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G-2015-14

February 2015

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La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2015.

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The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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A rolling horizon approach for stochastic mixed complementarity problems with endogenous learning: Application to natural gas markets

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February 2015

**Les Cahiers du GERAD
G-2015-14**

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Abstract: In this paper we present a new rolling horizon approach for solving stochastic mixed complementarity problems (MCPs). Such a scheme allows for decision-dependent probabilities, endogenous learning and closer realism to energy market MCPs. We also introduce a new concept, the Value of the Rolling Horizon (VoRH) to measure the closeness of different rolling horizon schemes to a perfect foresight benchmark. Lastly, theoretical and numerical results are presented with an application in natural gas markets to demonstrate the value of the proposed approach.

Key Words: Rolling horizon, stochastic mixed complementarity problem, natural gas market, game theory.

Acknowledgments: M.T. Devine is funded under Science Foundation Ireland award 09/SRC/E1780. Part of the work of S.A. Gabriel was done during a stay at GERAD as Trottier Senior Visiting Professor for 2014-2015, Institut de l'énergie Trottier, Polytechnique Montréal.

1 Introduction

Natural gas and electric power market modeling is a research area that has been fairly active in the last 25 years. The impetus for new and improved modeling has been both scientific as well as policy/market-driven. In the latter case, the liberalization of energy markets in both Europe and North America, combined with programs to promote renewable energy have led to development of models to better understand the potential impacts of such policies. The older, perfect-competition paradigm of models that maximize social welfare or one overall market objective function, have been replaced to some extent by more game theoretic ones (e.g., Nash-Cournot models) that depict often non-cooperating market participants in a more realistic way. An often-used format is the mixed complementarity problem (MCP) which generalizes single-optimization models via their associated Karush-Kuhn-Tucker (KKT) conditions, multiple optimization problems by market participants, and equilibrium problems not directly traceable to one or more optimization problems [22]. Formally stated, having a function $F : R^n \times R^m \rightarrow R^n \times R^m$, MCP(F) is to find a vector $(x, y) \in R^n \times R^m$ such that:

$$0 \leq F_x(x, y) \perp x \geq 0 \quad (1a)$$

$$0 = F_y(x, y), y \text{ free} \quad (1b)$$

Here x represents the nonnegatively constrained variables with associated nonnegative F components denoted F_x and complementarity between them given by \perp (i.e., $F_x(x, y)^T x = 0$). Also, y are the free variables with associated components F_y that must equal zero exactly [14, 22].

The current paper has several important contributions that are described more fully in the next subsections and sections. Namely: 1. the concept of a rolling horizon MCP as opposed to the more traditional perfect foresight approach, 2. the ability of rolling horizon MCPs to encode decision-dependent scenario trees (i.e., endogenous probabilities) combined with learning by the noncooperative players, 3. a new concept called the Value of the Rolling Horizon (VoRH) which measures the benefit of rolling horizon vs. perfect foresight and to some extent is related to online learning, and 4. some theoretical and numerical results to support the rolling horizon stochastic MCP concept for a natural gas network.

1.1 Rolling horizon stochastic MCP vs. perfect foresight MCP

To date, the majority of natural gas and electric power market models assume perfect foresight of the time horizon being considered which is less than realistic since market planners don't have perfect information for the entire time horizon. In fact, energy decisions are often made under uncertainty with hedging of worst-case scenarios. With deterministic, perfect-foresight models, such things as costs and other parameters are assumed to be known with 100% certainty. As such, these models while instructive to serve as base cases, are less realistic than ones that allow for stochastic elements and/or some rolling horizon foresight, more in line with the way markets work.

In the former case, there have been a number of stochastic MCPs. One problem with stochastic MCPs is that an already hard problem to solve (MCP) when large-scale, becomes prohibitively harder when even a small number of scenarios is considered. Two common approaches are to use scenario reduction [48, 38, 28, 23] or some sort of other strategy for stochastic equilibrium problems. For example, Haurie et al. [34] proposed a stochastic dynamic Nash-Cournot model with application in the European gas market. Ventosa et al. [53] used a traditional Stochastic Dynamic Programming (SDP) methodology to address the long term hydrothermal coordination of a generation company operating in a competitive market and at each SDP stage are stated as an MCP for a Cournot market equilibrium to represent the electricity market equilibrium. Dewolf and Smeers [12] built a stochastic Stackelberg game for the European gas market. Luna et al. [45] proposed two models for stochastic equilibrium: one based on the variational equilibrium of a generalized Nash game, and the other on the mixed complementarity formulation. The models differ in how the agents interpret their own actions on the market. Cabero et al. [7] developed a Benders approach for linear complementarity problems (LCP) for risk management in the Spanish power market. Demand, fuel prices and water inflow in the reservoirs are uncertain. Their work considered the master problem that determined output quantities and acceptable risk-levels (CVaR) in an oligopolistic setting among the producers while the subproblem

minimized cost for each producer to generate power. Gabriel and Fuller [23] developed a decomposition method for general stochastic MCP and applied it to an electricity market model with stochastic demand with consideration of market power aspects addressed in the subproblems. Lastly, Shanbhag et al. [51] considered a stochastic multi-leader, multi-follower equilibrium problem where the players competed in both the forward and spot markets in successive periods. A stochastic MCP was used to find an appropriate equilibrium and results for a power network were presented.

An alternative approach, and one of the contributions of this paper, is to use a rolling horizon, stochastic MCP in which only a subset of time periods (e.g., four future quarters of a year) are considered at a time. Once roll r has been solved for using H time periods in total including the current one, in the next roll $r + 1$, roll r 's decision are fixed as parameters. In this way, the associated scenario tree is greatly reduced leading to shorter solution times and also the behavior of the market participants more closely matches how decisions are made in practice and captures unexpected events more accurately. Rolling horizon optimization is not new (see for example, [49, 15, 6, 44, 2]) but no so often used in the energy literature. Several recent examples in energy system optimization are Tuohy et al. [52] for the Irish power sector, Guigues et. al. [32] who proposed a risk-average multistage stochastic program with robust rolling horizon approach to LNG (liquefied natural gas) contracts with a cancelation option and Devine et al. [11] who studied the UK natural gas market. By contrast, rolling horizon MCPs in energy or other disciplines is an unstudied area and thus the current paper offers something new in that domain.

It is important to note that the notion of perfect foresight vs. rolling horizons are not the same as including/excluding uncertainty. In perfect foresight (also called “open loop” equilibrium in the economics literature [18, 1]), the decisions for all the time periods are considered at one time by all the players. This is in contrast to the rolling horizon approach in which decisions are taken one time period at a time for all players with some sort of updating (potentially) of the parameters between these time periods. In addition, for a rolling horizon approach (also called “closed loop” in the economics literature [18]), it is understood that all decisions by all the players in prior periods are available. In both these cases the models can be deterministic. By contrast, a stochastic equilibrium involves either of these two notions: perfect foresight or rolling horizon but has some elements of the data uncertain and in the case of this paper, described by finite scenario trees.

1.2 Endogenous probabilities and learning

Besides the above-mentioned computational and realism improvements that come along with rolling horizon equilibrium problems (MCPs), there is another advantage that to our knowledge has not been well studied. More specifically, as each roll is a separate solving of an MCP, there is the opportunity to adjust inputs in between these rolls. For example, the new scenario tree for the next roll can be endogenously changed based on a solution from the previous roll so that the model has endogenous probabilities. In the case of optimization, decision-dependent uncertainty (endogenous probability) in stochastic programs gives rise to non-convex problems because the recourse model includes a probability multiplied by decision variables. Some relevant work that involves decision-dependent probabilities includes [47, 54, 39], and [40]. Pflug [47] applied a Robbins-Monro procedure for the optimization of simulated Markovian processes. Viswanath et al. [54] adjusted scenario probabilities based on an MILP and distribution selection with binary variables. Held and Woodruff [39] proposed the network interdiction problem where endogenous uncertainties were attached in the network. Heuristic solution approaches were also shown in this study. Hellemo et al. [40] provided several formulations for modifying probability distributions: 1. Scaling for uniform distribution, 2. Convex combination for discrete distribution, 3. Optimization over parameters of the distribution, and 4. Approximation of distributions.

Our approach is an improvement in the sense that the probabilities are potentially updated after each roll. Thus, despite assuming a finite probability mass function (scenario tree), this approach is quite general and allows for any probability distribution and any updating rule. The input data or scenario tree can be changed based on the previous rolls' solution, but it does not make the problem get more complicated to solve since there are not bilinear terms of the form: probability multiplied by decision variable.

This feature of a decision-dependent scenario tree represents a tremendous benefit to the model as it allows for endogenous learning and thus adds even more realism to the model as well as the potential for prescriptive guidance. Besides updating the scenario tree in between rolls, other data elements such as costs can also be modified. Consequently, the rolling horizon equilibrium perspective allows for endogenous spatial-temporal learning on the part of the market participants whose optimization problems make up the equilibrium problem. To be more specific, if for example the market share in profits for a particular gas producer has substantially decreased from one roll to a future one, the rolling horizon format then allows for lowering of costs by the producer involved to try to recover these profits in the next roll. Another example is adjusting in time period $t = r + 1$ for over- or under-estimating sales in time $t = r$. The temporal aspects consider how profits, production, etc. change over time in some relative way for just the producer's own decision variables. The spatial aspects for instance, could relate to how well the producer fares with respect to other producers in the same or other nodes.

In an abstract sense at roll r , player p is solving an optimization problem of the form:

$$\min_{x_{p,s}^r} \sum_s PROB_{p,r}^s(X^1, \dots, X^{r-1}) f_p(x_{p,s}^r; X^1, \dots, X^{r-1}) \quad (2a)$$

$$s.t. x_{p,s}^r \in S_p(X^1, \dots, X^{r-1}) \quad , \forall s \quad (2b)$$

where s is the index for scenarios, X^t , $t \in \{1, \dots, r-1\}$ are the vectors of solutions from prior rolls for player p and the other players and S_p is the feasible region for player p . Equation (2) shows that not only does the objective function f_p potentially change as a function of the previous roll's solutions X^t , $t \in \{1, \dots, r-1\}$, but so can the scenario tree probabilities $PROB_{p,r}^s(X^1, \dots, X^{r-1})$ and the feasible region $S_p(X^1, \dots, X^{r-1})$ as well. For example, if a particular pipeline is being used to its maximum capacity then, between rolls, the Transportation System Operator (TSO) may decide to increase the capacity of that pipeline, hence altering the feasible region of the problem. Within this rolling horizon format, clearly many possible endogenous learning functions are possible and later in the paper we describe one of them along with numerical results to indicate the richness of this approach.

1.3 Connection to sequential games and online optimization problems, the value of the rolling horizon for a stochastic MCP

From one perspective, this rolling horizon paradigm can be considered as an example of a sequential game [41, 3] where the rounds are the rolls themselves. If the time horizon is the full set of time periods then clearly one recovers the perfect foresight approach as a special case. In a market equilibrium characterized by a single-objective optimization problem without endogenous learning (e.g., maximizing social welfare), one would expect that the perfect foresight approach provides a better objective function value than a rolling horizon one and thus the former serves as some sort of benchmark. This is the benchmarking concept behind online optimization [50] for which different strategies to reoptimize after some new information is learned (e.g., in a vehicle routing problem) are compared against the optimization for which all information in the future is known in advance.

Even when no endogenous learning is included, such a straightforward comparison between an online strategy (relating to each roll in the current context), and the perfect-foresight equilibrium is not so obvious. This is because in the case of MCPs, there is no one objective function for all the equilibrium conditions.

A sometimes-used measure for MCPs which can help in comparing the rolling horizon MCP solutions to the perfect foresight ones is the following. The conditions in the MCP can be restated as finding the zero of the following function where $z = (x^T \ y^T)^T$:

$$H_i^{mid}(z) = z_i - mid(l_i, u_i, z_i - F_i(z)), \forall i \quad (3)$$

where $l_i = 0, u_i = +\infty$ for the nonnegative components (x in (1)) and $l_i = -\infty, u_i = +\infty$ for the other components (y in (1)) and $mid(a, b, c)$ is the median operator for the three scalars a, b, c [25, 21]. Consequently, z is a solution to the MCP if and only if $\|H(z)\| = 0$ for any vector norm $\|\cdot\|$ where $H(z) = (H_i(z), \forall i)$.

