under normal conditions  $p_t^i \in [0, 1]$  should be small. We linearized the model around the origin and study the problem of regulating the state of the following system to the origin:

$$\dot{p}_t^i = -\alpha_0 p_t^i + \bar{\eta} \frac{1}{N} \sum_{j=1}^N a_{ij} p_t^j + \beta_0 u_t^i, \quad t \in [0, T] \quad (26)$$

where  $u_t^i$  represents the control actions (via vaccinations or medications) at each node and  $\bar{\eta} = \eta N$ .

For finite networks this scaling by  $\frac{1}{N}$  makes no difference since it can be absorbed into  $a_{ij}$  or  $\eta$ . Since the averaged strength of the connection between any given group and the other groups in the network will adjust as new groups join the network, and since the model must retain the relative strengths of the various groups, the infection strength  $\eta$  is decreased at a rate  $1/N$  as the model size increases.

Let the quadratic cost associated with this problem be given by

$$J(u) = \sum_{i=1}^N \int_0^T (q_t(p_t^i)^2 + (u_t^i)^2 + (u_t^i - \frac{1}{N} \sum_{j=1}^N a_{ij} u_t^j)^2) dt + q_T(p_T^i)^2, \quad (27)$$

where  $q_t, q_T \geq 0$ . We want to reduce the infection level, limit the actions cost and make sure that subpopulations on the network receive almost the same amount of resources as their weighted neighbors for equity purposes. This forms a good example to illustrate the polynomial structures appearing in the cost functions in [38, 14].

The adjacency matrix  $A = [a_{ij}]$  of an undirected contact network has a spectral decomposition  $A = \sum_{\ell=1}^L \mu_\ell v_\ell v_\ell^\top$ , with  $v_\ell$  representing the normalized eigenvector of the non-zero eigenvalue  $\mu_\ell$ . The number  $L$  of non-zero eigenvalues can be much smaller than  $N$ . The optimal solution [38] to be executed at community  $i$  is given by

$$u_t^i = \frac{\beta_0}{2} \check{\Pi}_t p_t^i + \sum_{\ell=1}^L \left( \frac{\beta_0 \Pi_t^\ell}{(\frac{\mu_\ell}{N})^2 - 2\frac{\mu_\ell}{N} + 2} - \frac{\beta_0 \check{\Pi}_t}{2} \right) p_t^\top v_\ell v_\ell(i),$$

where  $\check{\Pi}_t$  and  $\Pi_t^\ell$  are given by

$$\begin{aligned} -\dot{\check{\Pi}}_t &= -2\alpha_0 \check{\Pi}_t - \frac{\beta_0^2 (\check{\Pi}_t)^2}{2} + q_t, \\ -\dot{\Pi}_t^\ell &= -2(\alpha_0 - \frac{\bar{\eta} \mu_\ell}{N}) \Pi_t^\ell - \frac{\beta_0^2 (\Pi_t^\ell)^2}{(\frac{\mu_\ell}{N})^2 - 2\frac{\mu_\ell}{N} + 2} + q_t, \end{aligned} \quad (28)$$

with  $\check{\Pi}_T = \Pi_T^\ell = q_T$ , and  $p_t = [p_t^1, \dots, p_t^N]^\top$ .

If the graphon limit  $\mathbf{A}$  exists, then the corresponding regulation problem for the graphon dynamical system, is given as follows: for a representative node  $\gamma \in [\underline{\gamma}, \bar{\gamma}] \subset [0, 1]$  on the graphon

$$\begin{aligned} \dot{\mathbf{p}}_t(\gamma) &= -\alpha_0 \mathbf{p}_t(\gamma) + \bar{\eta} \int_0^1 \mathbf{A}(\gamma, \rho) \mathbf{p}_t(\rho) d\rho + \beta_0 \mathbf{u}_t(\gamma), \\ J(\mathbf{u}) &= \int_0^T (\|\mathbf{p}_t\|_2^2 + \|\mathbf{u}_t\|_2^2 + \|(\mathbb{I} - \mathbf{A})\mathbf{u}_t\|_2^2) dt + \|\mathbf{p}_T\|_2^2 \end{aligned} \quad (29)$$

