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Mathematical optimization approaches for facility layout problems: The state-of-the-art and future research directions

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Abstract: Facility layout problems are an important class of operations research problems that has been studied for several decades. Most variants of facility layout are NP-hard, therefore global optimal solutions are difficult or impossible to compute in reasonable time. Mathematical optimization approaches that guarantee global optimality of solutions or tight bounds on the global optimal value have nevertheless been successfully applied to several variants of facility layout. This review covers three classes of layout problems, namely row layout, unequal-areas layout, and multifloor layout. We summarize the main contributions to the area made using mathematical optimization, mostly mixed integer linear optimization and conic optimization. For each class of problems, we also briefly discuss directions that remain open for future research.

Key Words: Facilities planning and design, unequal-areas facility layout, row layout, mixed integer linear optimization, semidefinite optimization.

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1 Introduction

Facility layout problems (FLPs) are a general class of operations research problems concerned with finding the optimal arrangement of a given number of non-overlapping indivisible departments within a given facility. The objective is to minimize the total expected cost of inter-departmental flows inside the facility, where the cost incurred for each pair of departments is equal to the rectilinear distance between the centroids of the departments multiplied by the projected flow between them. This projected flow accounts in the aggregate for all the relevant quantities; these may include adjacency preferences, transportation costs, the construction of a material-handling system, or the costs of connection wiring. The pairwise costs are normally assumed to be non-negative. The facility and the departments are rectangular, and the area of each department is specified, but finding the dimensions of each department at optimality is also part of the problem.

FLPs have a variety of applications. Much of the work was motivated by the physical organization of manufacturing systems, see e.g. Meller and Gau (1996). The FLP is particularly relevant in the area of flexible manufacturing systems: these are automated systems that produce an array of diferent parts. The layout of the machines and other production components has a significant impact on the costs and the productivity of these systems, see e.g. Hassan (1994). Other applications of FLPs include balancing hydraulic turbine runners (Laporte and Mercure, 1988), algorithm initalization in numerical analysis (Brusco and Stahl, 2000), VLSI fixed-outline floorplanning Luo et al. (2008), and optimal data memory layout generation for digital signal processors (Wess and Zeitlhofer, 2004).

FLPs have been extensively studied in the literature since the 1960s. Numerous variations on the basic problem described above have been considered, and different models have been proposed for each variation. Examples of such variations are: specially structured instances of the problem (e.g. layouts on rows or on loops); dynamic FLPs with time-dependencies; FLPs under uncertainty in the data; and multi-objective FLPs. We refer the reader to the survey papers of Meller and Gau (1996) and Singh and Sharma (2006) as well as the books of Kusiak (1990) and Heragu (2008) for more information about the FLP and its variations. A growing collection of FLP benchmark instances is available online Anjos (2015).

The FLP is NP-hard in general, so solving it to global optimality in reasonable time is often difficult. Indeed the restricted version where the dimensions of the departments are all equal and fixed, and the optimization is taken over a fixed finite set of possible locations for the departments, is already an NP-hard problem. This restricted version is known as the quadratic assignment problem, a combinatorial optimization problem well known for its computational difficulty, see e.g. Loiola et al. (2007).

The constraints of the basic FLP can be grouped into two sets:

- *Department shape requirements* include the required area, and restrictions on the dimensions (height and width) such as bounds on the ratios height/width and width/height, called aspect ratios. Although they typically lead to convex constraints, the shape requirements nevertheless pose certain challenges. While the area constraint traditionally required a careful linearization approach, it can be modelled exactly using conic optimization, see e.g. Anjos and Liers (2012). On the other hand, bounding the aspect ratios poses a greater challenge than it may appear because requiring small aspect ratios, while desirable in real-world applications, generally makes the problem harder.
- *Department location requirements* include the non-overlap of departments, fitting every department within the facility, assigning certain departments to, or forbidding them from, particular locations within the facility. The main challenge here are the non-overlap constraints that inherently non-convex and combinatorial; they greatly increase the complexity of the problem.

This review covers three broad classes of FLPs, namely row layout problems (Section 2), unequal-areas FLPs (Section 3), and multifloor FLPs (Section 4). The focus in this review is on mathematical optimization approaches, primarily those using mixed integer linear optimization (MILO) and semidefinite optimization (SDO). We also include brief discussions of symmetry reduction (Section 5) and valid inequalities (Section 6) as these are essential for solving the resulting relaxations efficiently. It is important to mention that there is a rich literature on heuristic algorithms for FLPs; these include genetic algorithms, tabu search, simulated annealing, and many others, see e.g. Mavridou and Pardalos (1997). We conclude this review with a summary of directions for future research in Section 7.

2 Row FLPs

Row FLPs share the following common problem statement: Given a set of rectangular departments, a number of rows, and a pairwise non-negative weight for each pair of departments, determine an assignment of departments to rows and the positions of the departments in each row so that the total of the weighted center-to-center distances is minimized.

Row FLPs are an important class of problems because in many practical contexts, the departments are to be placed in well-defined rows where the separation between the rows is predetermined according to the features of the material-handling system or the flows of people. In particular, a minimum space between the departments, the so-called clearance, is required to satisfy safety and operational considerations. For modelling purposes, we assume that this clearance is included in the lengths of the departments.

We assume here that the rows and the department all have the same height, that any department can be assigned to any row, and that the distances between adjacent rows are equal. Under these assumptions, solving an instance of the row FLP means resolving three questions:

1. Assign each department to exactly one row;
2. Express mathematically the center-to-center distance between departments (that may or may not be in the same row);
3. Handle possible empty space between departments.

Section 2.1 is concerned with the simplest version of row FLP, namely the single-row FLP. Section 2.2 covers the double-row FLP, and Section 2.3 extends the coverage to the general multirow FLP.

2.1 The Single-Row FLP

The Single-Row FLP (SRFLP) is concerned with the arrangement of a given number of rectangular departments in a line. An instance of the SRFLP consists of n one-dimensional departments with given positive lengths $\ell_1, \ldots, \ell_n$ and pairwise costs c_{ij}. The objective is to arrange the departments in a row so as to minimize the total weighted sum of the center-to-center distances. Figure 1 provides an illustration of the SRFLP in the context of placing the departments along the path of an automated guided vehicle (AGV); in this context the objective is typically to place the departments so as to minimize the distance travelled by the vehicle.

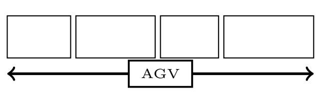

Figure 1: Illustration of a Single-Row Layout along the path of an AGV

Because there is only one row, there is no need to assign departments to rows; moreover, the assumption that $c_{ij} \geq 0$ ensures that there is no empty space between departments at optimality. Hence the remaining question is to express the center-to-center distance between departments.

The SRFLP is the most studied of the row FLPs. It is also known in the literature as the one-dimensional space allocation problem, and it has several interesting connections to other known combinatorial optimization problems such as the maximum-cut problem, the quadratic linear ordering problem, and the linear arrangement problem. These connections were explored in Anjos and Liers (2012).

The SRFLP consists of finding a permutation of the departments that minimizes the weighted sum of the pairwise distances. It can be succintly expressed as:

$$\min_{\pi\in\Pi_n} \sum_{i<j} c_{ij}\left[\frac{1}{2}\ell_i + D_\pi(i,j) + \frac{1}{2}\ell_j\right],$$

where Π_n denotes the set of all permutations of $\{1,2,\ldots,n\}$. The central modelling question is thus to express the quantity $D_\pi(i,j)$ in a way that allows efficient computation. This means that it is not necessary to know the position of each department; it suffices to know for each pair of objects which departments are *between* them. Hence the key concept here is that of *betweenness*. We present here two mathematical optimization models for the SRFLP, one based on mixed-integer linear optimization and the other based on semidefinite optimization, that model betweenness in different ways. Our presentation here is brief; for a much more detailed exposition on the state-of-the-art for the SRFLP, we refer the reader to the recent review paper in this journal by Keller and Buscher (2015).

2.1.1 MILO models

The approach sketched here was originally proposed in Amaral (2009). For three distinct departments i,j,k, define the betweenness variables ζ_{ijk} as:

$$\zeta_{ijk} = \begin{cases} 1, & \text{if department } k \text{ lies between departments } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases}$$

Using these variables, the objective function of the SRFLP is expressed as:

$$\sum_{i<j} c_{ij}\left[\frac{1}{2}(\ell_i+\ell_j) + \sum_{k\neq i,j} \ell_k\zeta_{ijk}\right] \tag{1}$$

and this is optimized subject to the following constraints:

$$\begin{aligned}
\zeta_{ijk}+\zeta_{ikj}+\zeta_{jki} &= 1, && \text{for all } \{i,j,k\}\subseteq\{1,\ldots,n\}, && (2)\\
\zeta_{ijd}+\zeta_{jkd}-\zeta_{ikd} &\geq 0, && \text{for all } \{i,j,k,d\}\subseteq\{1,\ldots,n\}, && (3)\\
\zeta_{ijd}+\zeta_{jkd}+\zeta_{ikd} &\leq 2, && \text{for all } \{i,j,k,d\}\subseteq\{1,\ldots,n\}, && (4)\\
0\leq \zeta_{ijk} &\leq 1, && \text{for all } \{i,j,k\}\subseteq\{1,\ldots,n\}. && (5)
\end{aligned}$$

Conditions (2) and (5) are specified for each triplet of distinct departments, whereas (3) and (4) are defined for each 4-tuple of departments.

When the ζ_{ijk} are relaxed to $0\leq\zeta_{ijk}\leq 1$, the resulting LO relaxation can be weak in practice. Amaral thus proposes an additional class of valid inequalities that improve the relaxation.

Proposition 1 *Amaral (2009) Let $\beta\leq n$ be a positive even integer and let $S\subseteq\{1,\ldots,n\}$ such that $|S|=\beta$. For each $r\in S$, and for any partition (S_1,S_2) of $S\backslash\{r\}$ such that $|S_1|=\frac{1}{2}\beta$, the inequality*

$$\sum_{t<q,t\in S_1,q\in S_1}\zeta_{tqr} + \sum_{t<q,t\in S_2,q\in S_2}\zeta_{tqr} - \sum_{t\in S_1,q\in S_2}\zeta_{\min\{t,q\},\max\{t,q\},r} \leq 0 \tag{6}$$

is valid for the above formulation of the SRFLP.

It is straightforward to check that for $\beta=4$, (6) is a triangle inequality of the form (3). Amaral (2009) also shows that the size of the LP relaxation can be reduced by projecting the feasible set into a lower-dimensional space.

2.1.2 SDO models

To introduce an SDO-based relaxation, we begin by introducing $\{\pm 1\}$ binary variables as in customary in SDO. For each pair of departments ij with $1 \le i < j \le n$, define

$$R_{ij} := \begin{cases} 1, & \text{if } i \text{ is to the right of } j, \\ -1, & \text{otherwise.} \end{cases}$$

In this definition, the order of the subscripts matters, and $R_{ij} = -R_{ji}$. In fact, R_{ij} determines a linear ordering on the objects.