To make the notion of the H -function described above more clear, consider the following small example. Find $x \in \mathbb{R}_+, y \in \mathbb{R}$ so that

$$\begin{aligned} 0 &\leq 10x + 2y - 7 \perp x \geq 0 \\ 0 &= 3x + 1y - 2 \quad y \text{ free} \end{aligned} \quad (4)$$

where

$$F_1(x, y) = 10x + 2y - 7 \quad (5a)$$

$$F_2(x, y) = 3x + 1y - 2 \quad (5b)$$

The unique solution to (4) is $(x^*, y^*) = (\frac{3}{4}, -\frac{1}{4})$ giving $H(x^*, y^*) = 0$ and for the non-solution $(x, y) = (1, -4)$, $\|H(1, -4)\|_1 = 4 > 0$. The H function (3) could be used to test the “efficiency” of the rolling horizon equilibrium solution vs. a perfect foresight approach for MCPs. More specifically, the rolling horizon equilibrium solution z^{rh} for each roll, after concatenating them together could then be evaluated against the vector associated with a perfect foresight solution z^{pf} but evaluated using this H function (or other ones) corresponding to the perfect foresight set of inputs. Such an analysis is done later in the paper using a new concept which we call the Value of the Rolling Horizon (VoRH) based on this H -function.

Thus, in summary, the rolling horizon equilibrium problem offers some both computational and modeling realism advantages over more traditional approaches. In the rest of the paper we compare analytically perfect foresight versus rolling horizon perspectives (Section 2), provide a specific natural gas formulation as an example (Section 3), provide selected computational results (Section 4) and conclude the contribution (Section 5).

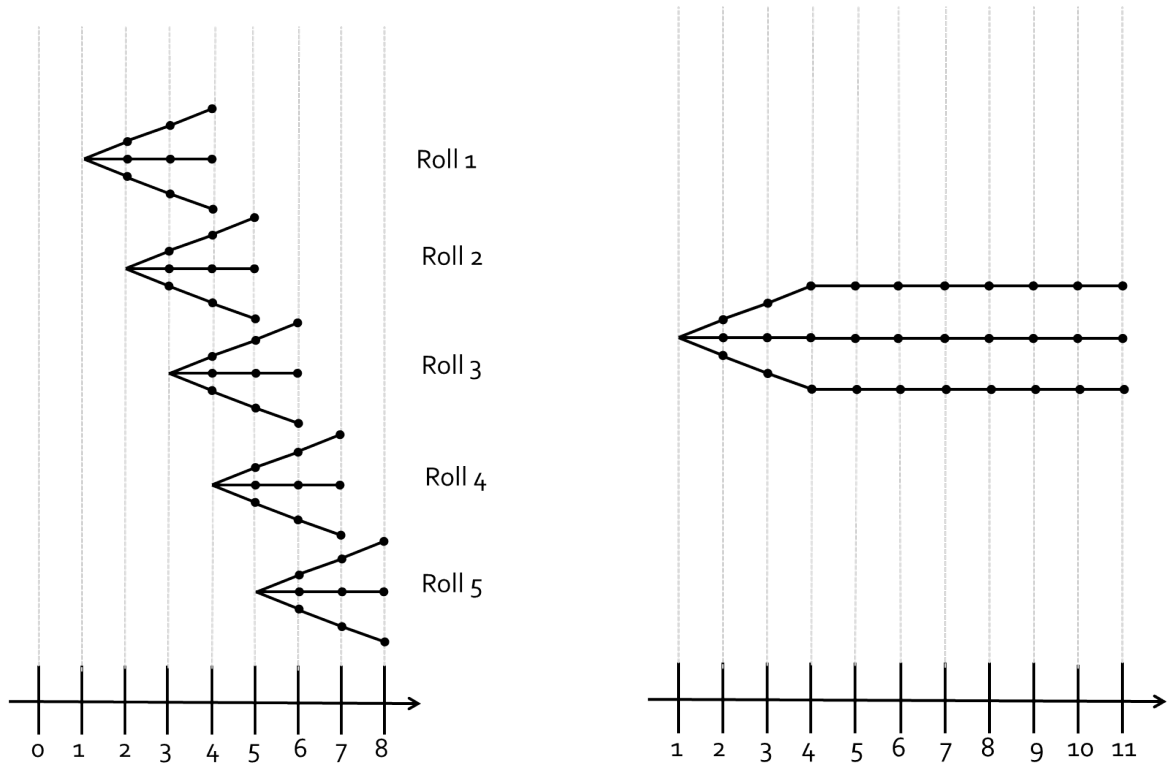
2 Perfect foresight vs. rolling horizon perspective with and without uncertainty

In this section we provide some analytical results and counter-examples to more clearly differentiate between perfect foresight and rolling horizon MCPs with and without uncertainty. In Figure 1a the rolling horizon stochastic trees are displayed. In roll 1, the demand for time period 1 is known exactly and is deterministic. For time periods 2-4, demand is stochastic with the variance of demand increasing with time as Figure 1a shows with widening of the branches in the stochastic demand trees.

In roll 2, the demand for time period 2 becomes exactly known and deterministic. For time periods 3 and 4 demand remains stochastic but with less variance from the previous roll. Figure 1a also illustrates how information regarding time period 5 is only fed into the model in roll 2. A similar pattern can also be seen for rolls 3-5 in Figure 1a.

Figure 1b displays the stochastic demand tree associated with the perfect foresight version of the model. This graph shows that only demand at time period 1 is deterministic while demand at all other time periods is stochastic. This is in contrast to Figure 1a. In addition, Figure 1b also show how information regarding demand for all time periods is known at the start of the model which is again in contrast to Figure 1a.

Now consider an MCP of the form (7) with specific time periods $t = 1, \dots, T$ and associated decision variables $(x^t, y^t), t = 1, \dots, T$. This assumes that each variable has a t subscript which in many instances is not overly restrictive. The first question is whether or not if a deterministic rolling horizon approach is applied to solve (7), one MCP for each time period, will the solution set of that set of problems be the same as if the perfect foresight approach is used. The answer depends on the separability of the MCP function F as shown in the next result.



(a) Rolling horizon of stochastic demand scenarios tree for rolls $r = 1$ to $r = 5$.

(b) Perfect foresight of stochastic demand tree.

Figure 1

Theorem 1 Let $F : R^n \times R^m \rightarrow R^n \times R^m$. Define the solution set SOL^{pf} to the perfect foresight MCP(F) as vectors of the form: (x^{pf}, y^{pf}) that satisfy

$$0 \leq F_x(x, y) \perp x \geq 0 \tag{6a}$$

$$0 = F_y(x, y), y \text{ free} \tag{6b}$$

Define the solution set SOL^{rh} to the rolling horizon MCP(F) version of the problem as sequentially solving for vectors (x^t, y^t) that satisfy

$$0 \leq F_x^t(x^t, y^t) \perp x^t \geq 0 \tag{7a}$$

$$0 = F_y^t(x^t, y^t), y^t \text{ free} \tag{7b}$$

for each $t = 1, \dots, T$. Then, $SOL^{pf} = SOL^{rh}$ if $F(x, y)$ is separable in t .

Proof. Suppose that $F(x, y)$ is separable in t . This means that

$$\begin{pmatrix} F_x(x, y) \\ F_y(x, y) \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \begin{pmatrix} F_x^1(x, y) \\ F_y^1(x, y) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_x^t(x, y) \\ F_y^t(x, y) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_x^T(x, y) \\ F_y^T(x, y) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_x^1(x^1, y^1) \\ F_y^1(x^1, y^1) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_x^t(x^t, y^t) \\ F_y^t(x^t, y^t) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_x^T(x^T, y^T) \\ F_y^T(x^T, y^T) \end{pmatrix} \end{pmatrix} \tag{9}$$

Thus, (x^{pf}, y^{pf}) solves the perfect foresight MCP if and only if for each $t = 1, \dots, T$ solves

$$0 \leq F_x^t(x, y) \perp x \geq 0 \quad (10a)$$

$$0 = F_y^t(x, y), y \text{ free} \quad (10b)$$

which by the separability assumption is the same as

$$0 \leq F_x^t(x^t, y^t) \perp x^t \geq 0 \quad (11a)$$

$$0 = F_y^t(x^t, y^t), y^t \text{ free} \quad (11b)$$

which is equivalent to solving the rolling horizon MCP. \square

Essentially this result is saying if there is no linkage between the time periods either in constraints or data updating (or otherwise), then the two modeling paradigms of perfect foresight and rolling horizon match up. For most models including the gas market one considered in this paper this no-linkage assumption is not present. For instance, in the numerical examples presented in Section 4, there is linkage between the time periods as, after each MCP, the amount of gas in storage is updated using the amount of gas injected/extracted in the previous roll. In addition, the stochastic demand tree in these examples is also updated between rolls as time new periods are added to the tree and the variance associated with demand for gas at time t reduces.

2.1 Value of rolling horizon

Besides comparing a rolling horizon vs. a perfect foresight solution to a deterministic MCP as was shown in the previous theorem, it is instructive to consider such a comparison when some of the data are stochastic. In that case, there are two effects to consider. First, the rolling horizon vs. perfect foresight aspects assuming nonseparability of the MCP function F in light of Theorem 1. Second, there are the stochastic aspects similar to the notion of value of the stochastic solution (VSS) for optimization problems as described in [5]. Different from an optimization problem, there is not one objective function to use but rather a number of merit functions such as the H function described earlier. Additionally, also different from optimization is the fact that due to the rolling horizon nature of the problem, the scenario tree itself may be endogenous changing due to equilibrium solutions at each roll. We introduce a new measure, the value of the rolling horizon (VoRH), to understand the importance of rolling horizon taking into account the two points just mentioned.

In order to explain VoRH, we first set up appropriate notation. First, we consider an MCP of the following form: find $x^t \in R^{t, n_x}, y^t \in R^{t, n_y}$ such that ¹

$$\begin{aligned} 0 &\leq F_x^t(x, y) \perp x^t \geq 0 \\ 0 &= F_y^t(x, y), y^t \text{ free} \end{aligned} \quad \text{for } t = 1, \dots, T \quad (12)$$

where

$$\begin{aligned} (x, y) &= \{(x^t, y^t), t = 1, \dots, T\} \\ F_x(x, y) &= F_x^t(x, y), t = 1, \dots, T \\ F_y(x, y) &= F_y^t(x, y), t = 1, \dots, T \\ F_x^t(x, y) &: R^{t, n_x} \rightarrow R^{t, n_x}, F_y^t(x, y) : R^{t, n_y} \rightarrow R^{t, n_y} \end{aligned}$$

Let SOL^{pf} be the solution set to (12) solving for all time periods at the same time, i.e., the perfect foresight perspective. For such a solution to (12), the H function defined earlier must necessarily satisfy $\|H_{(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{pf}, \bar{y}^{pf})\| = 0$ for any vector norm $\|\cdot\|$. Note that the H function now is made dependent on a particular perfect foresight solution $(\bar{x}^{pf}, \bar{y}^{pf}) \in SOL^{pf}$.

¹This notation implicitly assumes that there is the same set of variables for x and y , repeated for each time period t . This is not restrictive and can be relaxed by allowing n_x to vary by time period.

The VoRH concept is based on the notion of seeing how a solution to the rolling horizon set of problems compares with the perfect foresight one using the norm of this H function. More specifically, suppose that SOL^{rh} is the solution set to (12) but solving using a rolling horizon approach for each roll or time period. If $|SOL^{pf}| = |SOL^{rh}| = 1$, then a natural question is how far off from zero is using the rolling horizon solution in place of the perfect foresight one via the norm of the H function? Consequently, in the case of singleton solution sets, we have the following first definition of the VoRH.

Definition. Assume that $|SOL^{pf}| = |SOL^{rh}| = 1$. Then, the value of the rolling horizon (VoRH) for problem (12) is defined as

$$\left\| H_{(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{rh}, \bar{y}^{rh}) \right\| \quad (14)$$

where $(\bar{x}^{pf}, \bar{y}^{pf}) \in SOL^{pf}$ so that $\left\| H_{(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{pf}, \bar{y}^{pf}) \right\| = 0$.

Theorem 2 Assume that $|SOL^{pf}| = |SOL^{rh}| = 1$. Then, the VoRH ≥ 0 for problem (12).

Proof. The result follows since $\|H(z)\| \geq 0$ for any vector z and $\|H(z)\| = 0$ if and only if z solves (12). \square

Since the solution sets SOL^{pf} and SOL^{rh} may in fact not be singletons, a more generalized version of the above definition of VoRH is needed and is as follows.

Definition. The value of the rolling horizon (VoRH) for problem (12) is defined as

$$\inf_{(\bar{x}^{rh}, \bar{y}^{rh}) \in SOL^{rh}} \left\{ \sup_{(\bar{x}^{pf}, \bar{y}^{pf}) \in SOL^{pf}} \left\| H_{(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{rh}, \bar{y}^{rh}) \right\| \right\} \quad (15)$$

where $\left\| H_{(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{pf}, \bar{y}^{pf}) \right\| = 0$.