where  $\mathbf{p}_t, \mathbf{u}_t \in L^2[0, 1]$ . Let  $\mathbf{A} = \sum_{\ell=1}^\infty \lambda_\ell \mathbf{f}_\ell \mathbf{f}_\ell^\top$ . Then the optimal solution [14] is given by

$$\mathbf{u}_t(\gamma) = \frac{\beta_0}{2} \check{\Pi}_t \mathbf{p}_t(\gamma) + \sum_{\ell=1}^\infty \left( \frac{\beta_0 \Pi_t^\ell}{2 - 2\lambda_\ell + \lambda_\ell^2} - \frac{\beta_0 \check{\Pi}_t}{2} \right) \langle \mathbf{p}_t, \mathbf{f}_\ell \rangle \mathbf{f}_\ell(\gamma) \quad (30)$$

where

$$\begin{aligned} -\dot{\check{\Pi}}_t &= -2\alpha_0 \check{\Pi}_t - \frac{\beta_0^2 (\check{\Pi}_t)^2}{2} + q_t, \\ -\dot{\Pi}_t^\ell &= -2(\alpha_0 - \bar{\eta} \lambda_\ell) \Pi_t^\ell - \frac{\beta_0^2 (\Pi_t^\ell)^2}{(\lambda_\ell)^2 - 2\lambda_\ell + 2} + q_t, \end{aligned} \quad (31)$$

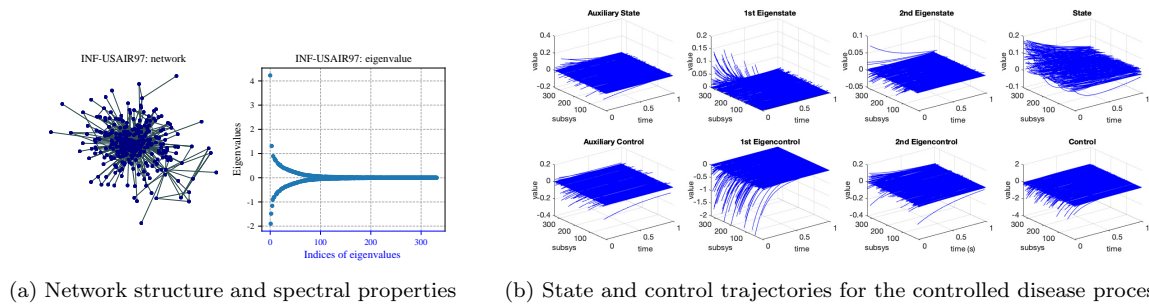
with  $\check{\Pi}_T = \Pi_T^\ell = q_T$ .

Note that the only difference between (28) and (31) lies in the eigenvalues  $\mu_\ell/N$  and  $\lambda_\ell$ . This is consistent with the discussion on the eigenvalues of step function graphons following Proposition 3.

The global state aggregates (i.e. projections of states in different eigendirections) instead of local neighboring states are used in the local control of each subsystem. If  $\mathbf{A}$  is an approximation of the underlying network, then the corresponding approximate control can be applied. See [12, 14] for more related discussions.