It is also clear that not every assignment of ± 1 values to the R_{ij} variables represents a permutation. For an assignment to represent a permutation, it is necessary to enforce

if i is to the right of j and j is to the right of k, then i is to the right of k.

Equivalently, if $R_{ij} = R_{jk}$ then $R_{ik} = R_{ij}$. This necessary transitivity condition can be formulated as a set of quadratic constraints:

$$R_{ij}R_{jk} - R_{ij}R_{ik} - R_{ik}R_{jk} = -1 \text{ for all triples } 1 \le i < j < k \le n. \tag{7}$$

Using the R_{ij} variables, it is straightforward to express betweenness after observing that $R_{ki}R_{kj} = -1$ if and only if facility k is between i and j. Hence the objective function can be expressed as

$$\sum_{i<j} c_{ij} \left[\frac{1}{2}(\ell_i + \ell_j) + \sum_{k \ne i,j} \ell_k \left(\frac{1 - R_{ki}R_{kj}}{2} \right) \right],$$

and the consequent formulation of SRFLP is:

$$\begin{array}{ll} \min & K - \sum\limits_{i<j} \frac{c_{ij}}{2} \left[\sum\limits_{k<i} \ell_k R_{ki}R_{kj} - \sum\limits_{i<k<j} \ell_k R_{ik}R_{kj} + \sum\limits_{k>j} \ell_k R_{ik}R_{jk} \right] \\ \text{s.t.} & \\ & R_{ij}R_{jk} - R_{ij}R_{ik} - R_{ik}R_{jk} = -1 \text{ for all triples } i<j<k \\ & R_{ij}^2 = 1 \text{ for all } i<j \end{array} \tag{8}$$

where $K := \left(\sum\limits_{i<j} \frac{c_{ij}}{2} \right) \left(\sum\limits_{k=1}^{n} \ell_k \right)$.

Applying standard techniques from SDO, this formulation leads to the following SDO relaxation Anjos et al. (2005):

$$\begin{array}{ll} \min & K - \sum\limits_{i<j} \frac{c_{ij}}{2} \left[\sum\limits_{k<i} \ell_k X_{ki,kj} - \sum\limits_{i<k<j} \ell_k X_{ik,kj} + \sum\limits_{k>j} \ell_k X_{ik,jk} \right] \\ \text{s.t.} & \\ & X_{ij,jk} - X_{ij,ik} - X_{ik,jk} = -1 \text{ for all triples } i<j<k \\ & X_{ii} = 1, \text{ for } i = 1, \dots, n \\ & X \succeq 0 \end{array} \tag{9}$$

where $X \succeq 0$ denotes that X is symmetric positive semidefinite. Note that $X \in \mathcal{S}^{\binom{n}{2}}$, the set of symmetric matrices of dimension $\binom{n}{2}$, and the interpretation of the entries of X is that $X_{p_i,p_j} = R_{p_i}R_{p_j}$ for any two pairs p_i, p_j.

Subsequent improvements to this relaxation were given in Hungerländer and Rendl (2013). We refer the reader to that paper and to Keller and Buscher (2015) for more details.

2.2 The Double-Row FLP

The Double-Row FLP (DRFLP) is an extension of the SRFLP in which departments can be placed on both sides of a central corridor. The distance between the two rows is assumed to be negligible, and thus the centre-to-centre distance between two departments is measured parallel to the corridor. Figure 2 illustrates this problem in the manufacturing context where the corridor is the operating space for an AGV transporting material between the departments. Another application for the DRFLP is the arrangement of rooms in buildings, see e.g. Ahonen et al. (2014).

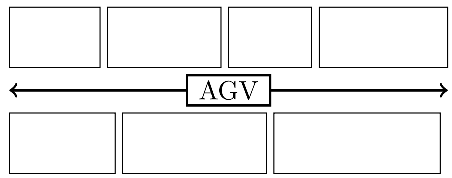

Figure 2: DR-FLP

The DRFLP has received much less attention in the literature. As far as we could ascertain, the first reference to double-row layouts is in Heragu and Kusiak (1988) where a nonlinear optimization model is proposed and used to find locally optimal solutions. Most of the subsequent mathematical optimization approaches in the literature are based on either MILO (Chung and Tanchoco (2010); Amaral (2013b)) or SDO Hungerländer and Anjos (2015).

Unlike in the SRFLP, there is in the DRFLP a need to address all three questions for row FLPs. The assignment of departments to rows is somewhat simplified by the fact that there are only two rows: it suffices to determine which departments are placed in the first row, because the remaining departments must be in the second row. On the other hand, betweenness no longer suffices to determine centre-to-centre distances, and the optimal layout may involve some empty space between departments.

2.2.1 MILO models

In this section we describe two approaches that extend in different ways the MILO models proposed for the SRFLP. Both extensions involve a combination of discrete and continuous variables, where the former represent the assignment of departments to rows and the relative position of two departments, and the latter give the positions of the department centers with respect to a fixed origin. Without loss of generality the corridor is placed along the x-axis, and the origin is at the left end of the corridor.

A model with $O(n^2)$ binary variables

Consider the binary vector $y = (y_{ij})_{1 \le i,j \le n}$ such that

$$y_{ij} = \begin{cases} 1, & \text{if department } i \text{ is to the left of department } j \\ & \text{ and both } i \text{ and } j \text{ are in the same row;} \\ 0, & \text{otherwise.} \end{cases}$$

Let the polytope P_n be the convex hull of all y-incidence vectors representing a partition of the set of n departments into two ordered subsets. The following inequalities are valid for P_n:

$$y_{ik} + y_{ki} + y_{jk} + y_{kj} - y_{ij} - y_{ji} \le 1, \quad 1 \le i,j,k \le n,\ i < j,\ k \ne i,j \tag{10}$$
$$y_{ik} + y_{ji} + y_{kj} - y_{ki} - y_{ij} - y_{jk} \le 1, \quad 1 \le i,j,k \le n,\ i,k < j,\ k \ne i \tag{11}$$
$$y_{ij} + y_{ik} + y_{jk} + y_{ji} + y_{ki} + y_{kj} \ge 1. \tag{12}$$

We gather these constraints in the definition of the polytope Q_n:

$$Q_n = \{y \in R^{n(n-1)} : (10), (11), (12), 0 \le y_{ij} \le 1,\ 1 \le i,j \le n,\ i \ne j\}.$$

Constraints (10) are transitivity constraints with respect to row assignments. They ensure that if departments i and k are in the same row ($y_{ik}+y_{ki}=1$) and departments k and j are in the same row ($y_{jk}+y_{kj}=1$), then

$$1+1-(y_{ij}+y_{ji}) \leq 1,$$

implying $y_{ij}+y_{ji} \geq 1$, i.e., departments i and j are in the same row.

Constraints (11) are three-cycle constraints. They forbids a solution where department k is placed to the right of department i, i is to the right of j, and j is to the right of k (thus forming an impossible cycle).

Constraints (12) require that the solution at least two departments of $\{i,j,k\} \subset \{1,\ldots,n\}$ have to be in the same row. It also ensures that the departments are partitioned into no more than two rows.

Next define the following continuous variables: x_i is the position of the center of department i ($1 \leq i \leq n$) along the corridor, and d_{ij} is the distance between (the centers of) departments i and j ($1 \leq i < j \leq n$) measured parallel to the corridor.

We now state the MILO model of Amaral (2013b):

$$\min \quad \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_{ij}d_{ij} \tag{13}$$

$$\text{s.t.} \quad d_{ij} \geq x_i - x_j,\ d_{ij} \geq x_j - x_i,\ 1 \leq i < j \leq n \tag{14}$$

$$x_i + \left(\frac{\ell_i+\ell_j}{2}\right) \leq x_j + L(1-y_{ij}),\ 1 \leq i,j \leq n,\ i \neq j \tag{15}$$

$$d_{ij} - \left(\frac{\ell_i+\ell_j}{2}\right) y_{ij} - \left(\frac{\ell_i+\ell_j}{2}\right) y_{ji} \geq 0,\ 1 \leq i < j \leq n \tag{16}$$

$$x_i^* \leq x_j^*,\ (i^*,j^*) = \arg\min_{1\leq i,j\leq n} c_{ij} \tag{17}$$

$$y \in Q_n \tag{18}$$

$$y_{ij} \in \{0,1\},\ 1 \leq i,j \leq n,\ i \neq j \tag{19}$$

$$\frac{\ell_i}{2} \leq x_i \leq L - \frac{\ell_i}{2},\ 1 \leq i \leq n \tag{20}$$

where $L = \sum_{i=1}^{n} \ell_i$.

Constraints (14) give the rectilinear distance between each pair of departments. Constraints (18) and (19) characterize the y-incidence vectors, and constraints (20) are bounds on the x variables.

Constraints (15) ensure that departments assigned to the same row do not overlap. Constraints (16) ensure that if department i is placed in the same row as department j, then the distance between their centers is at least $(\ell_i+\ell_j)/2$. Note that constraints (16) are redundant in the presence of constraints (14) and (15), but they may be helpful for a branching algorithm.

Constraints (17) eliminate symmetric solutions. The departments (i^*,j^*) correponding to the minimum cost $c_{i^*j^*}$ are required to have $x_i^* < x_j^*$. Reducing symmetry is useful from a computational viewpoint.

A model with $O(n^3)$ binary variables

For this model, we define two sets of binary variables:

$$y_{ik} = \begin{cases} 1, & \text{if department } i \text{ is assigned to row } k \\ 0, & \text{otherwise.} \end{cases}$$

$$z_{kij} = \begin{cases} 1, & \text{if department } j \text{ is placed to the right of department } i \text{ in row } k \\ 0, & \text{otherwise.} \end{cases}$$

As in the previous model, we use continuous variables to determine the location of the departments. Specifically we let x_{ik} denote the absolute location of department i in row k, and set it to zero if i is not assigned to row k.