3 Formulation

In this section, we describe the rolling horizon formulation for a stochastic natural gas market MCP. This formulation is then applied to a three-node system to provide insights into the rolling horizon approach.

3.1 Overview of rolling horizon and market participants

At each roll r the model looks at H time steps ahead: The first timestep is $t = r$ and the last one is $t = r + H - 1$. The set of timesteps for roll r is $T(r) = \{r, \dots, r + H - 1\}$. For example, if $H = 4$ then the time set for the first roll would be $T(1) = \{1, 2, 3, 4\}$. For the second roll it would be $T(2) = \{2, 3, 4, 5\}$ and so on. For each roll r , demand now (i.e., demand at time $t = r$) is known exactly and is scenario-independent. The first-stage decisions from the previous roll plus adjustments are used to meet this exactly known demand. For each roll r , first-stage decisions (gas production, injection, etc.) are made on how to meet the, at this point, uncertain demand for the next timestep ahead. These first-stage decisions are used in the next roll ($r + 1$) to meet the exactly known demand in that roll. For each roll r , hypothetical decisions are made on how to meet the uncertain demand for all time steps greater than one time step ahead. In total, over all rolls, there are $TT = |R| + H - 1$ timesteps where R is the set of rolls. Once the MCP for a given roll is solved, the model steps forward to the next MCP problem where the uncertain demand for the first time step ahead in the previous roll becomes known exactly. In this way, the model updates itself with decisions made in the previous roll.

In the gas market model to be presented, only two sets of players are modeled: producers and a transportation system operator (TSO). Clearly many more market players/functions are available but to keep the model illustrative yet computationally manageable, these two sets of players were chosen. The producer is endowed with decisions on production, storage, as well as exports so in some sense it is a generalized version

of actual ones. As the market being considered (roughly PJM) is small, we do not include liquefied natural gas (LNG) aspects as are considered in more global gas models such as the World Gas Model [26], Columbus [37], etc.

Tables 1–6 describe the variables in the model, the data, and the functions involved. The following conventions are used: lower-case Roman letters indicate indices or variables, upper-case Roman letters represent parameters (i.e., data, functions), Greek letters indicate endogenous or exogenous prices while thousand cubic meters and million cubic meters are represented by kcm and mcm respectively.

Table 1: Sets.

$a \in A$	Arcs (gas pipelines, LNG, other distribution).
$p \in P$	Producers.
$a \in A(p)$	Arcs that producer p is connected to.
$r \in R$	Rolls. For each roll an MCP is solved.
$t \in T(r) = \{r, \dots, r + H - 1\}$	Time set for roll r (e.g., quarters) where H is the time horizon, i.e., the number of timesteps in a roll.
$s(r) \in S(r)$	Scenario for stochastic elements, depending possibly on roll r .
$m \in M$	Gas node/market m .
$a^{in}(m)$	Arcs inward to market/node m .
$a^{out}(m)$	Arcs outward from node m .
$e \in \{r, \dots, t\}$	Dummy time index for storage constraint that represent timesteps from r to t .

Table 2: Parameters. *Note: Any variables with "previous" superscript are parameters. Also, the superscript * is a placeholder for multiple superscripts.*

H	The time horizon, i.e., the number of timesteps in a roll.
$TT = R + H - 1$	Total amount of timesteps over all rolls.
D_t	Discount factor for time t (%).
$DAYS_t$	Number of days in time period t .
τ_{at}^{REG}	Regulated price for using arc a at time t (\$/kcm).
DP_{pm}^{\max}	Maximum daily production capacity for producer p at node m (kcm/day).
DJ_{pm}^{\max}	Maximum daily storage injection rate for producer p at node m (kcm/day).
DX_{pm}^{\max}	Maximum daily storage extraction rate for producer p at market/node m (kcm/day).
DA_a^{\max}	Maximum arc capacity for arc a (mcm/day).
TP_{pmr}^{\max}	Total production capacity for producer p at node m over the whole time horizon (mcm).
$MINSTOR_{pm}$	Minimum amount of gas needed at storage facility at node/market m for producer p (mcm).
$INITSTOR_{pmr}$	Initial amount of gas in storage facility at node/market m for roll r for producer p (mcm).
$MAXSTOR_{pm}$	Maximum amount of gas allowed in storage facility at node/market m for producer p (mcm).
$LOSS_m$	Injection to storage loss factor for node m (%).
$LOSS_a$	Arc a loss factor (%).
$CONTRACTS_{pat}$	Contracted gas that producer p must provide through arc a at time t (mcm/day).
Z_{mr}^s	Fixed demand (demand curve intercept) at market/node m for roll r and scenario s (kcm).
B_{mr}^s	Demand curve slope at market/node m for roll r and scenario s (\$/kcm).
RU_{pmr}^*	Recourse/penalty cost associated with underestimating for producer p in market/node m for roll r (\$/kcm). There are different costs for each primal variable.
RO_{pmr}^*	Recourse/penalty cost associated with overestimating for producer p in market/node m for roll r (\$/kcm). There are different costs for each primal variable.
RU_{ar}^{flows}	Recourse/penalty cost for TSO associated with underestimating flows through arc a for roll r (\$/kcm).
RO_{ar}^{flows}	Recourse/penalty cost for TSO associated with overestimating flows through arc a for roll r (\$/kcm).
$PROB_{pr}^s$	probability producer p associates with scenario s at roll r .
$PROB_r^s$	probability TSO associates with scenario s at roll r .

Table 3: Functions.

$C_{pmtr}^{production}(\cdot)$	Production cost function for producer p at node m at time t roll r (\$/day).
$C_{pmtr}^{storage}(\cdot)$	storage cost function for producer p at node m at time t roll r (\$/day).
$C_{atr}^{arc}(\cdot)$	operations cost function for arc a at time t roll r (\$/day).

Table 4: Primal variables: Each of these primal variables have multiple superscripts which are described below. The superscript $*$ is a placeholder for the superscripts in Table 5.

$sales_{pmtr}^*$	Amount producer p , at node m , sells at time t . Decision made at roll r (kcm/day).
$prod_{pmtr}^*$	Amount produced by producer p , at node m , at time t . Decision made at roll r (kcm/day).
inj_{pmtr}^*	Amount injected into storage by producer p , at node m , at time t . Decision made at roll r (kcm/day).
xtr_{pmtr}^*	Amount extracted from storage by producer p , at mode m , for at time t . Decision made at roll r (kcm/day).
$flows_{patr}^{*,prod}$	Producer p 's flows through arc a at time t . Decision made at roll r (kcm/day).
$flows_{atr}^{*,tso}$	TSO flows through arc a at time t . Decision made at roll r (kcm/day).

Table 5: Superscripts.

s	Scenario s .
$previous$	Decisions made in the most immediately previous MCP, at roll $r - 1$. These represent parameters of the model.
$adj+$	Balancing decision made at roll r for time period $t = r$. Represents an increase.
$adj-$	Balancing decision made at roll r for time period $t = r$. Represents a decrease.
FS	First-stage decisions that are made at roll r for time period $t = r + 1$. These are scenario-independent.
$SS+$	Second-stage decisions that are made at roll r for time period $t = r + 1$. These represent an increase and are scenario-dependent.
$SS-$	Second-stage decisions that are made at roll r for time period $t = r + 1$. These represent an increase and are scenario-dependent.

Table 6: Dual variables.

τ_{atr}^s	price for arc a at time t at scenario s . Decision made at roll r (\$/kcm).
π_{mtr}^s	market-clearing price of gas for node m , time t , scenario s and roll r (\$/kcm).
$\lambda^{*,p\#}$	Lagrange multiplier associated with constraint $\#$ in producer's problem (unit depends on the constraint).
$\lambda^{*,tso\#}$	Lagrange multiplier associated with constraint $\#$ in TSO's problem (unit depends on the constraint).

3.2 Producer p 's problem for roll r

Producer p decides how much gas to sell, to produce, inject to storage, extract from storage and how much to flow to other nodes. Note that $E_{s(r)}$ for the producer and other players means a weighted summation. That is

$$E_s(x^s) = \sum_s PROB_r^s x^s$$

where $PROB_{pr}^s$ is the probability of scenario s for roll r and x^s is the particular variable whose expectation is being taken. Thus, when the KKT conditions are taken relative to this variable x^s , there will be a factor of $PROB_{pr}^s$ in front of the variable. The producer's problem is given below with the associated KKT conditions shown in the Appendix.

$$\begin{aligned}
& \max_{sales_{pmtr}^*, prod_{pmtr}^*, flows_{patr}^{s,prod}, inj_{pmtr}^*, xtr_{pmtr}^*} \\
& \sum_m \sum_{t=r}^{r+H-1} DAY S_t \left\{ E_{s(r)} \left[\pi_{mtr}^s sales_{pmtr}^s \right. \right. \\
& \quad - C_{pmtr}^{production}(prod_{pmtr}^s) \\
& \quad \left. \left. - \sum_{a \in A(p)} (\tau_{at}^{REG} + \tau_{atr}^s) flows_{patr}^{s,prod} - C_{pmtr}^{storage}(inj_{pmtr}^s, xtr_{pmtr}^s) \right] \right\} \\
& - D_{t=r} DAY S_{t=r} \left(RU_{pmr}^{prod} prod_{pm(t=r)r}^{adj+} + RO_{pmr}^{prod} prod_{pm(t=r)r}^{adj-} \right. \\
& \quad + RU_{pmr}^{sales} sales_{pm(t=r)r}^{adj+} + RO_{pmr}^{sales} sales_{pm(t=r)r}^{adj-} \\
& \quad + RU_{pmr}^{inj} inj_{pm(t=r)r}^{adj+} + RO_{pmr}^{inj} inj_{pm(t=r)r}^{adj-} \\
& \quad \left. + RU_{pmr}^{xtr} xtr_{pm(t=r)r}^{adj+} + RO_{pmr}^{xtr} xtr_{pm(t=r)r}^{adj-} \right) \tag{16} \\
& + \sum_{a \in A(p)} \left(RU_{par}^{flows} flows_{pa(t=r)r}^{adj+,prod} + RO_{par}^{flows} flows_{pa(t=r)r}^{adj-,prod} \right) \\
& - D_{t=r+1} DAY S_{t=r+1} E_{s(r)} \left[RU_{pmr}^{prod} prod_{pm(t=r+1)r}^{SS+,s} \right. \\
& \quad + RO_{pmr}^{prod} prod_{pm(t=r+1)r}^{SS-,s} \\
& \quad + RU_{pmr}^{sales} sales_{pm(t=r+1)r}^{SS+,s} + RO_{pmr}^{sales} sales_{pm(t=r+1)r}^{SS-,s} \\
& \quad + RU_{pmr}^{inj} inj_{pm(t=r+1)r}^{SS+,s} + RO_{pmr}^{inj} inj_{pm(t=r+1)r}^{SS-,s} \\
& \quad \left. + RU_{pmr}^{xtr} xtr_{pm(t=r+1)r}^{SS+,s} + RO_{pmr}^{xtr} xtr_{pm(t=r+1)r}^{SS-,s} \right] \\
& + \sum_{a \in A(p)} \left(RU_{par}^{flows} flows_{pa(t=r+1)r}^{SS+,s,prod} + RO_{par}^{flows} flows_{pa(t=r+1)r}^{SS-,s,prod} \right)
\end{aligned}$$

subject to:

$$\begin{aligned}
& prod_{pmtr}^s + \sum_{a \in a^{in}(m)} (1 - LOSS_a) flows_{patr}^{s,prod} + xtr_{pmtr}^s \\
= & sales_{pmtr}^s + \sum_{a \in a^{out}(m)} flows_{patr}^{s,prod} + inj_{pmtr}^s \quad \forall t, s, m \quad (\lambda_{pmtr}^{s,p1}) \tag{17a}
\end{aligned}$$

$$\begin{aligned}
MINSTOR_{pm} & \leq INITSTOR_{pmr} + \sum_{e=r}^t DAY S_e [(1 - LOSS_m) inj_{pmer}^s - xtr_{pmer}^s] \\
& \forall t, s, m \quad (\lambda_{pmtr}^{s,p2}) \tag{17b}
\end{aligned}$$

$$\begin{aligned}
INITSTOR_{pmr} & + \sum_{e=r}^t DAY S_e [(1 - LOSS_m) inj_{pmer}^s - xtr_{pmer}^s] \leq MAXSTOR_{pm} \\
& \forall t, s, m \quad (\lambda_{pmtr}^{s,p3}) \tag{17c}
\end{aligned}$$

$$flows_{patr}^{s,prod} \geq CONTRACTS_{pat} \quad \forall pat \quad (\lambda_{patr}^{s,p4}) \tag{17d}$$

$$prod_{pmtr}^s \leq DP_{pm}^{\max} \quad \forall s, t, m \quad (\lambda_{pmtr}^{s,p5}) \tag{17e}$$