## 10 Numerical Illustration

A numerical simulation is carried out for the controlled epidemic process with results shown in Figure 3. Each node represents a city. The contact network among cities is represented by the air traffic frequencies among the corresponding city airports. The network data set USAir97 in [36] is used in the simulation, which represents a network of 332 American airports in 1997. Note that this example is only for the purpose of illustration, and many other network factors and population sizes should be included to have a better representation of coupling strength among cities. The parameters for the numerical example are:  $\alpha_0 = -0.5$ ,  $\beta_0 = 1$ ,  $\eta = 1.5$ ,  $q_t = 2$ ,  $q_T = 4$ ,  $T = 1$  time unit. In Figure 3, the eigenstate and the eigencontrol in the  $v_\ell$  direction (i.e. the projections of states and controls into the  $v_\ell$  eigendirection) are given by  $p_t^\top v_\ell v_\ell$  and  $u_t^\top v_\ell v_\ell$ , respectively; the auxiliary states and controls are given by  $\check{p}_t = p_t - \sum_{\ell=1}^L p_t^\top v_\ell v_\ell$  and  $\check{u}_t = u_t - \sum_{\ell=1}^L u_t^\top v_\ell v_\ell$ , respectively.



**Figure 3: The simulation of the controlled disease process with couplings represented by the contact network corresponding to USAir97 [36]. In (b), the eigenstates and eigencontrols in only the two most significant eigendirections are shown due to space limitations.**

## 11 Conclusion

For controlling infinite dimensional systems and large-scale network systems, spectral properties are extremely useful in both control analysis and synthesis. Many important topics in the control of graphon dynamical systems still require further investigation. First, systematic procedures for specifying graphon labellings and relabellings (in general, measure preserving transformations, see [3]) need to be investigated. Furthermore, it is of great interest to develop low-complexity control solutions to control large-scale networks with nonlinear local dynamics. We note that the study of controllability in this work is limited to the class of graphon systems where  $\mathbb{A}$  and  $\mathbb{B}$  share the same spectral structure. Hence further investigation is required to extend this study more general graphon dynamical systems. Finally, future investigations need to include the control analysis and synthesis of network systems coupled over exchangeable random graphs.

## References

- [1] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztegombi, Convergent sequences of dense graphs i: Subgraph frequencies, metric properties and testing, *Advances in Mathematics*, 219(6):1801–1851, 2008.
- [2] —, Convergent sequences of dense graphs ii. multiway cuts and statistical physics, *Annals of Mathematics*, 176(1):151–219, 2012.
- [3] L. Lovász, *Large Networks and Graph Limits*. American Mathematical Soc., 2012, vol. 60.