These definitions support the model proposed in Chung and Tanchoco (2010). This model explicitly accounts for clearances between departments. As corrected by Zhang and Murray (2012), the model is:

$$\min \quad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(v_{ij}^{+} + v_{ij}^{-}\right) \tag{21}$$

$$\text{s.t.} \quad \sum_{k \in R} x_{ik} - \sum_{k \in R} x_{jk} + v_{ij}^{+} - v_{ij}^{-} = 0, \;\; i \in I_1, j \in I_2 \tag{22}$$

$$x_{ik} \leq M y_{ik}, \;\; i = 1, \ldots, n, \;\; k \in R \tag{23}$$

$$\sum_{k \in K} y_{ik} = 1, \;\; i = 1, \ldots, n \tag{24}$$

$$\frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} + a_{ik} z_{kji} \leq x_{ik} - x_{jk} + M(1 - z_{kji}), \;\; i \in I_1, j \in I_2, k \in K \tag{25}$$

$$\frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} + a_{ik} z_{kij} \leq x_{jk} - x_{ik} + M(1 - z_{kij}), \;\; i \in I_1, j \in I_2, k \in K \tag{26}$$

$$z_{kij} + z_{kji} \leq \frac{1}{2}(y_{ik} + y_{jk}), \;\; i \in I_1, j \in I_2, k \in K \tag{27}$$

$$z_{kij} + z_{kji} + 1 \geq y_{ik} + y_{jk}, \;\; i \in I_1, j \in I_2, k \in K \tag{28}$$

$$\left.\begin{array}{l} x_{ik} \geq 0, \;\; i \in I, k \in K \\ v_{ij}^{+}, v_{ij}^{-} \geq 0, i \in I_1, j \in I_2 \\ y_{ik} \in \{0,1\}, i \in I, k \in K \\ z_{kij} \in \{0,1\}, i \in I, j \in I \setminus \{i\}, k \in K \end{array}\right. \tag{29}$$

where a_{ij} is the required clearance between departments i and j, $I_1 = \{1, \ldots, n-1\}$, $I_2 = \{i+1, \ldots, n\}$, $K = \{1, 2\}$ is the set of rows, and the constant M is analogous to L in the previous model but also includes the clearances:

$$M = \sum_{i \in I} \left(\ell_i + \max_{j \in I} a_{ij} \right).$$

Constraints (22) compute the distances between departments. Constraints (23) set $x_{ik} = 0$ when department i is not assigned to row k. Constraints (24) ensure that a department is assigned to just one row. Constraints (25) and (26) prevent departments from overlapping if they are located in the same row.

Constraints (27) and (28) ensure consistency between the variables y and z as follows: If $y_{ik} = 1$ and $y_{jk} = 1$ then (27) and (28) together ensure that exactly one of z_{kij} and z_{kij} is equal to one. Otherwise, i.e., if at least one of y_{ik} and y_{jk} is equal to zero, then (27) sets both z_{kij} and z_{kij} to zero. Constraints (28) force either z_{kij} or z_{kji} to be 1 if i and j are both in row k.

Besides the fact that this model has significantly more binary variables, we also note that the meaning of the continuous variables x_{ik} is different, in the sense that $x_{ik} = 0$ does not mean that i is located at the origin of the x-axis.

Finally, it is important to observe that while the $O(n^2)$ model is specific to the DRFLP, the $O(n^3)$ model can be applied directly to the MRFLP by increasing the cardinality of K.

2.2.2 SDO model

Hungerländer and Anjos (2015) developed an SDO-based approach for the MRFLP that they also applied to the DRFLP. This approach is presented in Section 2.3.1.

2.3 Multirow layout

The MRFLP is the most general version of row layout problems. An instance of the MRFLP has a given number of rows to which the departments can be assigned, and departments can in general be assigned to any row, the departments all have the same height (equal to the row height, and the distances between adjacent rows are equal.

As examples of applications, MRFLP occur in flexible manufacturing systems, or more generally in pick and place applications, where gantry robots are used. These robots have linear axes of control and move up/down and left/right with the movement directions at right angles, as illustrated in Figure 3. This makes multi-row layouts particularly suitable, and the total weighted sum of the center-to-center rectilinear distances is a good measure of the total displacement of the robot to complete a given task.

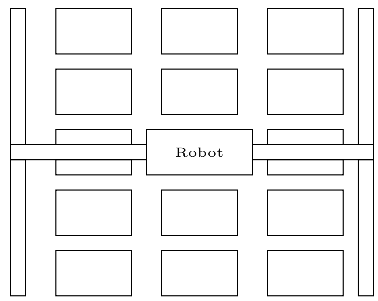

Figure 3: MRFLP

The MRFLP has also garnered very limited attention in the operations research literature to date. Heuristic algorithms were proposed in Heragu and Kusiak (1988), and a nonlinear formulation was given Gen and Cheng (1997) and solved using genetic algorithms.

In terms of approaches using MILO and SDO, as noted in Section 2.2.1, the $O(n^3)$ MILO formulation of Zhang and Murray (2012) for the DRFLP can be easily extended to the MRFLP, although this was not specifically done in that paper. More recently, an SDO-based approach was recently introduced in Hungerländer and Anjos (2015), and it is this approach that we present here. To the best of our knowledge, this is the only global optimization approach for the general row FLP with more than two rows. It is nevertheless clear that there is a need for more research on algorithms to solve the MRFLP.

2.3.1 SDO model

The SDO model presented in Hungerländer and Anjos (2015) for the MRFLP is based on the SDO formulation for the SRFLP presented in Section 2.1.2. The idea is to first assume that the assignment of departments to rows is fixed and that no spaces are allowed between departments in the same row. This restricted version of the MRFLP is called the k-Parallel Row Ordering Problem (k-PROP), see Section 2.3.2 and the references therein for more details.

Consider the k-PROP with n departments and m rows, and let $r : \{1,\dots,n\} \to \{1,\dots,m\}$ be the function that assigns departments to rows Let the binary variables R_{ij} be defined as in Section 2.1.2, and let d_{ij} represent the center-to-center distance between departments i and j measured parallel to the rows. If i and j are assigned to the same row, i.e., if $r(i) = r(j)$, then

$$d_{ij} = \frac{1}{2}(\ell_i + \ell_j) + \sum_{\substack{k\in N,\ k<i \\ r(k)=r(i)}} \ell_k \frac{1 - R_{ki}R_{kj}}{2} + \sum_{\substack{k\in N,\ i<k<j \\ r(k)=r(i)}} \ell_k \frac{1 + R_{ik}R_{kj}}{2} + \sum_{\substack{k\in N,\ k>j \\ r(k)=r(i)}} \ell_k \frac{1 - R_{ik}R_{jk}}{2}, \qquad (30)$$

while if $r(i) \neq r(j)$

$$
\begin{aligned}
d_{ij} = R_{ij} \Bigg[&\left(\frac{\ell_j}{2} + \sum_{\substack{k\in N,\ k<j \\ r(k)=r(j)}} \ell_k \frac{1+R_{kj}}{2} + \sum_{\substack{k\in N,\ k>j \\ r(k)=r(j)}} \ell_k \frac{1-R_{jk}}{2} \right) \\
&- \left(\frac{\ell_i}{2} + \sum_{\substack{k\in N,\ k<i \\ r(k)=r(i)}} \ell_k \frac{1+R_{ki}}{2} + \sum_{\substack{k\in N,\ k>i \\ r(k)=r(i)}} \ell_k \frac{1-R_{ik}}{2} \right) \Bigg].
\end{aligned} \tag{31}
$$

The above relations, plus the triangle inequalities relating the distances between every triplet of departments i, j, k:

$$z_{ij} + z_{ik} \geq z_{jk}, \ \ z_{ij} + z_{ik} \geq z_{jk}, \ \ z_{ik} + z_{jk} \geq z_{ij}, \ \ 1 \leq i < j < k \leq n, \tag{32}$$

are used in Hungerländer and Anjos (2015) to extend the SDO formulation for the SRFLP to an SDO formulation for the k-PROP. For the sake of brevity here, we refer the reader to Hungerländer and Anjos (2015) for the technical details.

Once an SDO formulation of the k-PROP is obtained, Hungerländer and Anjos (2015) allow the possibility of spaces using the following results:

Theorem 1 *If all the department lengths ℓ_i are integer, then there is always an optimal solution to the MRFLP on the half-integer grid.*

Corollary 1 *If all the department lengths ℓ_i are integer, then for each instance of the MRFLP, we obtain an equivalent instance of the k-PROP by adding spacing departments of length 0.5 such that the length of each row becomes equal to $M := \sum_{i=1}^{n} \ell_i$.*

The strategy is thus to add spacing departments of length 0.5 and with all involved connectivities equal to zero, and then apply the SDO approach for k-PROP. Because the number of spacing departments needed will normally be too large for practical computation, Hungerländer and Anjos (2015) also prove several results to reduce the number of spacing departments needed. One such result is the following:

Theorem 2 *Let p_i be the number of departments assigned to row i. Then an optimal solution to the MRFLP is preserved if we use p_i spaces of lengths $2^{k-1}, \ k = 0, 1, 2, \ldots, h_i$, where h_i is the smallest integer such that*

$$p_i \sum_{k=0}^{h_i} 2^{k-1} + \sum_{k=1,\ldots,n, \ r(k)=i} \ell_k \geq M.$$

Finally, to remove the restriction that the assignment of departments to rows is fixed, Hungerländer and Anjos (2015) obtain global optimal solutions (respectively bounds) by using this approach for all possible assignments (respectively for a subset of them).

2.3.2 Special cases of the MRFLP

The difficulty in solving the general MRFLP has motivated the study of a number of special cases with simplifying assumptions and/or specific structure that allow for more effective modelling and solution approaches. Some of those special cases are briefly presented here.

A first such special case is the equidistant MRFLP (MREFLP) in which all departments have the same length. This additional structure makes it possible to prove many interesting results. The case of the single-row MREFLP is the linear arrangement problem, see e.g. (Liu and Vannelli, 1995), and is well known to be NP-hard even if all the pairwise costs are binary Garey et al. (1974). The double-row MREFLP was considered in Amaral (2011) where an MILO formulation based on the quadratic assignment problem is given. For the general MREFLP, it is shown in Anjos et al. (2015) that the problem has an optimal solution on the integer grid (although the lengths of the spaces are in general continuous quantities). This implies that only spaces of unit length need to be used when modeling the MREFLP, and hence that the problem can be formulated as a purely discrete optimization problem, as is the case for the SRFLP in Section 2.1. Moreover, Anjos et al. (2015) prove exact results for the minimum number of spaces that need to be added so as to preserve at least one optimal solution. These results lead to both MILO and SDO models for the MREFLP.

Another important special case of the MRFLP is the Space-Free MRFLP (SF-MRFLP) in which no spaces are allowed within the rows, all rows have a common left origin, and the leftmost department in each row is flush with the left end of the row. Note that when there is only one row, the SF-MRFLP is equivalent to the SRFLP. Where there are two rows, the SF-MRFLP is also called the Space-Free DRFLP or the Corridor Allocation Problem, for which an MILO formulation is proposed in Amaral (2012). and an SDO approach is proposed in Hungerländer and Anjos (2012).

One more special case that has attracted attention in the literature is the k-Parallel Row Ordering Problem (k-PROP), for which as SDO approach was mentioned in Section 2.3.1 above. The k-PROP is a simplified version of the SF-MRFLP in which the assignment of departments to rows is given in advance, and no spaces are allowed between departments in the same row. The problem therefore reduces to finding the optimal permutation of the departments within each row to minimize the total weighted sum of the center-to-center distances. When the number of rows equals two, this problem is simply called PROP, and an MILO formulation for it was given in Amaral (2013a).