$$prod_{pmtr}^{FS} \leq DP_{pm}^{\max}, \quad t = r + 1, \forall m \quad (\lambda_{pmtr}^{p6}) \tag{17f}$$

$$\sum_{t=r}^{r+H-1} DAY S_t prod_{pmtr}^s \leq TP_{pmr}^{\max} \quad \forall s, m \quad (\lambda_{pmtr}^{s,p7}) \tag{17g}$$

$$inj_{pmtr}^s \leq DI_{pm}^{\max} \quad \forall s, t, m \quad (\lambda_{pmtr}^{s,p8}) \tag{17h}$$

$$inj_{pmtr}^{FS} \leq DI_{pm}^{\max}, \quad t = r + 1, \forall m, \quad (\lambda_{pmtr}^{p9}) \tag{17i}$$

$$xtr_{pmtr}^s \leq DX_{pm}^{\max} \forall s, t, m (\lambda_{pmtr}^{s,p10}) \quad (17j)$$

$$xtr_{pmtr}^{FS} \leq DX_{pm}^{\max}, t = r + 1, \forall m, (\lambda_{pmtr}^{p11}) \quad (17k)$$

$$\begin{aligned} sales_{pmtr}^s &= SALES_{pmt(r-1)}^{previous} + sales_{pmtr}^{adj+} - sales_{pmtr}^{adj-} \\ &t = r, \forall s, m (\lambda_{pmtr}^{s,p12}) \end{aligned} \quad (17l)$$

$$\begin{aligned} sales_{pmtr}^s &= sales_{pmtr}^{FS} + sales_{pmtr}^{SS+,s} - sales_{pmtr}^{SS-,s}, \\ &t = r + 1, \forall s, m (\lambda_{pmtr}^{s,p13}) \end{aligned} \quad (17m)$$

$$inj_{pmtr}^s = INJ_{pmt(r-1)}^{previous} + inj_{pmtr}^{adj+} - inj_{pmtr}^{adj-} \quad t = r, \forall s, m (\lambda_{pmtr}^{s,p14}) \quad (17n)$$

$$inj_{pmtr}^s = inj_{pmtr}^{FS} + inj_{pmtr}^{SS+,s} - inj_{pmtr}^{SS-,s}, t = r + 1, \forall s, m (\lambda_{pmtr}^{s,p15}) \quad (17o)$$

$$xtr_{pmtr}^s = XTR_{pmt(r-1)}^{previous} + xtr_{pmtr}^{adj+} - xtr_{pmtr}^{adj-} \quad t = r, \forall s, m (\lambda_{pmtr}^{s,p16}) \quad (17p)$$

$$xtr_{pmtr}^s = xtr_{pmtr}^{FS} + xtr_{pmtr}^{SS+,s} - xtr_{pmtr}^{SS-,s}, t = r + 1, \forall s, m (\lambda_{pmtr}^{s,p17}) \quad (17q)$$

$$\begin{aligned} flows_{patr}^{s,prod} &= FLOW_{pat(r-1)}^{previous,prod} + \\ &flows_{patr}^{adj+,prod} - flows_{patr}^{adj-,prod} \quad t = r, \forall s, a (\lambda_{patr}^{s,p18}) \end{aligned} \quad (17r)$$

$$\begin{aligned} flows_{patr}^{s,prod} &= flows_{patr}^{FS,prod} + \\ &flows_{patr}^{SS+,s,prod} - flows_{patr}^{SS-,s,prod}, t = r + 1, \forall s, a (\lambda_{patr}^{s,p19}) \end{aligned} \quad (17s)$$

$$\begin{aligned} prod_{pmtr}^s &= PROD_{pmt(r-1)}^{previous} + prod_{pmtr}^{adj+} - prod_{pmtr}^{adj-} \\ &t = r, \forall s, m (\lambda_{pmtr}^{s,p20}) \end{aligned} \quad (17t)$$

$$prod_{pmtr}^s = prod_{pmtr}^{FS} + prod_{pmtr}^{SS+,s} - prod_{pmtr}^{SS-,s} \quad t = r + 1, \forall s, m (\lambda_{pmtr}^{s,p21}) \quad (17u)$$

Producer p 's objective function (16) maximizes their expected profit in all time periods less the recourse cost of making adjustments to first-stage decisions from the previous roll (e.g., $SALES_{pmt(r-1)}^{previous}$ for sales) in $t = r$ and second-stage decisions in $t = r + 1$, across all markets. The expected profit of producers is the money they receive from sales less the cost of production, less the cost associated with flowing gas through pipelines and less the cost of injections and extractions to and from storage.

Constraint (17a) ensures that the amount of gas producer p has entering market m equals the amount of gas they have exiting that market. Lower and upper bounds for the amount of gas producer p can have in storage at time t is provided by constraints (17b) and (17c) respectively. Constraint (17d) ensures that producer p must meet any contract that requires them to have a (pre-defined) fixed amount of gas flowing through pipeline a . An upper bound on the daily amount of gas producer p can produce in market m is provided in constraints (17e) and (17f) whilst constraint (17g) ensures the total amount of gas produced by producer p in market m , across all time steps, is capped. Similar constraints for the daily injection and extractions rates to/from storage are given by constraints (17h)–(17k).

Constraint (17l) allows $SALES_{pmt(r-1)}^{previous}$ (a first-stage decision from the previous MCP) to be adjusted. Note: the right-hand-side of equation (17l) contains no superscript s ensuring $sales_{pmtr}^s$ is the same across all scenarios for $t = r$. In contrast, constraint (17m) allows $sales_{pmtr}^{FS}$ to be adjusted, by second-stage decisions, for each scenario s . Equations (17n)–(17u) provide similar constraints for injections variables, extractions variables, flows through pipelines variables as well as production variables. The variable in the parentheses, alongside constraints (17a)–(17u), represent the Lagrange multipliers associated with that constraint.

Finally, all primal variables in producer p 's problem are constrained to be non-negative. The superscripts $adj+$ and $adj-$ are associated with variables in time period $t = r$ only while the superscripts FS , $SS+$ and $SS-$ are associated with variables in time period $t = r + 1$ only.

3.3 Transporation system operator (TSO) problem for roll r

The transmission system operator provides an economic mechanism to efficiently allocate transport capacity to producers. The TSO's problem is the following with the associated KKT conditions shown in the Appendix.

$$\begin{aligned} \max_{flow_{atr}^{s,tso}} \sum_a \left\{ \sum_{t=r}^{r+H-1} D_t DAY S_t E_{s(r)} \left[(\tau_{atr}^s + \tau_{at}^{REG}) flow_{atr}^{s,tso} - C^a(flow_{atr}^{s,tso}) \right] \right. \\ \left. - D_{t=r} DAY S_{t=r} (RU_{ar}^{flows} flow_{a(t=r)r}^{adj+,tso} + RO_{ar}^{flows} flow_{a(t=r)r}^{adj-,tso}) \right. \\ \left. - D_{t=r+1} DAY S_{t=r+1} E_{s(r)} \left[RU_{ar}^{flows} flow_{a(t=r+1)r}^{SS+,s,tso} + RO_{ar}^{flows} flow_{a(t=r+1)r}^{SS-,s,tso} \right] \right\} \end{aligned} \quad (18)$$

subject to:

$$\begin{aligned} flow_{atr}^{s,tso} &= FLOW_{at(r-1)}^{previous,tso} + flow_{atr}^{adj+,tso} - flow_{atr}^{adj-,tso}, \\ &t = r, \forall s, a (\lambda_{atr}^{s,tso1}) \end{aligned} \quad (19a)$$

$$\begin{aligned} flow_{atr}^{s,tso} &= flow_{atr}^{FS,tso} + flow_{atr}^{SS+,s,tso} - flow_{atr}^{SS-,s,tso}, \\ &t = r + 1, \forall s, a (\lambda_{atr}^{s,tso2}) \end{aligned} \quad (19b)$$

$$flow_{atr}^{s,tso} \leq DA_a^{\max} \quad \forall s, t, a (\lambda_{atr}^{s,tso3}) \quad (19c)$$

$$flow_{atr}^{FS,tso} \leq DA_a^{\max}, t = r + 1, \forall a (\lambda_{atr}^{tso4}) \quad (19d)$$

In a similar manner to the producer's problem their objective function is to maximize their expected profits less the recourse cost of making adjustments to first-stage decisions from the previous roll in $t = r$ and second-stage decisions in $t = r + 1$. The TSO's expected profit is the money they receive from producers less the cost associated with flowing gas through each pipeline a . The TSO receives two payments for each unit of gas flown through one of their pipelines; a pre-defined regulated charge (τ_{at}^{REG}) and a market price (τ_{atr}^s).

Constraint (19a) allows $FLOW_{at(r-1)}^{previous,tso}$ (a first-stage decision from the previous MCP) to be adjusted. In a comparison to similar constraints presented in producer p 's problem, note that the right-hand-side of equation (19a) contains no superscript s ensuring $flow_{atr}^{s,tso}$ is the same across all scenarios for $t = r$. In contrast, constraint (19b) allows $flow_{atr}^{FS,tso}$ to be adjusted, by second-stage decisions, for each scenario s . Constraints (19c) and (19d) provide an upper bound on the daily amount of gas that can flow through each pipeline a . The variable in the parentheses, alongside constraints (19a)–(19d), represent the Lagrange multipliers associated with that constraint. In addition all primal variables in the TSO's problem are constrained to be non-negative.

3.4 Market-clearing conditions for roll r

Besides optimization problems for each of the producers and the TSO, there are market-clearing conditions. The market-clearing conditions for the producers balance supply and demand for gas by market and are shown in (20a). This equation ensures that the price of gas in each scenario s , market m and time period t (π_{mtr}^s) is equal to an inverse demand curve. Similarly for the TSO, the flows requested by the producers and allowed on the pipelines by the TSO are balanced in (20b). The Lagrange multipliers for these two sets of market-clearing conditions, respectively, π_{mtr}^s and τ_{atr}^s , are the marginal supply and marginal transportation prices.

$$\pi_{mtr}^s = Z_{mtr}^s - B_{mtr}^s \sum_p DAY S_t sales_{pmtr}^s \quad \forall mtr, s, \pi_{mtr}^s \text{ free} \quad (20a)$$

$$flow_{atr}^{s,tso} = \sum_p flow_{patr}^{s,prod}, \quad \forall atr, s, \tau_{atr}^s \text{ free} \quad (20b)$$

3.5 The complete MCP at roll r

The Karush-Kuhn-Tucker (KKT) conditions for both producers and the TSO are presented in the Appendix. Assuming that all cost functions are convex these conditions are both necessary and sufficient for optimally

for both players. The MCP for roll r consists of these KKT conditions in addition to the market clearing conditions (20).

3.6 Update rules after each MCP (or roll)

In natural gas markets many market parameters update over time, for example, information regarding demand. In this section we show how the rolling horizon MCP captures this behavior. After each roll of the model, the following parameter changes take place:²

- All first-stage decisions in the previous MCP become the *previous* parameters for the next MCP, e.g., $flows_{at(r-1)}^{FS} \rightarrow FLOWS_{atr}^{previous}$
- The initial amount of gas in storage is updated based on the actual decisions (i.e., decisions made in time period $t = r$) made in the previous roll for injections and extractions for storage:
- Similarly, the total production capacity for producer p in market m (TP_{pmr}^{max}) reduces by the actual amount of gas produced for $t = r$ in the previous roll ensuring that production capacities deplete over time.
- The parameters associated with the stochastic demand tree update reflecting how information regarding demand updates throughout time, i.e., demand becomes less uncertain and eventually known actually as time moves forward. The stochastic demand for $t = r + 1$ now becomes deterministic and known exactly. The stochastic demands for $t = r + 1, \dots, r + H - 1$ remain uncertain but with reduced variance while information regarding stochastic demand for time period $t = r + H$ now becomes available to the model.
- The index for roll increases by one: $r \rightarrow r + 1$

4 Numerical results for three-node model

4.1 Data

In this section numerical examples of the rolling horizon MCP are presented. The model is formulated with $|P| = 3$ producers, $|M| = 3$ markets, $|S| = 3$ demand scenarios and with $|R| = 8$ rolls. The three markets are connected via $|A| = 4$ pipelines as shown in Figure 2. For each roll/MCP solved, the time horizon is $H = 4$ which means that the total number of time steps is $TT = |R| + H - 1 = 11$. Each time step represents a season, i.e., spring, summer, fall or winter. As a result, $DAYS_t = 89$ for $t = 1, 5, 9$ and $DAYS_t = 92$ for $t = 2, 3, 4, 6, 7, 8, 10, 11$. As the model is being solved over a relatively short timescale, no discount factor is considered, i.e., $D_t = 1 \forall t$. Each of the producers plus the TSO assign equal probabilities to the demand

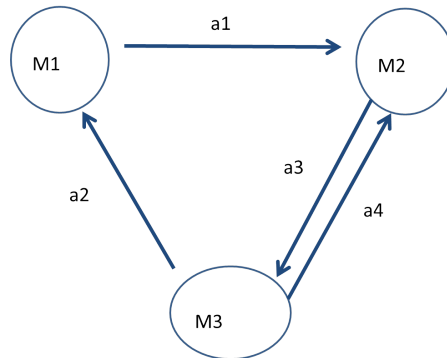


Figure 2: Pipelines connecting three markets.