- [4] G. S. Medvedev, The nonlinear heat equation on dense graphs and graph limits, *SIAM Journal on Mathematical Analysis*, 46(4):2743–2766, 2014.
- [5] H. Chiba and G. S. Medvedev, The mean field analysis of the Kuramoto model on graphs I. the mean field equation and transition point formulas, *Discrete and Continuous Dynamical Systems-Series A*, 39(1):131–155, 2019.
- [6] C. Kuehn and S. Throm, Power network dynamics on graphons, arXiv preprint arXiv:1807.03573, 2018.
- [7] M. Avella-Medina, F. Parise, M. Schaub, and S. Segarra, Centrality measures for graphons: Accounting for uncertainty in networks, *IEEE Trans. Netw. Sci. Eng.*, 2018.
- [8] F. Parise and A. Ozdaglar, Graphon games, arXiv preprint arXiv:1802.00080, 2018.
- [9] P. E. Caines and M. Huang, Graphon mean field games and the GMFG equations, in *Proc. Conf. Decision and Control*, December 2018, pp. 4129–4134.
- [10] S. Gao and P. E. Caines, The control of arbitrary size networks of linear systems via graphon limits: An initial investigation, in *Proc. Conf. Decision and Control*, Melbourne, Australia, December 2017, pp. 1052–1057.
- [11] —, Graphon linear quadratic regulation of large-scale networks of linear systems, in *Proc. Conf. Decision and Control*, Miami Beach, FL, USA, December 2018, pp. 5892–5897.
- [12] —, Graphon control of large-scale networks of linear systems, arXiv preprint arXiv:1807.03412, 2018.
- [13] S. Gao, Graphon control theory for linear systems on complex networks and related topics, Ph.D. dissertation, McGill University, 2019.
- [14] S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Accepted by the 58th IEEE Conference on Decision and Control (CDC), December 2019.
- [15] M. Aoki, Control of large-scale dynamic systems by aggregation, *IEEE Trans. Autom. Control*, 13(3):246–253, 1968.
- [16] J. Swigart and S. Lall, Optimal controller synthesis for decentralized systems over graphs via spectral factorization, *IEEE Trans. Autom. Control*, 59(9):2311–2323, 2014.
- [17] F. M. Callier and J. Winkin, LQ-optimal control of infinite-dimensional systems by spectral factorization, *Automatica*, 28(4):757–770, 1992.
- [18] F. Sauvigny, *Partial Differential Equations 2: Functional Analytic Methods*. Springer Science & Business Media, 2012.
- [19] W. Rudin, *Functional Analysis*, ser. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
- [20] B. J Mercer, Functions of positive and negative type, and their connection the theory of integral equations, *Phil. Trans. R. Soc. Lond. A*, 209(441-458):415–446, 1909.
- [21] B. Szegedy, Limits of kernel operators and the spectral regularity lemma, *European Journal of Combinatorics*, 32(7):1156–1167, 2011.
- [22] P. Orbanz and D. M. Roy, Bayesian models of graphs, arrays and other exchangeable random structures, *IEEE Trans. Pattern Anal. Mach. Intell.*, 37(2):437–461, 2015.
- [23] C. Borgs, J. T. Chayes, H. Cohn, and Y. Zhao, An  $L^p$  theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions, arXiv preprint arXiv:1401.2906, 2014.
- [24] J. B. Conway, *A Course in Functional Analysis*, 2nd ed. Springer-Verlag New York, 1990, vol. 96.
- [25] S. Janson, Graphons, cut norm and distance, couplings and rearrangements, arXiv preprint arXiv:1009.2376, 2010.
- [26] F. Chung and M. Radcliffe, On the spectra of general random graphs, *the electronic journal of combinatorics*, 18(1):215, 2011.
- [27] K. Saxe, *Beginning Functional Analysis*. Springer Science & Business Media, 2013.
- [28] R. M. Gray et al., Toeplitz and circulant matrices: A review, *Foundations and Trends in Communications and Information Theory*, 2(3):155–239, 2006.
- [29] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*. Springer Science & Business Media, 2007.
- [30] M. Vidyasagar, On the controllability of infinite-dimensional linear systems, *Journal of Optimization Theory and Applications*, 6(2):171–173, 1970.
- [31] R. Triggiani, On the lack of exact controllability for mild solutions in banach spaces, *Journal of Mathematical Analysis and Applications*, 50(2):438–446, 1975.

- 
- [32] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, The large-scale organization of metabolic networks, *Nature*, 407(6804):651, 2000.
  - [33] SocioPatterns, Infectious contact networks, <http://www.sociopatterns.org/datasets/>. Accessed 09/12/12.
  - [34] W. W. Zachary, An information flow model for conflict and fission in small groups, *Journal of Anthropological Research*, 33(4):452–473, 1977.
  - [35] R. A. Rossi and N. K. Ahmed, The network data repository with interactive graph analytics and visualization, in *Proceedings of the 29th AAAI Conference on Artificial Intelligence*, 2015. [Online]. Available: <http://networkrepository.com>.
  - [36] V. Batagelj and A. Mrvar, Pajek datasets, 2006, <http://vlado.fmf.uni-lj.si/pub/networks/data>.
  - [37] C. Nowzari, V. M. Preciado, and G. J. Pappas, Analysis and control of epidemics: A survey of spreading processes on complex networks, *IEEE Control Syst. Mag.*, 36(1):26–46, 2016.
  - [38] S. Gao and A. Mahajan, Networked control of coupled subsystems: Spectral decomposition and low-dimensional solutions, Accepted by the 58th IEEE Conference on Decision and Control (CDC), December 2019.