2.4 Computational performance of the models

Row layout FLP remain highly challenging problems. We summarize here the state-of-the-art in terms of the computational performance of the approaches preserved above.

For both the SRFLP and the single-row MREFLP, the largest instances solved to optimality had 42 departments, as reported in Hungerländer and Rendl (2013) and Hungerländer (2014) respectively.

For the DRFLP, Amaral (2013b) used the $O(n^2)$ model to obtain solutions of instances with up to 12 departments within one hour. Amaral (2013b) also tested the $O(n^3)$ model, but was unable to solve instances with more than 10 departments within three hours. Murray et al. (2013) used the corrected model of Zhang and Murray (2012) for asymmetric flows. The constraints are (23)–(29), and the objective function is

$$\sum_{i\in I_1}\sum_{j\in I_2}(c_{ij}+c_{ji})\left(v_{ij}^{+}+v_{ij}^{-}\right).$$

The conclusion of the computational tests is that with a run-time limit of 10 minutes, most of the heuristic algorithms perform better than CPLEX for instances with more than 20 departments.

Finally for the MRFLP, Hungerländer and Anjos (2015) computed tight global bounds for instances with up to 12 departments. They adapted an approach originally proposed by Fischer et al. (2006) that was successful for the max-cut problem and several ordering problems. The SDO-based approach was applied to instances with up to 5 rows and up to 8 departments. The results show that the SDO approach is most effective for 4 or 5 rows. There may be an intuitive explanation for this: as an extreme example, note that it is easier to partition 5 departments into 5 rows than into 2 rows. This is in part because the model does not take into account the distance between rows, so assigning department 1 to row 1 is exactly the same as assigning it to row 4. Accounting for the distances between rows may change the nature of the results, but has not yet been done to the best of our knowledge.

3 Unequal-Areas FLP

The Unequal-Areas FLP (UA-FLP) is concerned with finding the optimal arrangement of a given number of nonoverlapping indivisible departments with varying areas so as to minimize the total expected cost of flows inside the facility. Unlike in the row FLPs, the dimensions of each department are optimized (subject to the area requirement).

The UA-FLP, sometimes called the single-floor FLP, has received much attention in the literature. It was originally formulated by Armour and Buffa (1963), and Montreuil (1991) proposed one of the first MILO formulations on the plane, using binary variables to prevent overlap. Among the subsequent MILO approaches, we mention those of (Sherali et al., 2003), Castillo and Westerlund (2005), and Meller et al. (2007). This last approach successfully solved instances with up to 11 departments to global optimality, the largest such results to date.

Most of the approaches reviewed here are two-stage methodologies, where the first stage determines the relative location of the departments, and the second stage obtains a final layout via a mathematical optimization model. The main differences between different approaches are in the first-stage algorithms. Some researchers (Anjos and Vannelli, 2002, 2006; Jankovits et al., 2011; Anjos and Vieira, 2015) apply a mathematical optimization model where overlapping is allowed but penalized. Recently, Gonçalves and Resende (2015) successfully applied a random-key genetic algorithm (GA); this is to date the most successful GA approach. Xie and Sahinidis (2008) present a branch and bound algorithm that also uses the sequence-pair representation, and Bozer and Wang (2012) present a heuristic based on a graph-pair representation and a simulated annealing technique.

A number of heuristics for the UA-FLP are used with a slicing-tree structure. This structure represents the floor plan by recursively partitioning it in such a way that each rectangular partition in the slicing-tree structure corresponds to the space allocated to a facility. The slicing tree is a binary tree that shows the recursive partitioning process applied. Each slicing tree corresponds to a particular layout, and the nodes of the tree contain either a department or a cut operator. This strategy was first used by Otten (1982) in the context of VLSI design and later extended to the UA-FLP by Tam (1992). It is used by Shayan and Chittilappilly (2004), Diego-Mas et al. (2008), Scholz et al. (2009), Komarudin and Wong (2010), and Chang and Ku (2013).

We will describe several approaches from the last decade; many of them are two-stage approaches as these seem to be the most promising for attacking large-scale instances of UA-FLP.

3.1 An exact formulation of the UA-FLP

We begin by presenting an exact formulation that uses only continuous variables. The reasons for doing so are two-fold: we establish some notation that will be common for the remainder of this section, and we explicit point out where the difficulties lie in solving UA-FLP, thus motivating the solution approaches subsequently presented.

We label the departments $i = 1, ..., n$, where n is the total number of departments. We assume that we are given the height and width of the facility as h_F and w_F respectively, and that for each department i we have lower and upper bounds $w_i^{\min}$ and $w_i^{\max}$ on its width, and $h_i^{\min}$ and $h_i^{\max}$ on its height. We also assume that β_i, an upper bound on the aspect ratio of department i, is given for each department i. It is necessary that $\beta_i \geq 1$, and the closer β_i is to unity, the closer the shape of department i will be to a square.

With this notation, the UA-FLP can be formulated as follows:

$$\min_{x_i, y_i, h_i, w_i} \quad \sum_{1 \leq i < j \leq N} c_{ij}(|x_i - x_j| + |y_i - y_j|) \tag{33}$$

$$\text{s.t.} \quad w_i^{\min} \leq w_i \leq w_i^{\max}, \text{ for } i = 1, \ldots, n \tag{34}$$

$$h_i^{\min} \leq h_i \leq h_i^{\max}, \text{ for } i = 1, \ldots, n \tag{35}$$

$$w_i h_i = A_i, \quad \text{for } i = 1, \ldots, n \tag{36}$$

$$\max\left\{\frac{w_i}{h_i}, \frac{h_i}{w_i}\right\} \leq \beta_i, \text{ for } i = 1, \ldots, n \tag{37}$$

$$x_i + \frac{1}{2}w_i \leq \frac{1}{2}w_F \quad \text{and} \quad \frac{1}{2}w_i - x_i \leq \frac{1}{2}w_F, \text{ for } i = 1, \ldots, n \tag{38}$$

$$y_i + \frac{1}{2}h_i \le \frac{1}{2}h_F \quad \text{and} \quad \frac{1}{2}h_i - y_i \le \frac{1}{2}h_F, \text{ for } i = 1, \ldots, n \tag{39}$$

$$\begin{aligned} &|x_i - x_j| \ge \frac{1}{2}(w_i + w_j) \quad \text{or} \quad |y_i - y_j| \ge \frac{1}{2}(h_i + h_j), \\ &\qquad \text{for all } 1 \le i < j \le n. \end{aligned} \tag{40}$$

The first four sets of constraints enforce the shape requirements. Constraints (34) and (35) enforce the bounds on the width and height of each department. Constraints (36) enforce the area requirement for each department. Note that these constraints can be relaxed to $w_i h_i \ge A_i$. This relaxed form has the advantage of being convex, and in fact it can be formulated as a conic constraint as shown in Section 3.2. Because the optimization of the objective function will push this relaxed form of the constraint towards equality, it floows that in general $w_i h_i$ will equal A_i at optimality. Moreover Theorem 3.1 in Takouda et al. (2005) states that if $\sum_{i=1}^n A_i = h_F w_F$ then the constraints (36) must hold at every feasible solution. Constraints (37) limit the maximum aspect ratio; it is straightforward to write them as two linear inequality constraints.

The last two sets of constraints enforce the location requirements. Constraints (38)–(39) ensure that the departments are inside the facility, Finally, Constraints (40) prevent overlapping; these constraints are disjunctive and non-convex, and are the hardest ones to handle. If the relative position of each pair of departments is known, then the constraints (40) can be written as linear inequalities, and the formulation becomes a (convex) conic optimization problem that is straightforward to solve. This observation motivates the two-stage philosophy in several of the approaches in the literature; we present the most prominent in Sections 3.3 and 3.4.

Note that this formulation locates the center of the facility at the origin, while some of the models below locate the origin at the bottom left-hand corner of the facility. This difference is otherwise of no consequence.

3.2 MILO models

We begin with the MILO model introduced by Meller et al. (1999) and enhanced by Sherali et al. (2003). Define the binary variables

$$z_{ij}^h = \begin{cases} 1 & \text{if } i \text{ must precede } j \text{ horizontally,} \\ 0 & \text{otherwise,} \end{cases}$$

$$z_{ij}^v = \begin{cases} 1 & \text{if } i \text{ must precede } j \text{ vertically,} \\ 0 & \text{otherwise.} \end{cases}$$

The MILO formulation is as follows:

$$\min \quad \sum_{1 \le i < j \le n} c_{ij}(u_{ij} + v_{ij}) \tag{41}$$

$$\text{s.t.} \quad \begin{aligned} &u_{ij} \ge x_i - x_j \text{ and } u_{ij} \ge x_j - x_i, \ 1 \le i < j \le n \\ &v_{ij} \ge y_i - y_j \text{ and } v_{ij} \ge y_j - y_i, \ 1 \le i < j \le n \end{aligned} \tag{42}$$

$$\begin{aligned} &\frac{1}{2}w_i \le x_i \le w_F - \frac{1}{2}w_i, \ i = 1, \ldots, n \\ &\frac{1}{2}h_i \le y_i \le h_F - \frac{1}{2}h_i, \ i = 1, \ldots, n \end{aligned} \tag{43}$$

$$\begin{aligned} &w_i^{\min} \le w_i \le w_i^{\max}, \ i = 1, \ldots, n \\ &h_i^{\min} \le h_i \le h_i^{\max}, \ i = 1, \ldots, n \end{aligned} \tag{44}$$

$$\begin{aligned} &a_i w_i + 4\left(w_i^{\min} + \frac{\lambda}{\Delta - 1}(w_i^{\max} - w_i^{\min})\right)^2 h_i \ge \\ &\quad 2a_i\left(w_i^{\min} + \frac{\lambda}{\Delta - 1}(w_i^{\max} - w_i^{\min})\right), \ \lambda = 0, 1, \ldots, \Delta - 1 \end{aligned} \tag{45}$$

$$z_{ij}^h + z_{ji}^h + z_{ij}^v + z_{ji}^v = 1, \ 1 \le i < j \le n \tag{46}$$

$$x_i + \frac{1}{2}w_i \leq x_j - \frac{1}{2}w_j + w_F(1 - z^h_{ij}),\ i \neq j$$
$$y_i + \frac{1}{2}h_i \leq y_j - \frac{1}{2}h_j + h_F(1 - z^v_{ij}),\ i \neq j \tag{47}$$
$$z^h_{ij},\ z^v_{ij} \in \{0,1\},\ i,j \in N. \tag{48}$$

Constraints (42) provide a linearization of the objective function (33) above. Constraints (43) ensure that each department is within the facility; they differ from (38)–(39) because this formulation places the origin at the bottom left-hand corner of the facility. Constraints (44) are lower and upper bounds for the widths and heights of the departments.