²When a learning algorithm is used in the model, further model parameters (e.g., probabilities, production costs or recourse costs) are also updated. How this is implemented is described in detail in Section 4.3. The update rules presented in this section are in place in the model regardless of whether a learning algorithm is used or not.

scenarios. Hence, $PROB_{pr}^s = \frac{1}{|S|} = \frac{1}{3} \forall p, s$ and $PROB_r^s = \frac{1}{|S|} = \frac{1}{3} \forall s$. Note: in some of the example presented in Section 4.3.1, these probabilities change between rolls.

The maximum daily production rates (DP_{pm}^{\max}) for producer p in market m are given in Table 7 while, for the first roll, the total production capacity (TP_{pmr}^{\max}) is given by the following equation:

$$TP_{pmr}^{\max} = (365000)(DP_{pm}^{\max}). \quad (21)$$

Note: as Section 3.6 describes, TP_{pmr}^{\max} , reduces between rolls when there has been production in that field. The values for DP_{pm}^{\max} are based on information provided by the United States' Energy Information Authority³ while the values for TP_{pmr}^{\max} are arbitrarily chosen. The cost function $C_{pmtr}^{\text{production}}(x)$ for producer p in market m is a Golombek cost function [26] and is given by the following equation:

$$C_{pmtr}^{\text{production}}(x) = (\alpha_{p,m} - \gamma_{p,m})x + \frac{1}{2}\beta_{p,m}x^2 + \gamma_{p,m}(DP_{pm}^{\max} - x) \ln \left(\frac{DP_{pm}^{\max} - x}{DP_{pm}^{\max}} \right) \quad \forall t, r, \quad (22)$$

where the parameters $\alpha_{p,m}$, $\beta_{p,m}$ and $\gamma_{p,m}$ are given in Tables 8–10 respectively.

Table 7: Values for DP_{pm}^{\max} (mcm/day).

	$m = 1$	$m = 2$	$m = 3$
$p = 1$	3.5	4.5	3.9
$p = 2$	10.7	39	4.5
$p = 3$	8.9	0.5	26.9

Table 8: Values for α_{pm} .

	$p = 1$	$p = 2$	$p = 3$
$m = 1, 2, 3$	60	48	60

Table 9: Values for β_{pm} .

	$m = 1$	$m = 2$	$m = 3$
$p = 1$	2.2	0.25	1.8
$p = 2$	122	60	2.6
$p = 3$	0.42	0.75	0.4

Table 10: Values for γ_{pm} .

	$m = 1$	$m = 2$	$m = 3$
$p = 1$	6.9	6.9	6.9
$p = 2$	6.8	6.6	6.8
$p = 3$	6.9	6.9	6.9

The maximum daily injection to and extraction from storage rates are displayed in Tables 11 and 12 respectively. The minimum ($MINSTOR_{pm}$) and maximum ($MAXSTOR_{pm}$) amount of gas allowed in storage for each producer p in market m is zero and 10^6 , mcm respectively. For the first roll of the model the initial amount of gas in storage ($INTSTOR_{pm}$) is set to zero for each producer p in market m . For subsequent rolls, the initial amount is determined by the update rules described in Section 3.6. The storage cost function is assumed linear as follows:

$$C_{pmtr}^{\text{storage}}(inj_{pmtr}^s, xtr_{pmtr}^s) = 1.7(inj_{pmtr}^s + xtr_{pmtr}^s), \quad \forall p, m, t, r, s. \quad (23)$$

³<http://www.eia.gov>

The value of 1.7 is arbitrary but illustrative. The maximum daily capacity for each pipeline a (DA_a^{\max}) is given in Table 13 while the TSO's pipeline cost function is also assumed to be linear as follows:

$$C^a(\text{flows}_{atr}^{s,tso}) = (C_a^{MARG})(\text{flows}_{atr}^{s,tso}), \quad \forall t, r, s, \quad (24)$$

where C_a^{MARG} is given in Table 14. The regulated pipeline tariff that the TSO receives is set equal to the marginal pipeline cost, i.e., $\tau_{at}^{REG} = C_a^{MARG}$, $\forall t$. Again, these costs are both arbitrary and illustrative.

Table 11: Values for DI_{pm}^{\max} (mcm/day).

	$m = 1$	$m = 2$	$m = 3$
$p = 1, 2, 3$	150.1	423.2	91.7

Table 12: Values for DX_{pm}^{\max} (mcm/day).

	$m = 1$	$m = 2$	$m = 3$
$p = 1, 2, 3$	300.2	846.4	183.4

Table 13: Values for DA_a^{\max} (mcm/day).

$a = 1$	$a = 2$	$a = 3$	$a = 4$
13	241	46	78

Table 14: Values for C_a^{MARG} (\$/kcm).

$a = 1$	$a = 2$	$a = 3$	$a = 4$
1	2	2	2

The loss factors associated with injections to storage and flows through pipelines are both set to 5%, i.e., $LOSS_a = 0.05, \forall a$ and $LOSS_m = 0.05, \forall m$. In the numerical examples described in this section, no contract flows are assumed meaning $CONTRACTS_{pat} = 0, \forall p, a, t$ while all recourse costs are given a value of 0.2.

The slope values associated with the inverse demand function (see equation (20a)), are given by the following equation:

$$B_{mtr}^s = \bar{B}_{mtr}, \quad \forall m, t, s, \quad (25)$$

where \bar{B}_{mtr} , is the deterministic demand slope for market m and time t with values displayed in Table 15. The corresponding demand intercepts are given as follows:

$$Z_{mtr}^s = VAR_t^s \bar{Z}_{mtr}, \quad \forall m, t, s, \quad (26)$$

where \bar{Z}_{mtr} are the deterministic demand intercepts for market m at time t and VAR_t^s are the multipliers applied to the deterministic demand giving increased/decreased demand to scenario s in time t . The values for \bar{Z}_{mtr} are 723.481, 643.743 and 483.49 for markets $m = 1, 2, 3$, respectively, $\forall t, r$. Table 16 displays the values for VAR_t^s . In this numerical example, demand scenario $s = 3$ is deemed to be the *high* demand scenario, scenario $s = 2$ the *low* demand scenario and $s = 1$ the *middle* demand scenario.

Table 15: Values for the inverse demand function slope \bar{B}_{mtr} (\$/kcm).

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$
$m = 1$	0.024	0.028	0.029	0.022	0.024	0.028	0.029	0.022
$m = 2$	0.018	0.02	0.021	0.016	0.018	0.02	0.021	0.016
$m = 3$	0.013	0.015	0.015	0.012	0.013	0.015	0.015	0.012

Table 16: Values for VAR_t^s for the numerical example described in Section 4.1.

	$s = 1$	$s = 2$	$s = 3$
$t = 1$	1	1	1
$t = 2$	1	0.95	1.05
$t = 3$	1	0.9	1.1
$t = 4$	1	0.8	1.2

4.2 Perfect foresight vs. rolling horizon foresight

In this example, we show how the rolling horizon MCP format captures unexpected events more realistically than a perfect foresight approach. In particular we show how and why the price spike, as a result of the unexpected event, reduces as information regarding the event becomes known earlier. The unexpected event in this example is increased demand for $t = 7$. Increased demand is obtained by increasing the deterministic demand intercept (\bar{Z}_{mtr}) levels, described in Section 4.1, by 50%. In the *Base Case*, the rolling horizon MCP is run without any increased demand and with all parameters as described in Section 4.1. In the *No Foresight Case*, none of the players have information about the increased demand until roll 7; see Figure 3b for the increase in demand for $t = 7$ in roll $r = 7$ only, compared with the *Base Case* in Figure 3a.

In the *1 Period Ahead Foresight Case*, each player can see the increased demand one period (roll) ahead, i.e., in roll 6. See Figure 3c for the increase in demand for $t = 7$ in rolls $r = 6$ and $r = 7$ only, compared with the base case in Figure 3a. Note: while the values for the stochastic demand increase for $t = 7$ in roll 6 for the *1 Period Ahead Foresight case*, the probabilities associated with the high, low and medium demand scenarios remain the same.

In the *3 Period Ahead Foresight Case*, each player can see the increase in demand three periods (rolls) ahead, i.e., in roll 4. See Figure 3d for the increase in demand (compared with the base case in Figure 3a) for $t = 7$ for all rolls from $r = 4$ to $r = 7$. Again, while the values for the stochastic demand increase for $t = 7$ in rolls 4-6 for the *3 Period Ahead Foresight case*, the probabilities associated with the high, low and medium demand scenarios remain the same.

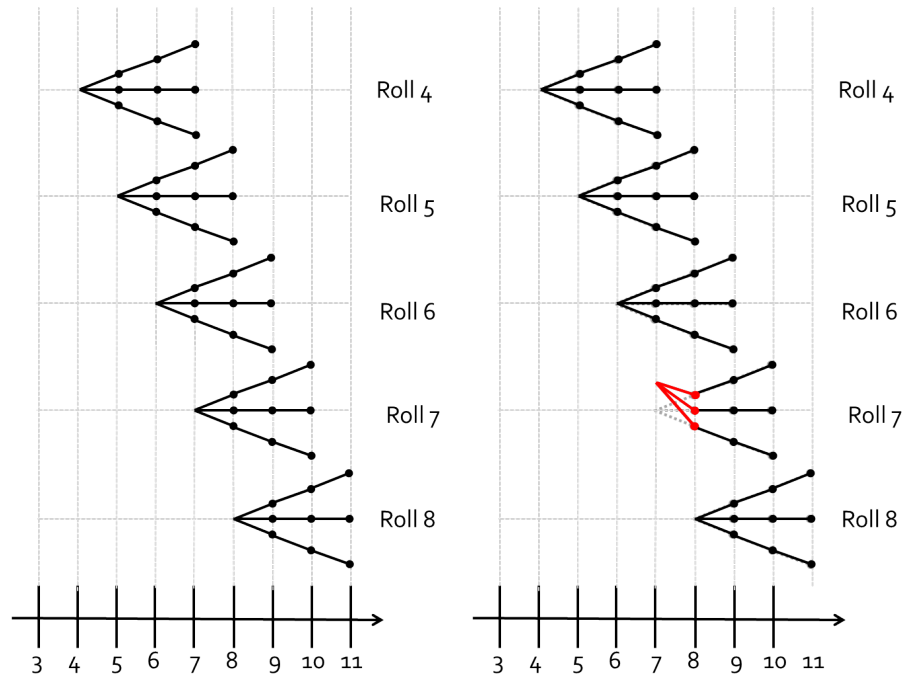
In the *Perfect Foresight Case*, there is only one roll of the model and each player can see all time periods ahead at the start of the model. As a result, each player can see the increased stochastic demand from the start of the model; again, see Figure 4 for the increase in demand for $t = 7$ compared with Figure 1b.

Figure 5 shows how increased information allows for smaller price spikes for the prices in market $m = 1$ ($\pi_{m=1,t=1,r}$) while Figure 6 displays the net injections of gas storage across all markets, i.e, the sum of all injections to storage less the sum of all extractions. In the *No Foresight Case*, Figure 6 explains how only a small amount of gas is extracted from storage in roll 7 because, not knowing about the increased demand in previous rolls, a relatively small amount of gas is in storage prior to roll 7. As a result, Figure 5 shows how this case has the largest spike in price for roll 7.

In the *One Period Foresight* and *Three Period Foresight Cases*, there is prior knowledge of the stressed demand. As a result, Figure 6 shows increases in the amount of gas injected to storage in rolls 5 and 6 which in turn allows for an increased amount of gas to be withdrawn from storage in roll 7. Hence, as Figure 5 shows, the price spikes in roll 7 for these two cases is not as big as in the *No Foresight Case*. However, because of the increase in the amount of gas injected into storage, there is increased prices seen in the rolls prior to roll 7 for these two cases (relative to the *Base Case* and *No Foresight Case*).