Constraints (45) are the discretized polyhedral outer-approximation introduced by Sherali et al. (2003) (and also used by Meller et al. (2007) and Liu and Meller (2007), see Section 3.2.1 below) in lieu of the exact area constraint $w_ih_i = A_i$, This approximation is effective in practice but less efficient that using the aforementioned convex conic relaxation that is supported by most current MILO solvers. This is because $w_ih_i \geq A_i$ is equivalent to a second-order cone constraint:

$$\begin{bmatrix} h_i & \sqrt{A_i} \\ \sqrt{A_i} & w_i \end{bmatrix} \succeq 0 \Leftrightarrow w_i + h_i \geq \left\| \begin{pmatrix} w_i - h_i \\ 2\sqrt{A_i} \end{pmatrix} \right\|_2 \tag{49}$$

(because $w_i \geq 0$ and $h_i \geq 0$).

Finally, constraints (46)–(48) use the relative-location variables z^h_{ij} and z^v_{ij} to prevent overlapping of the departments. These constraints separate departments i and j or horizontally or vertically, and i is to the left (or right, but noth both) of j or i is below (or above, but not both) of j, accordingly to which one of the variables $z^h_{ij}, z^v_{ij}, z^h_{ji}, z^v_{ji}$ is set to 1.

An alternative MILO representation of the relative positions of department is given in the next section.

3.2.1 Sequence-pair formulations

Sequence-pair approaches determine the relative positions of the departments using the so-called *sequence-pair representation*, and combine this representation with a MILO model similar to the one above to obtain the optimal layout. The sequence-pair representation was first used for VLSI design by Murata et al. (1995), and for the FLP by Meller et al. (2007) and Liu and Meller (2007).

A sequence-pair is a pair of department sequences, denoted Γ_+ and Γ_-, that together encode the relative location of the departments. The following theorem indicates how to translate a sequence pair into a layout.

Theorem 3 (Meller et al. (2007)) *Given a sequence-pair* (Γ_+, Γ_-) *and two departments* i *and* j *in* (Γ_+, Γ_-)*,* i *and* j *satisfy the following horizontal/vertical relationship in the FLP:*

- *if* j *succeeds* i *in both* Γ_+ *and* Γ_-*, then department* j *is to the right of department* i*;*
- *if* j *precedes* i *in both* Γ_+ *and* Γ_-*, then department* j *is to the left of department* i*;*
- *if* j *precedes* i *in* Γ_+ *and succeeds* i *in* Γ_-*, then department* j *is above department* i*;*
- *if* j *succeeds* i *in* Γ_+ *and precedes* i *in* Γ_-*, then department* j *is below department* i*.*

The sequence-pair structure can be incorporated in an MILO model as follows. Given a sequence pair (Γ_+, Γ_-) and departments i and j, define the binary variables:

$$z^+_{ij} = \begin{cases} 1 & \text{if } i \text{ precedes } j \text{ in } \Gamma_+, \\ 0 & \text{otherwise,} \end{cases}$$
$$z^-_{ij} = \begin{cases} 1 & \text{if } i \text{ precedes } j \text{ in } \Gamma_-, \\ 0 & \text{otherwise.} \end{cases}$$

These definitions lead to Theorem 4:

Theorem 4 (Meller et al. (2007)) *For any two departments i and j in facility layout, the following relationships hold:*

- *if $z^+_{ij} = 1$ and $z^-_{ij} = 1$, then department i precedes j horizontally;*
- *if $z^+_{ij} = 0$ and $z^-_{ij} = 0$, then department j precedes i horizontally;*
- *if $z^+_{ij} = 0$ and $z^-_{ij} = 1$, then department i precedes j vertically;*
- *if $z^+_{ij} = 1$ and $z^-_{ij} = 0$, then department j precedes i vertically.*

A new MILO model for the FLP can now be formulated:

$$\min \sum_{1\le i<j\le n} c_{ij}(u_{ij} + v_{ij}) \tag{50}$$

$$\text{s.t.} \quad (42)\text{–}(45) \tag{51}$$

$$\begin{aligned} &z^+_{ij} + z^+_{ji} = 1,\ 1 \le i < j \le n \\ &z^-_{ij} + z^-_{ji} = 1,\ 1 \le i < j \le n \end{aligned} \tag{52}$$

$$\begin{aligned} &z^+_{ik} + z^+_{kj} - z^+_{ij} \le 1,\ 1 \le i < j \le n \\ &z^-_{ik} + z^-_{kj} - z^-_{ij} \le 1,\ 1 \le i < j \le n \end{aligned} \tag{53}$$

$$\begin{aligned} &x_i + \frac{1}{2}w_i \le x_j - \frac{1}{2}w_j + w_F(2 - z^+_{ij} - z^-_{ij}),\ i,j \in N,\ i \ne j \\ &y_i + \frac{1}{2}h_i \le y_j - \frac{1}{2}h_j + h_F(1 + z^+_{ij} - z^-_{ij}),\ i,j \in N,\ i \ne j \end{aligned} \tag{54}$$

$$z^+_{ij},\ z^-_{ij} \in \{0,1\},\ i,j \in N. \tag{55}$$

Constraints (52) ensure that every department appears exactly once in each sequence, and constraints (53) are transitivity constraints for the two sequences. Together, these constraints ensure that the binary variables represent valid sequences. Constraints (54) are the non-overlapping expressed in terms of the sequence-pair variables.

Using the sequence-pair MILO model above with additional valid inequalities that include the position $p - q$ symmetry-breaking constraints (see Section 5), Meller et al. (2007) solved instances of UA-FLP with up to 11 departments to global optimality. They used CPLEX with a branching priority measure; the computational time reached almost 17 hours for the 11-department instance.

Castillo and Westerlund (2005) proposed a MILO model that satisfies the area requirements within a given accuracy ε using cutting planes. We omit the details for this because the area constraints can be handled more effectively using conic optimization, as mentioned above. We point out however that Castillo and Westerlund (2005) used the following formulation of the non-overlap constraints:

$$\frac{1}{2}(w_i + w_j) - (x_i - x_j) \le w_F(X_{ij} + Y_{ij}),\ 1 \le i < j \le n \tag{56}$$

$$\frac{1}{2}(w_i + w_j) - (x_j - x_i) \le w_F(1 + X_{ij} - Y_{ij}),\ 1 \le i < j \le n \tag{57}$$

$$\frac{1}{2}(h_i + h_j) - (y_i - y_j) \le h_F(1 - X_{ij} + Y_{ij}),\ 1 \le i < j \le n \tag{58}$$

$$\frac{1}{2}(h_i + h_j) - (y_j - y_i) \le h_F(2 - X_{ij} - Y_{ij}),\ 1 \le i < j \le n \tag{59}$$

$$X_{ij}, Y_{ij} \in \{0,1\},\ 1 \le i \le n. \tag{60}$$

and that this representation is essentially the sequence-pair representation.

Liu and Meller (2007) implemented a GA where each sequence-pair is a chromosome. The sequence-pair gives the relative position of the departments (first stage) and a linear optimization model is then solved to find the best layout (second stage). Their work empirically proves the efficiency of the sequence-pair approach. They achieved the best results up to then for instances with up to 35 departments; the computational time was 26 hours for the 35-department instance.

Xie and Sahinidis (2008) present a branch and bound algorithm that uses the sequence-pair representation. They solve a minimum-cost network flow problem to obtain a feasible layout from the sequence-pair representation of the relative position layout. However, this is valid for a restricted version of the FLP in which the department widths and heights are fixed, and the facility has no limitations. They transform an UA-FLP with just the non-overlapping constraints into a network flow problem. The advantage is that network flow algorithms can be several orders of magnitude faster than general linear optimization algorithms.

It was observed by Meller et al. (2007) and Liu and Meller (2007) that the MILO model with constraints (46) and (48) has $2^{n(n-1)}$ possible combinations of the binary variables, while the sequence-pair-based MILO formulation has $(n!)^2$ sequences generating the same set of relative-position combinations. They claim that this difference is key for the effectiveness of the sequence-pair approach. The difference is indeed significant: Using Stirling's approximation, we have that $(n!)^2 = \theta(e^{2(n\ln(n)-n)})$ while $2^{n(n-1)} = e^{0.693(n^2-n)}$. However, a comparison of the two formulations shows that another important difference is the presence of the transitivity constraints (53) in the sequence-pair model. However, it is not entirely clear to what extent each of these differences contributes to the efficiency of the sequence-pair approach. It would be interesting to carry out a computational study of the MILO models to clarify their impacts.

3.3 Attractor-repeller approaches

Anjos and Vannelli (2002) introduce a two-stage model based on the attractor-repeller (AR) technique for VLSI floorplanning. The first stage uses the AR technique to establish the relative position of the departments, and the second stage finds a feasible layout satisfying the relative positions specified by the solution to the first stage. The objective of this approach is not to achieve proven global optimality but rather to efficiently compute competitive solutions to large-scale instances of UA-FLP.

The AR model approximates each department by a circle with radius r_i, with the radius proportional to the square root of A_i. The AR places the circles inside the facility while allowing some overlapping; the amount of overlapping is controlled via a so-called *target distance*. Given $\alpha > 0$, the target distance t_{ij} for circles i and j is:

$$t_{ij} = \alpha(r_i + r_j)^2,\ 1 \le i < j \le n.$$

The AR model is as follows:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{1\le i<j\le n} c_{ij}D_{ij} + f\left(\frac{D_{ij}}{t_{ij}}\right) \\
\text{s.t.} \quad & x_i + r_i \le \frac{1}{2}w_F,\ i = 1,\dots,n \\
& x_i - r_i \ge -\frac{1}{2}w_F,\ i = 1,\dots,n \\
& y_i + r_i \le \frac{1}{2}h_F,\ i = 1,\dots,n \\
& y_i - r_i \ge \frac{1}{2}h_F,\ i = 1,\dots,n
\end{aligned}$$

where $D_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$ and f is a penalty function. The constraints ensure that the circles are placed inside the facility; here the origin is the centre of the facility. The location of the circles is determined by the trade-off in the objective function between the attractor term $c_{ij}D_{ij}$ and the repeller term $f\left(\frac{D_{ij}}{t_{ij}}\right)$, where $f(z) = \frac{1}{z} - 1$.

While the constraints are linear, the objective function is nonlinear and non-convex. Anjos and Vannelli (2002) propose to convexify the objective function by replacing the term $c_{ij}D_{ij} + f\left(\frac{D_{ij}}{t_{ij}}\right)$ with the following piecewise function:

$$f_{ij}(x_i, x_j, y_i, y_j) = \begin{cases} c_{ij}z + \frac{t_{ij}}{z} - 1, & z \ge \sqrt{\frac{t_{ij}}{c_{ij}}} \\ 2\sqrt{c_{ij}t_{ij}} - 1, & 0 \le z < \sqrt{\frac{t_{ij}}{c_{ij}}} \end{cases} \tag{61}$$

where $z = (x_i - x_j)^2 + (y_i - y_j)^2$, and it is assumed that $c_{ij} > 0$. Note that the second branch of f_{ij} is constant.

By construction, f_{ij} attains its minimum whenever the positions of circles i and j satisfy $D_{ij} \leq \sqrt{t_{ij}/c_{ij}}$. This includes the case where $D_{ij} = 0$, i.e., the two circles completely overlap. Of course, such a placement is undesirable. The ideal arrangement of the circles has $D_{ij} \approx \sqrt{t_{ij}/c_{ij}}$, i.e., close to the boundary of the flat portion of f_{ij}. At these points, the minimum value of f_{ij} is attained at the same time as the resulting overlap is minimized. This motivates Anjos and Vannelli (2006) to introduce a *generalized target distance*:

$$T_{ij} = \sqrt{\frac{t_{ij}}{c_{ij} + \epsilon}},\ 1 \leq i, j \leq n, \tag{62}$$

where $\epsilon > 0$ is chosen to be sufficiently small so that $T_{ij} \approx \sqrt{t_{ij}/c_{ij}}$. This modification also eliminates the need for the assumption that $c_{ij} > 0$.

In practice, achieving a solution with $D_{ij} \approx T_{ij}$ is not easy to achieve. Thus Anjos and Vannelli (2006) again sacrifice convexity and propose a modified AR model with the following objective function:

$$\min_{(x_i, y_i, w_i, h_i)} \quad \sum_{1 \leq i < j \leq n} f_{ij}(x_i, x_j, y_i, y_j) - K \log \left\{ \frac{D_{ij}}{T_{ij}} \right\}, \tag{63}$$

where

$$f_{ij}(x_i, x_j, y_i, y_j) = \begin{cases} c_{ij} z + \frac{t_{ij}}{z} - 1, & z \geq T_{ij} \\ 2\sqrt{c_{ij} t_{ij}} - 1, & 0 \leq z < T_{ij} \end{cases},$$

T_{ij} and the non-convex logarithmic term steers the optimization away from solutions with $D_{ij} \approx 0$. Indeed the minima of this function satisfy $D_{ij} \approx T_{ij}$. This can be viewed as a compromise in the sense that convexity is lost, but computational efficiency is gained.

The optimal solution of the modified AR model gives coordinates for the centers of the departments. In the second stage, the nonoverlapping constraints (40) are formulated as complementarity constraints. For each pair i, j, we introduce new variables X_{ij} and Y_{ij} satisfying

$$\begin{aligned} X_{ij} &\geq \frac{1}{2}(w_i + w_j) - |x_i - x_j|,\ X_{ij} \geq 0, \\ Y_{ij} &\geq \frac{1}{2}(h_i + h_j) - |y_i - y_j|,\ Y_{ij} \geq 0, \\ X_{ij} Y_{ij} &= 0 \end{aligned}$$

This last constraint requires that at least one of X_{ij} and Y_{ij} equal zero, thus enforcing non-overlap. The resulting UA-FLP model is a mathematical programming problem with complementarity constraints. Using the coordinates for the centers of the circles as the initial colution for a non-linear optimization solver, this approach improved on the solutions from earlier techniques, in particular for the Armour-Buffa 20-department instance.

There is some awkwardness in these models. Anjos and Vannelli (2002) develop a nonconvex model with the repeller function $\frac{1}{z} - 1$. The model is then modified to achieve convexity, but Anjos and Vannelli (2006) add a new penalty term, with the consequent loss of convexity. Moreover, the optimization problem with complementarity constraints is difficult to solve for large-scale instances.

The AR approach was significantly improved in the work of Jankovits et al. (2011). For the first stage, they replace the function $f_{ij}(x_i, x_j, y_i, y_j)$ by a more complicated expression, and also attempt to integrate information about the aspect ratio constraints. We refer the reader to their paper for details, and do not give them here because of the subsequent improvement by Anjos and Vieira (2015) mentioned below.

The major improvement is in the second stage, and the linkage of the two stages. Indeed Jankovits et al. (2011) use the following convex second-stage model that can be solved efficiently:

$$\min_{(x_i,y_i),w_i,h_i} \quad \sum_{1\le i<j\le n} (u_{ij}+v_{ij}) \tag{64}$$

$$\text{s.t.} \quad u_{ij} \ge x_i - x_j, \text{ for } 1 \le i < j \le n \tag{65}$$

$$u_{ij} \ge x_j - x_i, \text{ for } 1 \le i < j \le n \tag{66}$$

$$v_{ij} \ge y_i - y_j, \text{ for } 1 \le i < j \le n \tag{67}$$

$$v_{ij} \ge y_j - y_i, \text{ for } 1 \le i < j \le n \tag{68}$$

$$w_i^{\min} \le w_i \le w_i^{\max}, \text{ for } 1 \le i < j \le n \tag{69}$$

$$h_i^{\min} \le h_i \le h_i^{\max}, \text{ for } 1 \le i < j \le n \tag{70}$$

$$w_i h_i \ge A_i, \text{ for } 1 \le i < j \le n \tag{71}$$

$$\beta_i w_i - h_i \ge 0, \text{ for } 1 \le i < j \le n \tag{72}$$

$$\beta_i h_i - w_i \ge 0, \text{ for } 1 \le i < j \le n, \tag{73}$$

plus appropriately chosen linear inequality constraints to ensure non-overlap. These constraints are obtained as follows. Consider the coordinates of the centers of the circles in the optimal solution to the first stage as a set of points on the plane, and compute their Delaunay triangulation. This is the triangulation with the property that no point is inside the circumcircle of any triangle. One of its features is that is maximizes the minimum angle over all the angles of the triangles; in practice this means that thin triangles are less likely. The edges of the Delaunay triangulation are taken to represent the relative positions of the departments, and these positions are then enforced by the appropriate linear constraints. For example, if j is to the right of i, then the constraint $x_j - x_i \ge \frac{1}{2}(w_i + w_j)$ is added to the model. The result is a second stage model that is a conic optimization problem and can be solved efficiently.

The overall approach in Jankovits et al. (2011) provided further improved layouts for the classical Armour-Buffa instance, and computed high-quality layouts for several 30-department instances in 5 minutes or less of computation time.

Most recently, Anjos and Vieira (2015) further developed the AR methodology. As the second stage of Jankovits et al. (2011) is highly effective, the novelty in Anjos and Vieira (2015) is the formulation of the first stage. Specifically they propose a more precise formulation that models the departments as rectangles instead of approximating them by circles. The aspect ratio constraints can therefore be exactly enforced at the first stage, instead of being approcimated as in Jankovits et al. (2011). Furthermore they forego convexity and use the simple objective function

$$c_{ij} D_{ij}^2 + K \frac{\theta_{ij}^2}{D_{ij}^2} - 1,$$

where $\theta_{ij}^2 = \frac{1}{4}\left((w_i + w_j)^2 + (h_i + h_j)^2\right)$. Note that $D_{ij}/\theta_{ij} \approx 1$ indicates that some of the borders of the rectangles are close, whether or not the rectangles are overlapping by a small amount.

The resulting first-stage model is:

$$\min_{x_i,y_i,h_i,w_i} \quad \sum_{1\le i<j\le n} \left(c_{ij} D_{ij}^2 + K \frac{\theta_{ij}^2}{D_{ij}^2} - 1 \right) \tag{74}$$

$$\text{s.t.} \quad x_i + \frac{1}{2} w_i \le \frac{1}{2} w_F \text{ and } \frac{1}{2} w_i - x_i \le \frac{1}{2} w_F, \quad \text{for } i = 1, \dots, n, \tag{75}$$

$$y_i + \frac{1}{2} h_i \le \frac{1}{2} h_F \text{ and } \frac{1}{2} h_i - y_i \le \frac{1}{2} h_F, \quad \text{for } i = 1, \dots, n, \tag{76}$$

$$w_i h_i \ge A_i, \quad \text{for } i = 1, \dots, n, \tag{77}$$

$$\beta w_i - h_i \ge 0, \quad \text{for } i = 1, \dots, n, \tag{78}$$

$$\beta h_i - w_i \ge 0, \quad \text{for } i = 1, \dots, n, \tag{79}$$

$$w_i^{\min} \le w_i \le w_i^{\max}, \quad \text{for } i = 1, \dots, n, \tag{80}$$

$$h_i^{\min} \le h_i \le h_i^{\max}, \quad \text{for } i = 1, \dots, n. \tag{81}$$

The value of K is varied according to the following formula:

$$K = \alpha \sum_{1 \le i < j \le n} c_{ij}, \quad \text{with} \quad 0 < \alpha \le 1. \tag{82}$$

By solving for different choices of α, and hence of K, Anjos and Vieira (2015) improved on the best solutions by earlier techniques, in little computational time. Moreover, they computed layouts for instances with up to 100 departments in less than 15 min of computation time. This is the only approach entirely based on mathematical optimization models that has been able to reach such large-scale instances of UA-FLP.

The question of computing global bounds for instances of UA-FLP remains open.

3.4 Other two-stage approaches

Bozer and Wang (2012) present a heuristic based on a graph-pair representation and a simulated annealing technique. One graph of the pair represents the horizontal separation of the departments, and the other represents the vertical separation. Their results are generally good, but the authors make changes in the areas of the facilities or departments. These modifications are reasonable from a practical point of view, but they make it difficult to compare with other techniques.

Kulturel-Konak and Konak (2013) introduce a linear optimization-based GA approach, which differs from the approach of Liu and Meller (2007) in the chromosome coding. In particular it does not use sequence-pairs. It outperforms the previous approaches: the cost function is reduced and the computational time is lower. The GA searches for the relative locations of the departments, and the linear optimization model determines their exact locations and shapes.

Kulturel-Konak and Konak (2013) propose a new representation, called *location/shape representation*, to represent the relative department locations. Like the sequence-pair representation, the location/shape representation always generates a consistent assignment of the binary decision variables. Basically, the relative location of department i is represented as (x_i, y_i, α_i), where $\alpha_i = h_i/w_i$. The chromosomes are encoded as $(x_1, y_1, \alpha_1), \ldots, (x_n, y_n, \alpha_n)$. For each (x_i, y_i, α_i), we define two straight lines, one passing through (x_i, y_i) and the upper right corner $(x_i + w_i/2, y_i + h_i/2)$, and the other passing through (x_i, y_i) and the upper left corner $(x_i - w_i/2, y_i + h_i/2)$. These lines split the facility into four regions with reference to department i, so every other department is above or below or left or right of i. Note that the MILO-model does not contain the transitivity constraints for the integer variables, but this encoding based on the continuous variables encapsulates transitivity.