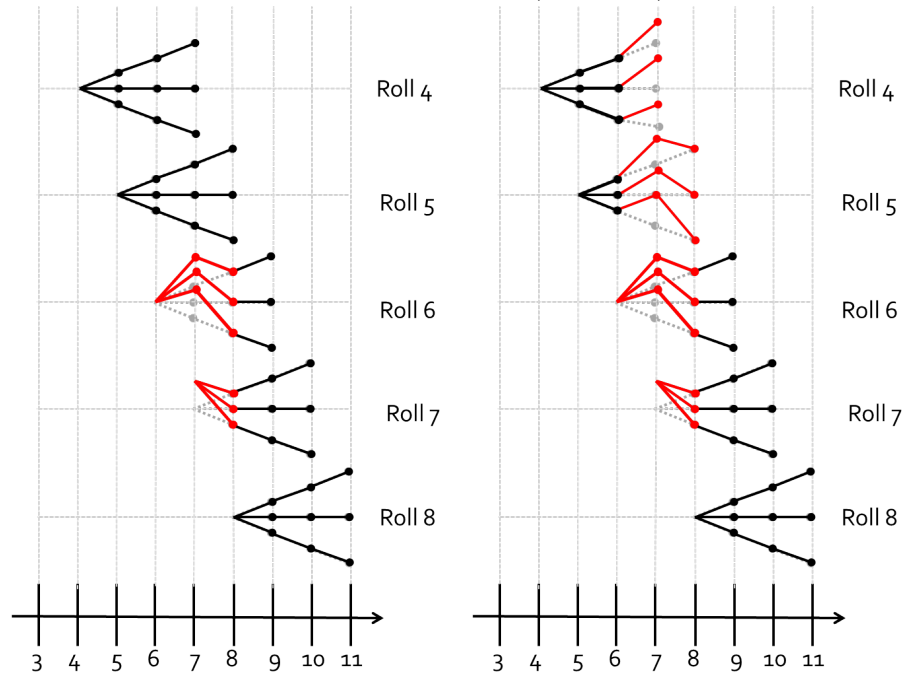
As Figure 6 shows, the *Perfect Foresight Case* has the largest amount of injections (cumulatively) to storage before roll 7, leading to the largest amount of extractions from storage in roll 7. This allows the price spike in roll 7 to be the smallest but there are increased prices (relative to the *Base Case*) seen from roll 1.

In the *One Period Foresight*, *Three Period Foresight* and *Perfect Foresight Cases*, the market is able to prepare for the increased demand by injecting gas into storage prior to roll 7. The more gas that is injected to storage prior roll 7 ensures smaller prices spikes in roll 7.



(a) Base case (no increase in demand).

(b) No foresight of increased demand for time $t = 7$ (continuous line) with Base Case (dotted line).



(c) One period ahead foresight of increased demand for time $t = 7$ (continuous line) with Base Case (dotted line).

(d) Three period ahead foresight of increased demand for time $t = 7$ (continuous line) with Base Case (dotted line).

Figure 3: Stochastic demand scenario trees for rolls $r = 4$ to $r = 8$ for increased demand example.

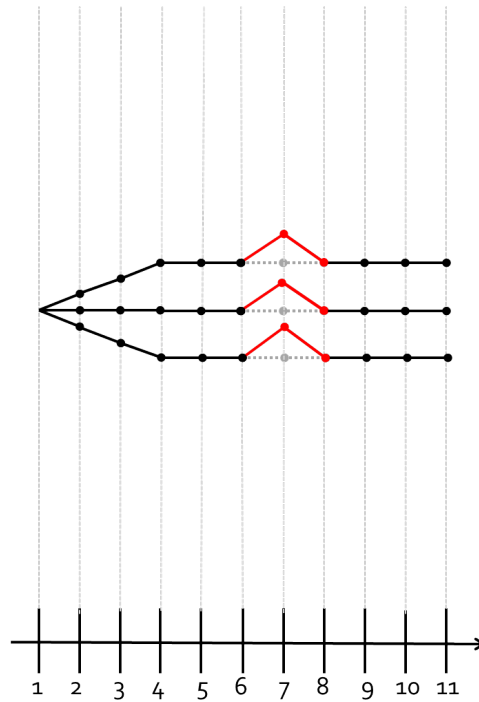


Figure 4: Perfect foresight of increased stochastic demand for $t = 7$ (continuous line) with Base Case (dotted line).

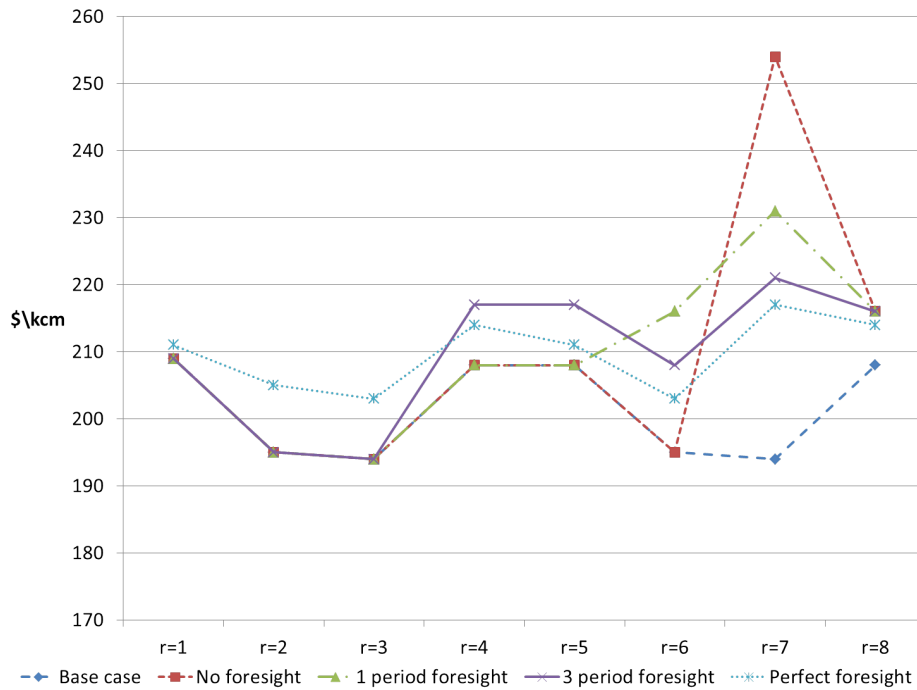


Figure 5: Prices in market $m = 1$.

Withdrawals from storage in roll 8 can only be seen in the *Base Case* and in the *Perfect Foresight Case*. This is because, in all other cases, storage facilities are emptied to meet the increased demand in roll 7.

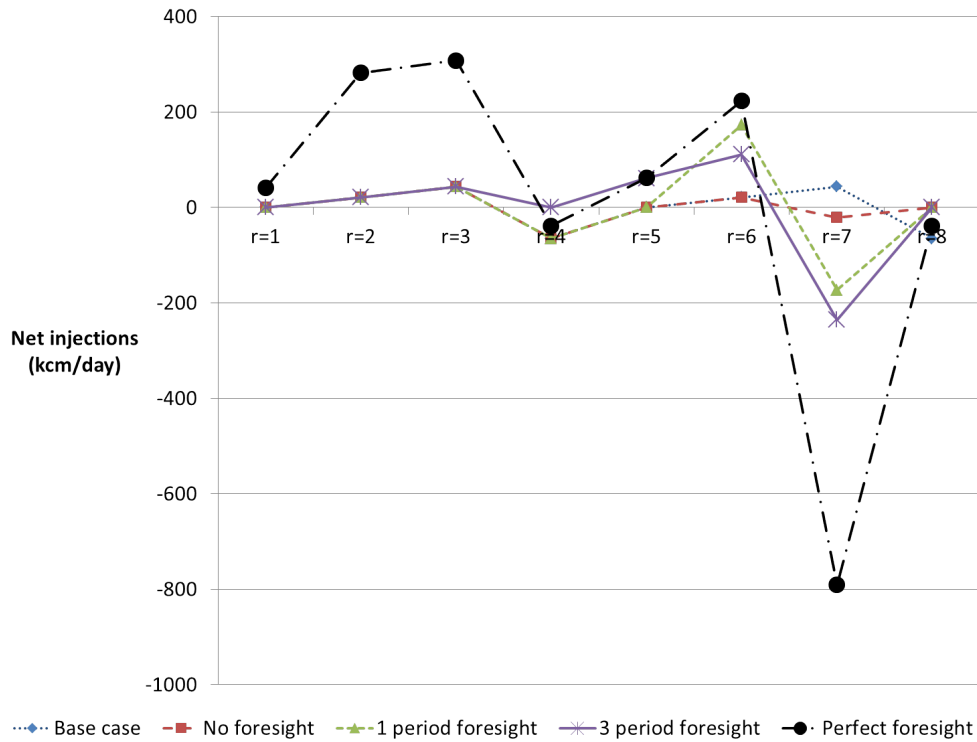


Figure 6: Injections less extractions (cumulative across all markets).

Similar gas price trends and spikes were found for markets $m = 2$ and $m = 3$ as those presented for $m = 1$ in Figure 5, i.e., the sooner the market knows about the increased demand, the smaller the price spike in roll 7 is relative to the *Base Case*.

4.2.1 VoRH results for increased demand example

Table 17 displays the VoRH results associated with the examples described above in Section 4.2. Here the solutions from the *No Foresight*, *One Period Foresight* and *Three Period Foresight* cases are compared with the *Perfect Foresight* solutions.^{4,5} The second column of Table 17 displays the VoRH results as calculated by equation (14) while in the third column these VoRH figures relative to the *Base Case* VoRH are presented.

Table 17: VoRH values for increased demand examples in Section 4.2.

	VoRH	Relative VoRH	VoRH without large values	Relative VoRH without large values
No foresight	292616.95	2.69	93.40	0.90
One period ahead foresight	216355.14	1.99	83.07	0.80
Three periods ahead foresight	230268.63	2.12	65.00	0.63

The *No Foresight* case has the largest VoRH indicating that the solutions arising from this case are furthest away from the *Perfect Foresight* solutions. This is not surprising because, as explained previously for Figures 5 and 6, the *No Foresight* case has the largest price spike for $t = 7$ and market participants only

⁴The *Base Case* VoRH is excluded from this analysis as this case does not contain any increased demand and is thus not comparable with any of the other cases.

⁵This analysis assumes locally unique solutions in each case.

adjust the storage levels when they become aware of the increased demand in roll $r = 7$, which is in contrast to the *Perfect Foresight* case.

However, the *One Period Foresight* case has a smaller VoRH value compared with the *Three Period Foresight* case suggesting that the solutions from the *One Period Foresight* case are closer to the *Perfect Foresight* solutions than those from the *Three Period Foresight* case. This is a surprising result because one would expect, as more foresight is added to the model, that the solutions would become closer to those from the *Perfect Foresight* case, all things being equal.

When these results were analyzed more closely it was found that the median value for $|H_{i,(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{rh}, \bar{y}^{rh})|$ (see equation (14)), for each of the cases, was zero suggesting that there are a few $|H_{i,(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{rh}, \bar{y}^{rh})|$ values that are skewing the VoRH results. As a result, the VoRH figures were calculated again with $|H_{i,(\bar{x}^{pf}, \bar{y}^{pf})}(\bar{x}^{rh}, \bar{y}^{rh})|$ values greater than two excluded; see column four of Table 17. As before, column five of Table 17 displays these VoRH figures relative to the *Base Case* VoRH. The results in these columns show that as more foresight of the increased demand is added to the model, the VoRH figures decrease informing us that the solutions to the rolling horizon MCP get closer to those from the *Perfect Foresight* case as more information regarding demand is added to the model. Thus, these re-calculated results show the benefit of using VoRH in comparing solutions from perfect foresight with those from rolling horizon. Finally, it must be noted that this analysis is an initial study of the VoRH and future work will examine the metric further.

4.3 Learning algorithm and endogenous probabilities

In this section a learning algorithm is introduced. In particular we show how producers who continuously make the mistake of over/underestimating demand (and hence their sales) adjust the probabilities they associate with demand scenarios in order to decrease or eliminate the effects of the mistake. The changes to probabilities occur between rolls. For a producer to do this in the algorithm, they must overestimate sales in at least L rolls (not necessarily consecutive). Specifically, overestimation is deemed to have occurred if the producer's adjustments for overestimating sales ($\sum_m sales_{pmtr}^{adj-} \geq 0$) less its adjustments for underestimating sales ($\sum_m sales_{pmtr}^{adj+} \geq 0$), relative to its total sales, are above a certain threshold:

$$\frac{\sum_m sales_{pmtr}^{adj-} - sales_{pmtr}^{adj+}}{\sum_m sales_{pm,t=1,r}^s} = \alpha_p \geq THRESHOLD. \quad (27)$$

When this happens, the probability associated with the high demand scenario ($s = 3$) is reduced by the value $PROB_ADJ$ and hence, to ensure probabilities sum to one, the probabilities of the other two scenarios are increased by $\frac{PROB_ADJ}{2}$. Similarly, if a producer underestimates sales (i.e., $-\alpha \geq THRESHOLD$) in at least L rolls then the probability associated with the low demand scenario ($s = 2$) is reduced by $PROB_ADJ$. The following is a description of the algorithm.

Algorithm:

$learning_counter_p \rightarrow 0$

for $p \in P$ **do**

if $\alpha_p \geq THRESHOLD$ **then**

$learning_counter_p \rightarrow learning_counter_p + 1$

else

if $-\alpha_p \geq THRESHOLD$ **then**

$learning_counter_p \rightarrow learning_counter_p - 1$

end if

end if

if $learning_counter_p = L$ **then**

$PROB_{pr}^{s=3} \rightarrow PROB_{pr}^{s=3} - PROB_ADJ$

$PROB_{pr}^{s=1} \rightarrow PROB_{pr}^{s=1} + PROB_ADJ \setminus 2$

$PROB_{pr}^{s=2} \rightarrow PROB_{pr}^{s=2} + PROB_ADJ \setminus 2$

```

learning_counterp → 0

else
  if learning_counterp = -L then
    PROBprs=2 → PROBprs=2 - PROB_ADJ
    PROBprs=1 → PROBprs=1 + PROB_ADJ\2
    PROBprs=3 → PROBprs=3 + PROB_ADJ\2
    learning_counterp → 0
  end if
end if
end for

```

4.3.1 Results of learning algorithm

We now compare the results of when the learning algorithm is used for the rolling horizon MCP against the case when a learning algorithm isn't used, i.e., an example similar to the *Base Case* presented Section 4.2. The analysis shows the benefit of using a learning algorithm because when a learning algorithm isn't used, players make the same mistakes (e.g., over or underestimating demand) throughout the time period of model.

Except for the producer's probabilities, all the same data as described in Section 4.1 are used for these examples. The probabilities each of producers assign to the three demand scenarios, for the first roll of the model, are given in Table 18. The values for *THRESHOLD*, *L* and *PROB_ADJ* are 0.05, 2 and $\frac{1}{6}$ respectively for this example.

Table 18: Values for $PROB_{pr}^s$ for results associated with the Learning Algorithm in Section 4.3.1.

	<i>s</i> = 1	<i>s</i> = 2	<i>s</i> = 3
<i>p</i> = 1	1/6	1/6	4/6
<i>p</i> = 2	1/3	1/3	1/3
<i>p</i> = 3	1/6	4/6	1/6

The bar charts in Figure 7 display by how much producer $p = 1$ overestimates sales ($\sum_m sales_{pmt_r}^{adj-}$) at time $t = r$ when the learning algorithm is used and when no learning algorithm is used for each roll of the rolling horizon MCP. The line chart in Figure 7 shows how the probability producer $p = 1$ associates with the high demand scenario changes over the course of the model when the learning algorithm is utilized. For $r = 1$, producer $p = 1$ does not over or underestimate its sales (i.e., $-0.05 < \alpha_{p=1} < 0.05$).⁶ Because $p = 1$ assigns a high probability with the high demand scenario, the producer overestimates its sales (i.e., $\alpha_{p=1} \geq 0.05$) for $r = 2$ and $r = 3$. Consequently, because this overestimation occurs twice, producer $p = 1$ reduces the probability it associates with the high demand scenario; see the line chart for roll $r = 4$ in Figure 7.

For $r = 4$, sales are not overestimated (i.e., $\alpha_{p=1} < 0.05$) by enough for the producer $p = 1$'s learning counter ($learning_counter_{p=1}$) to be increased. For $r = 5$ and $r = 6$ sales are overestimated again (i.e., $\alpha_{p=1} \geq 0.05$) and hence, producer $p = 1$ reduces the probability it associates with the high demand scenario again. For $r = 7$ and $r = 8$, Figure 7 shows that the amount producer $p = 1$ overestimates sales by are greatly reduced compared with when a learning algorithm isn't used, thus showing the benefit of using a learning algorithm for the rolling horizon MCP. As a result and because $\alpha_{p=1} < 0.05$, producer $p = 1$'s probabilities do not change after these rolls.

Similar results were found for producer $p = 3$ who, because it associates a high probability with the low demand scenario, has a problem with underestimating sales. Producer $p = 2$ assigns equal probabilities to each of the demand scenarios. As a result, it does not over or underestimate demand for any roll of model. Hence its probabilities are not adjusted at any stage of the model.

⁶This is because of a start-of-model effect, where $SALES_{pmt(r-1)}^{previous}$ is set by the model user for roll $r = 1$. For all other rolls its values are chosen from the previous MCP (see Section 3.6).

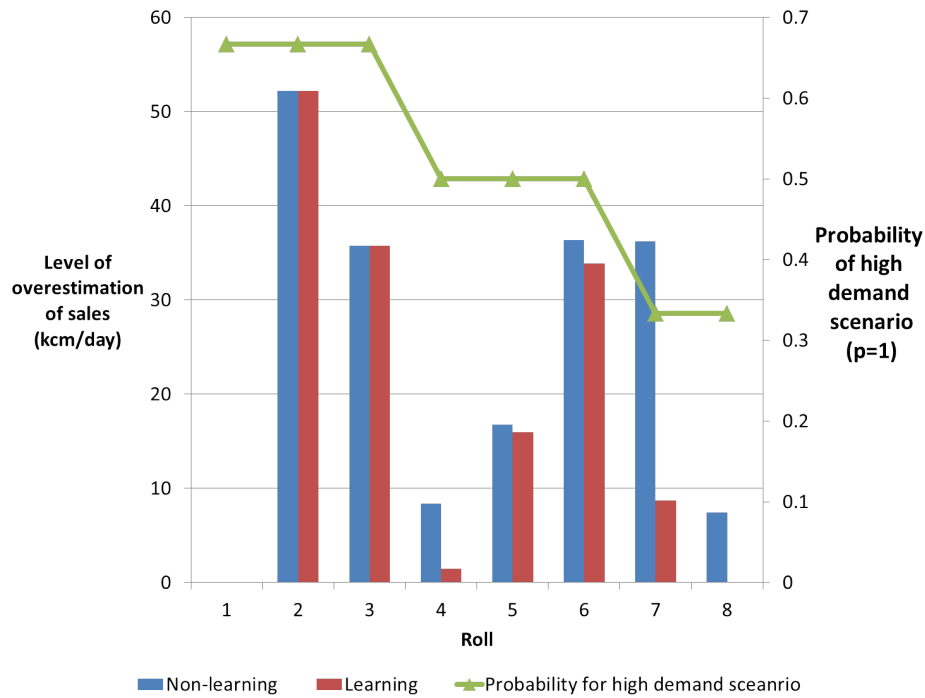


Figure 7: Amount by which producer $p = 1$ overestimates sales (bar charts and left vertical axis) for when a learning algorithm is and isn't used with the rolling horizon MCP plus probability (line chart and right vertical axis) associated with high demand scenario for producer $p = 1$.

Finally, Table 19 displays the VoRH and relative VoRH results for when the solutions of the two examples considered in this section are compared with the perfect foresight version of the model.⁷ The VoRH figure for when a learning algorithm isn't used is smaller than the figure from when a learning algorithm is used. This indicates that the solutions from this non-learning example are closest to the perfect foresight solutions. This result is not surprising because in the perfect foresight case there is only one roll which means there is no opportunity for endogenous learning, as is the case when a learning algorithm isn't used.

Table 19: VoRH values for learning algorithm example in Section 4.3.1.

	VoRH	Relative VoRH	VoRH without large values	Relative VoRH without large values
No learning algorithm	396432.63	1.00	66.18	1.00
Learning algorithm	787902.35	1.99	75.28	1.14

As Figure 7 shows, endogenous probabilities allows the different players in the model to adjust the decisions throughout the model. As a result, the solutions from this case are much further away from the perfect foresight solutions.

5 Conclusions

In this paper we have introduced and developed the concept of a stochastic mixed complementarity problem (MCP) for a rolling horizon with a natural gas market application. This format, while more closely match-

⁷The perfect foresight case considered here is similar to that presented in Section 4.2 but with the probabilities as described by Table 18.

ing how energy other markets function, also allows for decision-dependent scenario-tree probabilities and endogenous learning by the market participants. Several theoretical concepts (e.g., the Value of the Rolling Horizon) and numerical results were developed and shown to validate the rolling horizon approach.

6 Appendix

6.1 KKT Conditions for All the Producers' problems

6.1.1 Sales Variables

$$0 \leq -D_t DAY S_t (PROB_{pr}^s \pi_{mtr}^s) - \lambda_{pmtr}^{s,p1} + \begin{cases} \lambda_{pmtr}^{s,p12}, & t = r \\ \lambda_{pmtr}^{s,p13}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} \quad \perp sales_{pmtr}^s \geq 0, \forall pmts \quad (28a)$$

$$0 \leq D_t DAY S_t RU_{pmr}^{sales} - \sum_s \lambda_{pmtr}^{s,p12} \perp sales_{pmtr}^{adj+} \geq 0, \forall pm, t = r \quad (28b)$$

$$0 \leq D_t DAY S_t RO_{pmr}^{sales} + \sum_s \lambda_{pmtr}^{s,p12} \perp sales_{pmtr}^{adj-} \geq 0, \forall pm, t = r \quad (28c)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RU_{pmr}^{sales} - \lambda_{pmtr}^{s,p13} \perp sales_{pmtr}^{SS+,s} \geq 0, \forall pms, t = r + 1 \quad (28d)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RO_{pmr}^{sales} + \lambda_{pmtr}^{s,p13} \perp sales_{pmtr}^{SS-,s} \geq 0, \forall pms, t = r + 1 \quad (28e)$$

$$0 \leq - \sum_s \lambda_{pmtr}^{s,p13} \perp sales_{pmtr}^{FS} \geq 0, \forall pm, t = r + 1 \quad (28f)$$

6.1.2 Production Variables

$$0 \leq D_t DAY S_t (PROB_{pr}^s) \frac{\partial C_{pmtr}^{production}(prod_{pmtr}^s)}{\partial prod_{pmtr}^s} + \lambda_{pmtr}^{s,p1} + \lambda_{pmtr}^{s,p5} + DAY S_t \lambda_{pmtr}^{s,p7} + \begin{cases} \lambda_{pmtr}^{s,p20}, & t = r \\ \lambda_{pmtr}^{s,p21}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} \perp prod_{pmtr}^s \geq 0, \forall pmts \quad (29a)$$

$$0 \leq D_t DAY S_t RU_{pmr}^{prod} - \sum_s \lambda_{pmtr}^{s,p20} \perp prod_{pmtr}^{adj+} \geq 0, \forall pm, t = r \quad (29b)$$

$$0 \leq D_t DAY S_t RO_{pmr}^{prod} + \sum_s \lambda_{pmtr}^{s,p20} \perp prod_{pmtr}^{adj-} \geq 0, \forall pm, t = r \quad (29c)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RU_{pmr}^{prod} - \lambda_{pmtr}^{s,p21} \perp prod_{pmtr}^{SS+,s} \geq 0, \forall pms, t = r + 1 \quad (29d)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RO_{pmr}^{prod} + \lambda_{pmtr}^{s,p21} \perp prod_{pmtr}^{SS-,s} \geq 0, \forall pms, t = r + 1 \quad (29e)$$

$$0 \leq \lambda_{pmtr}^{p6} - \sum_s \lambda_{pmtr}^{s,p21} \perp prod_{pmtr}^{FS} \geq 0, \forall pm, t = r + 1 \quad (29f)$$

6.1.3 Flow Variables

$$0 \leq D_t DAY S_t (PROB_{pr}^s) (\tau_{at}^{REG} + \tau_{atr}^s) + \begin{cases} (1 - LOSS_a) \lambda_{pmtr}^{s,p1} & a \in a^{in}(m) \\ -\lambda_{pmtr}^{s,p1} & a \in a^{out}(m) \end{cases} - \lambda_{patr}^{s,p4} + \begin{cases} \lambda_{patr}^{s,p18}, & t = r \\ \lambda_{patr}^{s,p19}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} \perp flows_{patr}^{s,prod} \geq 0, \forall pts, a \in A(p) \quad (30a)$$

$$0 \leq D_t DAY S_t RU_{par}^{flows} - \sum_s \lambda_{patr}^{s,p18} \perp flows_{patr}^{adj+,prod} \geq 0, \forall p, a \in A(p), t = r \quad (30b)$$

$$0 \leq D_t DAY S_t RO_{par}^{flows} + \sum_s \lambda_{patr}^{s,p18} \perp flows_{patr}^{adj-,prod} \geq 0, \forall p, a \in A(p), t = r \quad (30c)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RU_{par}^{flows} - \lambda_{patr}^{s,p19} \perp flows_{patr}^{SS+,s,prod} \geq 0, \forall ps, a \in A(p), t = r + 1 \quad (30d)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RO_{par}^{flows} + \lambda_{patr}^{s,p19} \perp flows_{patr}^{SS-,s,prod} \geq 0, \forall ps, a \in A(p), t = r + 1 \quad (30e)$$

$$0 \leq \sum_s -\lambda_{patr}^{s,p19} \perp flows_{patr}^{FS,prod} \geq 0, \forall p, a \in A(p), t = r + 1 \quad (30f)$$