Gonçalves and Resende (2015) successfully apply a random-key GA as a first stage and a linear optimization model as a second stage. They report results on several instances of UA-FLP from the literature, and find slightly better solutions in a considerably shorter computational time, in comparison with the other GA approaches. They also applied their approach to larger instances with up to 125 departments, but without restrictions on the dimensions of the facility. The computational time is reduced because they do not solve all the linear optimization problems originating from the relative-position solutions. (Each linear problem is large: in particular there are $\binom{n}{2}$ nonoverlapping constraints.) Therefore, they solve only the problems that provably yield a feasible solution with a cost not exceeding 40% of that of the previous best solution. They also do not enforce constraints on the dimensions of the facility.

3.5 Flexible bay structure

A flexible bay structure is a continuous layout where the departments are located in parallel bays with flexible widths. This is a special case of the UA-FLP that commonly arises in manufacturing facilities (Meller, 1997). The bay structure is similar to the row structure in row FLPs, but a fundamental difference is that the width of each bay depends on the total area of the departments in that bay, whereas in row FLPs, the heights of the rows and of the departments are equal and fixed. The bays have straight aisles on both sides, and departments are not allowed to span multiple bays. This structure restricts the set of feasible solutions, but it has advantages in practice. The bay boundaries form the basis of an aisle structure that facilitates the transfer of the layout solution to an actual facility design.

Konak et al. (2006) present a MILO formulation for this problem. The variables x_i, y_i determine the location of department i, and the binary variables are defined as follows:

$$z_{ik} = \begin{cases} 1, & \text{if department } i \text{ is assigned to bay } k \\ 0, & \text{otherwise;} \end{cases}$$
$$r_{ij} = \begin{cases} 1, & \text{if department } i \text{ is above department } j \text{ in the same bay} \\ 0, & \text{otherwise;} \end{cases}$$
$$\delta_k = \begin{cases} 1, & \text{if bay } k \text{ is occupied} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the continuous variable h_{ik} determines the height of department i in bay k.

The MILO model is:

$$\min \quad \sum_{1\le i<j\le n} c_{ij}(u_{ij}+v_{ij}) \tag{83}$$
$$\text{s.t.} \quad \begin{aligned} & u_{ij} \ge x_i - x_j \text{ and } u_{ij} \ge x_j - x_i,\ 1\le i<j\le n \\ & v_{ij} \ge y_i - y_j \text{ and } v_{ij} \ge y_j - y_i,\ 1\le i<j\le n \end{aligned} \tag{84}$$
$$\sum_{k\in K} z_{ik} = 1,\ i\in\mathcal{N} \tag{85}$$
$$w_k = \frac{1}{h_F}\sum_{i\in\mathcal{N}} z_{ik}A_i,\ k\in\mathcal{K} \tag{86}$$
$$w_i^{\min} z_{ik} \le w_k \le w_i^{\max} + w_F(1-z_{ik}),\ k\in K, i\in\mathcal{N} \tag{87}$$
$$\begin{aligned} & x_i \ge \sum_{j\le k} w_j - 0.5w_k - (w_F - w_i^{\min})(1-z_{ik}),\ k\in\mathcal{K}, i\in\mathcal{N} \\ & x_i \le \sum_{j\le k} w_j - 0.5w_k + (w_F - w_i^{\min})(1-z_{ik}),\ k\in\mathcal{K}, i\in\mathcal{N} \end{aligned} \tag{88}$$
$$\begin{aligned} & \frac{h_{ik}}{A_i} - \frac{h_{jk}}{a_j} - \max\left\{\frac{\ell_i^{\max}}{A_i}, \frac{\ell_j^{\min}}{a_j}\right\}(2-z_{ik}-z_{jk}) \le 0,\ i<j \\ & \frac{h_{ik}}{A_i} - \frac{h_{jk}}{a_j} + \max\left\{\frac{\ell_i^{\max}}{A_i}, \frac{\ell_j^{\min}}{a_j}\right\}(2-z_{ik}-z_{jk}) \le 0,\ i<j \end{aligned} \tag{89}$$
$$\sum_{i\in\mathcal{N}} h_{ik} = h_F\delta_k,\ k\in\mathcal{K} \tag{90}$$
$$h_i^{\min} z_{ik} \le h_{ik} \le h_i^{\max} z_{ik},\ i\in\mathcal{N}, k\in\mathcal{K} \tag{91}$$
$$\sum_{i\in\mathcal{N}} h_{ik} = h_i,\ i\in\mathcal{N} \tag{92}$$
$$y_i - 0.5h_i \ge y_j + 0.5h_j - w_H(1-r_{ij}),\ i\ne j \tag{93}$$
$$r_{ij} + r_{ji} = 1,\ 1\le i<j\le n \tag{94}$$
$$r_{ij} + r_{ji} \ge z_{ik} + z_{jk} - 1,\ 1\le i<j\le n, k\in\mathcal{K} \tag{95}$$
$$0.5h_i \le y_i \le w_H - 0.5h_i,\ i\in\mathcal{N}. \tag{96}$$

where $\mathcal{K}$ is the set of bays, and $\mathcal{N}$ is the set of departments.

Constraints (85) ensure that each department is assigned to a single bay. Constraints (86) calculate the width of each bay as the total area of the departments assigned to that bay divided by the facility height. Note that under the assumption that $\sum_{i\in\mathcal{N}} A_i \le w_F h_F$, we have $\sum_{k\in\mathcal{K}} w_k \le w_F$. Constraints (87) impose bounds on the bay widths, based on the width bounds of the departments assigned to each bay. Constraints (88) determine the horizontal locations of the department centroids. In this model, the x-coordinate is located at the middle of the bays. Therefore, if department i is assigned to bay k, x_i is calculated as

$$x_i = \sum_{j=1}^{k} w_j - 0.5 w_k.$$

This agrees with constraints (88) with $z_{ik} = 1$. If i and j are in the same bay k, then constraints (89) ensure that the widths of the two departments are the same and equal to the bay width. Constraints (90) set the total heights of the departments in a bay equal to h_F if the bay is used, and zero if the bay is empty. Constraints (91) are bounds on the department heights. They also enforce $h_{ik} = 0$ when department i is not located in bay k. Constraints (92) define the heights of the departments. Constraints (94)–(95) ensure that department i is either above or below department j. Constraints (93) prevent departments in the same bay from overlapping. Constraints (96) ensure that the departments are inside the facility.

By adding symmetry-reduction constraints and valid inequalities, Konak et al. (2006) could solve to optimality instances with up to 14 departments. This last case needed around 120 hours of computational time.

Constraints (90) seem to be restrictive for the case where $\sum A_i < w_F h_F$, but this is not addressed in Kulturel-Konak and Konak (2013). Moreover, the authors (Kulturel-Konak and Konak (2011a,b)) present heuristic-based solutions where these constraints are not enforced.

4 Multifloor FLP

The multifloor FLP (MF-FLP) involves finding the optimal arrangement of departments in a facility with multiple floors. The practical applications include layout in production facilities, hotels, office buildings, and hospitals. This problem has added complexity in comparison to the UA-FLP because we must also consider the interactions between departments on different floors. Furthermore, elevators and/or stairwells are required to transfer people and/or material between the floors, and these need to be placed in coherent locations in every floor that they reach.

Globally optimal algorithms for MF-FLP work in general only for small instances; see Hahn et al. (2010). Meller and Bozer (1997) were the first to investigate the problem but most of the subsequent models in the literature are designed for specific types of MF-FLP, as the literature survey in Section 4.1 shows. Indeed there is no commonly agreed definition of the MF-FLP because different authors make their own assumptions about the structure of the problem. This lack of a common definition makes it is hard to compare the approaches. We therefore propose in Section 4.2 a general formulation for the MF-FLP that we hope will gain acceptance as a standard formulation, and will lead to increased research activity on this problem.

4.1 Survey of the literature

Some approaches first distribute the departments over the floors, minimizing the vertical interaction costs. This is essentially the first stage of a two-stage approach, where the second stage then optimizes the layout of each floor independently; see Meller and Bozer (1997) and Bernardi and Anjos (2013). Given the difficulty of the problem, this is a reasonable approach. Specifically Meller (1997) use the following MILO formulation to assign departments to floors:

$$\min \quad \sum_{1 \le i < j \le n} c_{ij} d_{ij}^v \tag{97}$$

$$\text{s.t.} \quad \sum_{k=1}^{p} z_{ik} = 1, \ 1 \le i \le n \tag{98}$$

$$d_{ij}^v \ge \delta \sum_{k=1}^{p} k(z_{ik} - z_{jk}), \ 1 \le i < j \le n \tag{99}$$

$$d_{ij}^v \ge \delta \sum_{k=1}^{p} k(z_{jk} - z_{ik}), \ 1 \le i < j \le n \tag{100}$$

$$\sum_{k=1}^{p} a_i z_{ik} \leq w_F \times h_F, \ 1 \leq i \leq n \tag{101}$$

$$z_{ik} \in \{0, 1\}, \ 1 \leq i < j \leq n, \ 1 \leq k \leq p. \tag{102}$$

Each floor then becomes an instance of UA-FLP with some additional constraints to ensure coherence in the location of the elevators. Computing the vertical costs still remains a challenge and was addressed in Bernardi and Anjos (2013).

Another possible simplification is to restrict all the departments to have the same shape and to require that they be assigned to specific locations in the building. This reduces the problem to a quadratic assignment problem. Such a formulation was used by Hahn et al. (2010), and was solved using the RLT linearization technique (Adams and Sherali, 1986; Sherali and Adams, 1990) within a branch-and-bound algorithm.

Patsiatzis and Papageorgiou (2002) present a mathematical formulation for process plant layout. Their objective function considers the construction and land costs to decide the number of floors and the floor area.

Lee et al. (2005) use a GA for layout with inner walls and passages. The connections between the departments, passages, and elevators are represented as an adjacency graph, and the distances are calculated using Dijkstra's shortest-path algorithm. This representation allows them to measure the distances of paths that use corridors and elevators. Their model minimizes the total cost of transporting the materials and maximizes the adjacency achieved. They apply it to a multideck ship layout with inner walls.

Defersha and Chen (2006) propose a model for a processing plant that incorporates many structural and operational issues; however the model becomes unwieldy. The same is true for the models in Khaksar-Haghani et al. (2013) and Kia et al. (2014), where GAs are applied.

Hathhorn et al. (2013) consider an MF-FLP similar to the one we present here. However, the overall length and width of the facility, the number of elevators, and the number of floors are modeled as decision variables. They propose a biobjective optimization: they minimize the material handling costs (as usual) and the facility construction costs.

Izadinia et al. (2014) propose a robust model in which some of the usual parameters are considered to be uncertain.

Park et al. (2011) take into account safety distances in the event of an explosion.

4.2 A MF-FLP formulation

We assume that the following parameters are given: the number of departments and their areas, the number of floors, the dimensions and height of the floors, the interconnection costs, and the number and size of the elevators. We consider the elevators to be a general system (incorporating elevators, stairs, pipes, etc.) for vertical movement. We want to determine the locations of the elevators and the locations and dimensions of the departments. The horizontal distance is the rectilinear distance (which is a reliable measure, as in the single-floor case), and the vertical distance will be measured using the elevators. This makes the formulation complex. We use the following notation:

w_F: width of floor;
h_F: length of floor;
a_i: area of department i;
δ: floor height;
c_{ij}: interconnection cost between i and j;
p: number of floors;
e: number of elevators.