6.1.4 Storage Injection Variables

$$0 \leq D_t DAY S_t (PROB_{pr}^s) \frac{\partial C_{pmtr}^{storage}(inj_{pmtr}^s)}{\partial inj_{pmtr}^s} - \lambda_{pmtr}^{s,p1} + \lambda_{pmtr}^{s,p8} + \begin{cases} \lambda_{pmtr}^{s,p14}, & t = r \\ \lambda_{pmtr}^{s,p15}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} \\ - \sum_{e=t}^{r+H-1} DAY S_e (1 - LOSS_m) \lambda_{pmer}^{s,p2} \quad (31a)$$

$$+ \sum_{e=t}^{r+H-1} DAY S_e (1 - LOSS_m) \lambda_{pmer}^{s,p3} + \perp inj_{pmtr}^s \geq 0, \forall pmts \quad (31a)$$

$$0 \leq D_t DAY S_t RU_{pmr}^{inj} - \sum_s \lambda_{pmtr}^{s,p14} \perp inj_{pmtr}^{adj+} \geq 0, \forall pm, t = r \quad (31b)$$

$$0 \leq D_t DAY S_t RO_{pmr}^{inj} + \sum_s \lambda_{pmtr}^{s,p14} \perp inj_{pmtr}^{adj-} \geq 0, \forall pm, t = r \quad (31c)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RU_{pmr}^{inj} - \lambda_{pmtr}^{s,p15} \perp inj_{pmtr}^{SS+,s} \geq 0, \forall pms, t = r + 1 \quad (31d)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RO_{pmr}^{inj} + \lambda_{pmtr}^{s,p15} \perp inj_{pmtr}^{SS-,s} \geq 0, \forall pms, t = r + 1 \quad (31e)$$

$$0 \leq \lambda_{pmtr}^{p9} - \sum_s \lambda_{pmtr}^{s,p15} \perp inj_{pmtr}^{FS} \geq 0, \forall pm, t = r + 1 \quad (31f)$$

6.1.5 Storage Extraction Variables

$$0 \leq D_t DAY S_t (PROB_{pr}^s) \frac{\partial C_{pmtr}^{storage}(xtr_{pmtr}^s)}{\partial xtr_{pmtr}^s} + \lambda_{pmtr}^{s,p1} + \lambda_{pmtr}^{s,p10} + \begin{cases} \lambda_{pmtr}^{s,p16}, & t = r \\ \lambda_{pmtr}^{s,p17}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} \\ + \sum_{e=t}^{r+H-1} DAY S_e \lambda_{pmer}^{s,p2} - \sum_{e=t}^{r+H-1} DAY S_e \lambda_{pmer}^{s,p3} \perp xtr_{pmtr}^s \geq 0, \forall pmts \quad (32a)$$

$$0 \leq D_t DAY S_t RU_{pmr}^{xtr} - \sum_s \lambda_{pmtr}^{s,p16} \perp xtr_{pmtr}^{adj+} \geq 0, \forall pm, t = r \quad (32b)$$

$$0 \leq D_t DAY S_t RO_{pmr}^{xtr} + \sum_s \lambda_{pmtr}^{s,p16} \perp xtr_{pmtr}^{adj-} \geq 0, \forall pm, t = r \quad (32c)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RU_{pmr}^{xtr} - \lambda_{pmtr}^{s,p17} \perp xtr_{pmtr}^{SS+,s} \geq 0, \forall pms, t = r + 1 \quad (32d)$$

$$0 \leq D_t DAY S_t (PROB_{pr}^s) RO_{pmr}^{xtr} + \lambda_{pmtr}^{s,p17} \perp xtr_{pmtr}^{SS-,s} \geq 0, \forall pms, t = r + 1 \quad (32e)$$

$$0 \leq \lambda_{pmtr}^{p11} - \sum_s \lambda_{pmtr}^{s,p17} \perp xtr_{pmtr}^{FS} \geq 0, \forall pm, t = r + 1 \quad (32f)$$

6.1.6 Storage Constraints

$$0 \leq INITSTOR_{mr} + \sum_{e=r}^t DAYSe[(1 - LOSS_m)inj_{pmer}^s - xtr_{pmer}^s] - MINSTOR_m \perp \lambda_{pmtr}^{s,p2} \geq 0, \forall pmts, \quad (32g)$$

$$0 \leq MAXSTOR_m - INITSTOR_{mr}$$

$$-\sum_{e=r}^t DAYSe[(1 - LOSS_m)inj_{pmer}^s - xtr_{pmer}^s] \perp \lambda_{pmtr}^{s,p3} \geq 0, \forall pmts, \quad (32h)$$

$$0 \leq -inj_{pmtr}^s + DI_{pm}^{\max} \perp \lambda_{pmtr}^{s,p8} \geq 0, \forall pmts, \quad (32i)$$

$$0 \leq -inj_{pmtr}^{FS} + DI_{pm}^{\max} \perp \lambda_{pmtr}^{p9} \geq 0, \forall pm, t = r + 1, \quad (32j)$$

$$0 \leq -xtr_{pmtr}^s + DX_{pm}^{\max} \perp \lambda_{pmtr}^{s,p10} \geq 0, \forall pmts, \quad (32k)$$

$$0 \leq -xtr_{pmtr}^{FS} + DX_{pm}^{\max} \perp \lambda_{pmtr}^{p11} \geq 0, \forall pm, t = r + 1, \quad (32l)$$

$$0 = inj_{pmtr}^s - INJ_{pmt(r-1)}^{previous} - inj_{pmtr}^{adj+} + inj_{pmtr}^{adj-}, \lambda_{pmtr}^{s,p14} \text{ free}, \forall pms, t = r, \quad (32m)$$

$$0 = inj_{pmtr}^s - inj_{pmtr}^{FS} - inj_{pmtr}^{SS+,s} + inj_{pmtr}^{SS-,s}, \lambda_{pmtr}^{s,p15} \text{ free}, \forall pms, t = r + 1, \quad (32n)$$

$$0 = xtr_{pmtr}^s - XTR_{pmt(r-1)}^{previous} - xtr_{pmtr}^{adj+} + xtr_{pmtr}^{adj-}, \lambda_{pmtr}^{s,p16} \text{ free}, \forall pms, t = r, \quad (32o)$$

$$0 = xtr_{pmtr}^s - xtr_{pmtr}^{FS} - xtr_{pmtr}^{SS+,s} + xtr_{pmtr}^{SS-,s}, \lambda_{pmtr}^{s,p17} \text{ free}, \forall pms, t = r + 1. \quad (32p)$$

6.1.7 Sales, Flow, and Production Constraints

$$0 = sales_{pmtr}^s - SALES_{pmt(r-1)}^{previous} - sales_{pmtr}^{adj+} + sales_{pmtr}^{adj-}, \lambda_{pmtr}^{s,p12} \text{ free}, \forall pms, t = r, \quad (32q)$$

$$0 = sales_{pmtr}^s - sales_{pmtr}^{FS} - sales_{pmtr}^{SS+,s} + sales_{pmtr}^{SS-,s}, \lambda_{pmtr}^{s,p13} \text{ free}, \forall pms, t = r + 1, \quad (32r)$$

$$0 = flows_{patr}^{s,prod} - FLOW_{pat(r-1)}^{previous,prod} - flows_{patr}^{adj+,prod} + flows_{patr}^{adj-,prod}, \lambda_{patr}^{s,p18} \text{ free}, \forall pas, t = r, \quad (32s)$$

$$0 = flows_{patr}^{s,prod} - flows_{patr}^{FS,prod} - flows_{patr}^{SS+,s,prod} + flows_{patr}^{SS-,s,prod}, \lambda_{patr}^{s,p19} \text{ free}, \forall pas, t = r + 1, \quad (32t)$$

$$0 = prod_{pmtr}^s - PROD_{pmt(r-1)}^{previous} - prod_{pmtr}^{adj+} + prod_{pmtr}^{adj-}, \lambda_{pmtr}^{s,p20} \text{ free}, \forall pms, t = r, \quad (32u)$$

$$0 = prod_{pmtr}^s - prod_{pmtr}^{FS} - prod_{pmtr}^{SS+,s} + prod_{pmtr}^{SS-,s}, \lambda_{pmtr}^{s,p21} \text{ free}, \forall pms, t = r + 1, \quad (32v)$$

$$0 \leq flows_{patr}^{s,prod} - CONTRACTS_{pat} \perp \lambda_{patr}^{s,p4} \geq 0 \forall pats, \quad (32w)$$

$$0 \leq prod_{pmtr}^s - DP_{pm}^{\max} \perp \lambda_{pmtr}^{s,p5} \geq 0 \forall pmts, \quad (32x)$$

$$0 \leq prod_{pmtr}^{FS} - DP_{pm}^{\max} \perp \lambda_{pmtr}^{p6} \geq 0 \forall pm, t = r + 1. \quad (32y)$$

6.1.8 Balance Constraints

$$0 = prod_{pmtr}^s + \sum_{a \in a^{in}(m)} (1 - LOSS_a) flows_{patr}^{s,prod} + xtr_{pmtr}^s - sales_{pmtr}^s - \sum_{a \in a^{out}(m)} flows_{patr}^{s,prod} - inj_{pmtr}^s, \lambda_{pmtr}^{s,p1} \text{ free } \forall pmts. \quad (33a)$$

6.2 KKT Conditions for TSO

6.2.1 Flow Variables

$$0 \leq D_t DAY S_t (PROB_r^s) \left[-(\tau_{atr}^s + \tau_{at}^{REG}) + \frac{\partial C^a (flows_{atr}^{s,tso})}{\partial flows_{atr}^{s,tso}} \right] + \begin{cases} \lambda_{atr}^{s,tso1}, & t = r \\ \lambda_{atr}^{s,tso2}, & t = r + 1 \\ 0, & t > r + 1 \end{cases} + \lambda_{atr}^{s,tso3} \perp flows_{atr}^{s,tso} \geq 0, \forall ats \quad (34a)$$

$$0 \leq D_t DAY S_t RU_{ar}^{flows} - \sum_s \lambda_{atr}^{s,tso1} \perp flows_{atr}^{adj+,tso} \geq 0, \forall a, t = r \quad (34b)$$

$$0 \leq D_t DAY S_t RO_{ar}^{flows} + \sum_s \lambda_{atr}^{s,tso1} \perp flows_{atr}^{adj-,tso} \geq 0, \forall a, t = r \quad (34c)$$

$$0 \leq \lambda_{atr}^{tso4} - \sum_s \lambda_{atr}^{s,tso2} \perp flows_{atr}^{FS,tso} \geq 0, \forall a, t = r + 1 \quad (34d)$$

$$0 \leq D_t DAY S_t (PROB_r^s) RU_{ar}^{flows} - \lambda_{atr}^{s,tso2} \perp flows_{atr}^{SS+,s,tso} \geq 0, \forall as, t = r + 1 \quad (34e)$$

$$0 \leq D_t DAY S_t (PROB_r^s) RO_{ar}^{flows} + \lambda_{atr}^{s,tso2} \perp flows_{atr}^{SS-,s,tso} \geq 0, \forall as, t = r + 1 \quad (34f)$$

6.2.2 TSO Constraints

$$0 = flows_{atr}^{s,tso} - FLOW S_{at(r-1)}^{previous,tso} - flows_{atr}^{adj+,tso} + flows_{atr}^{adj-,tso}, \lambda_{atr}^{s,tso1} \text{ free}, t = r, \forall as, \quad (35a)$$

$$0 = flows_{atr}^{s,tso} - flows_{atr}^{FS,tso} - flows_{atr}^{SS+,s,tso} + flows_{atr}^{SS-,s,tso}, \lambda_{atr}^{s,tso2} \text{ free}, t = r + 1, \forall as, \quad (35b)$$

$$0 \leq DA_a^{\max} - flows_{atr}^{s,tso} \perp \lambda_{atr}^{s,tso3} \geq 0, \forall ats, \quad (35c)$$

$$0 \leq DA_a^{\max} - flows_{atr}^{FS,tso} \perp \lambda_{atr}^{tso4} \geq 0, t = r + 1, \forall a. \quad (35d)$$

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