We define the following variables:

$z_{ik} = 1$	if department i is assigned to floor k, 0 otherwise;
$Z_{ij} = 1$	if departments i and j are allocated to the same floor, 0 otherwise;
X_{ij}, Y_{ij}:	nonoverlapping binary variables;
(x_i, y_i):	coordinates of the centroid of department i;
d^v_{ij}:	vertical distance between i and j;
d^h_{ij}:	horizontal distance between i and j located on the same floor;
d^e_{ij}:	horizontal distance between i and j located on different floors, where the path includes an elevator.

If $n+1 \leq i \leq n+e$ the coordinates (x_i, y_i) refer to elevators. The number of floors and elevators is assumed to be fixed; if necessary, we could run the model for several different options. The floor dimensions are fixed, but they could easily become decision variables. The formulation is as follows:

$$\min \quad \sum_{1 \leq i < j \leq n} c_{ij}(d^e_{ij} + d^v_{ij}) \tag{103}$$

$$\text{s.t.} \quad \sum_{k=1}^{p} z_{ik} = 1,\ 1 \leq i \leq n \tag{104}$$

$$\begin{aligned} & d^v_{ij} = \delta \left| \sum_{k=1}^{p} k(z_{ik} - z_{jk}) \right|,\ 1 \leq i < j \leq n \\ & d^h_{ij} = |x_i - x_j| + |y_i - y_j|,\ 1 \leq i < j \leq n \\ & d^e_{ij} \geq d^h_{ij},\ 1 \leq i < j \leq n \\ & d^e_{ij} \geq |x_i - x_\ell| + |y_i - y_\ell| + |x_j - x_\ell| + |y_j - y_\ell| - MZ_{ij}, \\ & \ 1 \leq i < j \leq n,\ n+1 \leq \ell \leq n+e \end{aligned} \tag{105}$$

$$\begin{aligned} & Z_{ij} \geq z_{ik} + z_{jk} - 1,\ 1 \leq i < j \leq n, k = 1, \ldots, p \\ & Z_{ij} \leq 1 - z_{ik} + z_{jk},\ 1 \leq i < j \leq n, k = 1, \ldots, p \\ & Z_{ij} \leq 1 + z_{ik} - z_{jk},\ 1 \leq i < j \leq n, k = 1, \ldots, p \end{aligned} \tag{106}$$

$$\begin{aligned} & x_i + \frac{1}{2}w_i \leq \frac{1}{2}w_F,\ x_i - \frac{1}{2}w_i \geq -\frac{1}{2}w_F,\ 1 \leq i \leq n+e \\ & y_i + \frac{1}{2}h_i \leq \frac{1}{2}h_F,\ y_i - \frac{1}{2}h_i \geq -\frac{1}{2}h_F,\ 1 \leq i \leq n+e \end{aligned} \tag{107}$$

$$\begin{aligned} & w_i h_i = a_i,\ 1 \leq i \leq n \\ & w_i - \beta h_i \leq 0,\ h_i - \beta w_i \leq 0,\ 1 \leq i \leq n \end{aligned} \tag{108}$$

$$\begin{aligned} & x_i - x_j \geq \frac{1}{2}(w_i + w_j) - w_F(1 - Z_{ij} + X_{ij} + Y_{ij}),\ 1 \leq i < j \leq n+e \\ & x_j - x_i \geq \frac{1}{2}(w_i + w_j) - w_F(2 - Z_{ij} - X_{ij} + Y_{ij}),\ 1 \leq i < j \leq n+e \\ & y_i - y_j \geq \frac{1}{2}(h_i + h_j) - h_F(2 - Z_{ij} + X_{ij} - Y_{ij}),\ 1 \leq i < j \leq n+e \\ & y_j - y_i \geq \frac{1}{2}(h_i + h_j) - h_F(3 - Z_{ij} - X_{ij} - Y_{ij}),\ 1 \leq i < j \leq n+e \end{aligned} \tag{109}$$

$$\begin{aligned} & z_{ik} = 1,\ n+1 \leq i \leq n+e,\ 1 \leq k \leq p \\ & Z_{ij} = 1,\ n+1 \leq i < j \leq n+e \end{aligned} \tag{110}$$

$$X_{ij}, Y_{ij}, Z_{ij}, z_{ik} \in \{0,1\},\ 1 \leq i < j \leq n+e,\ 1 \leq k \leq p \tag{111}$$

$$h_i, w_i, \geq 0,\ 1 \leq i \leq n, \tag{112}$$

$$\tag{113}$$

where $M = w_F + h_F + \delta p$.

Constraints (104) allocate each department to exactly one floor. Constraints (105) compute the distances between each pair of departments; if two departments are on different floors, the distance depends on the elevator position. Constraints (106) set $Z_{ij} = 1$ if i and j are on the same floor, and 0 otherwise. Constraints (109) prevent the overlapping of departments and elevators on the same floor. Constraints (106) and (109) have been taken from Patsiatzis and Papageorgiou (2002). Constraints (110) ensure that each elevator covers all the floors and every pair of elevators shares the same floor.

4.3 Final comment on MF-FLP

As can be seen, there has been increased interest in the MF-FLP in different forms. The variety of definitions and assumptions makes it impossible to appropriately evaluate the effectiveness of the various approaches that have been developed.

We hope that our review prompts a discussion of which standard assumptions should be made in defining a standard version of the problem. This could then lead not only to the development of novel models and solution techniques, but also to more effective comparisons of them, which is essential to help the research community make further progress on this difficult but important problem.

5 Symmetry-breaking constraints

Many versions of the FLP have symmetric solutions. For example, it is clear that flipping a solution to UA-FLP by 180 degrees gives exactly the same solution. This matters because the presence of symmetry is often problematic for MILO solvers. For this reason, several symmetry-reduction strategies have been proposed in the literature, see e.g. Sherali and Smith (2001).

To break the symmetry for the UA-FLP, Meller et al. (1999) require some department k to be located in a specific quarter of the facility via constraints such as

$$x_k \leq 0.5w_F, \; y_k \leq 0.5h_F.$$

These constraints assume that the lower left corner of the facility is located at the origin of the cartesian plan. Sherali et al. (2003) call this the position k method. However, if department k has its centroid located at the facility centroid, then this strategy does not work. Sherali et al. (2003) propose a strategy called the position $p - k$ method that considers a pair of critical departments p and k, chosen to satisfy

$$c_{pk} = \max_{i,j \in N} c_{ij}.$$

This strategy requires the centroid of p to be south and west of the centroid of k. This is modeled by

$$\begin{aligned}
&x_p \leq x_k, \; y_p \leq y_k \\
&z^h_{kp} = z^v_{kp} = 0 \\
&(x_k - x_p) + (y_k - y_p) \geq \min\{w_k^{\min} + w_p^{\min}, h_k^{\min} + h_p^{\min}\}.
\end{aligned}$$

Castillo and Westerlund (2005) claim that simply choosing departments 1 and 2 works just as well. From Sherali et al. (2003), it is not clear whether the position k method or the position $p - k$ method is better. Amaral (2013b) applied the $p - k$ method to the DR-FLP, where p and k are chosen such that $c_{pk} = \min_{i,j \in N} c_{ij}$.

In the context of the UA-FLP, Castillo and Westerlund (2005) add the constraints $x_1 - x_2 \geq 0$ and $y_2 - y_1 \geq 0$, ensuring that department 1 is located to the southeast of department 2.

Sherali and Smith (2001) propose several classes of hierarchical constraints that are applicable to general symmetric MIPs. Sherali et al. (2003) study the effect of one such class of constraints:

$$\begin{aligned}
4\sum_{i=1}^{n} i x_i &\leq n(n+1)w_F \\
4\sum_{i=1}^{n} y_i &\leq n(n+1)h_F.
\end{aligned}$$

They find that these constraints can break the symmetry effectively, but their dense structure renders the CPLEX enumeration procedure relatively ineffective.

6 Valid inequalities

As already mentioned, valid inequalities are essential for solving mathematical optimization models efficiently in practice. First among them are the transitivity constraints, often called triangle inequalities, that apply to nearly every variant of the FLP and have been mentioned throughout this review, see e.g. constraints (10) and (53) above.

Meller et al. (1999) were the first to investigate valid inequalities for the UA-FLP. The inequalities reduced the number of nodes in the branch and bound tree but increased the computational time. Sherali et al. (2003) determined that the best results were obtained by incorporating only the B2 and V2 constraints of Meller et al. (1999). Using the notation of model (41)–(48), these inequalities are

$$\begin{aligned}
&\text{(B2)} \quad u_{ij} \geq (w_i^{\min} + w_j^{\min})(z_{ij}^h + z_{ji}^h)\\
&\text{(B2)} \quad v_{ij} \geq (h_i^{\min} + h_j^{\min})(z_{ij}^v + z_{ji}^v)\\
&\text{(V2)} \quad u_{ij} \geq (w_i + w_j) - \min\{w_i^{\max} + w_j^{\max}, w_F\}(1 - z_{ij}^h - z_{ji}^h)\\
&\text{(V2)} \quad v_{ij} \geq (w_i + w_j) - \min\{h_i^{\max} + h_j^{\max}, w_F\}(1 - z_{ij}^v - z_{ji}^v).
\end{aligned}$$

These constraints do not reduce the feasible set of the relaxed MILO model because they are redundant, and they do not enforce the separation of the departments. They are useful in branch and bound algorithms because they improve the lower bounds. In the linear relaxation, if

$$z_{ij}^h, z_{ji}^h = (w_F - w_i - w_j)/w_F \text{ and } z_{ij}^v, z_{ji}^v = (h_F - h_i - h_j)/h_F$$

then (47) leads to $x_i \approx x_j$, $y_i \approx y_j$, i.e., departments i and j overlap. Thus the root lower bound of (41)–(48) is typically zero.

Taking this into account, Sherali et al. (2003) model the constraint $u_{ij} = |x_i - x_j|$ in an unusual way. Define the variables

$$t_{ij}^h = \begin{cases} 1 & \text{if } x_i \leq x_j,\\ 0 & \text{if } x_i \geq x_j, \end{cases}$$

where the choice of 0 or 1 is inconsequential when $x_i = x_j$. An upper bound on u_{ij} is

$$U_{ij} = w_F - w_i^{\min} - w_j^{\min}.$$

Sherali et al. (2003) prove that $u_{ij} = |x_i - x_j|$ can be modeled by

$$\begin{aligned}
&0 \leq u_{ij} + x_i - x_j \leq 2U_{ij}(1 - t_{ij}),\ i < j\\
&0 \leq u_{ij} - x_i + x_j) \leq 2U_{ij}t_{ij},\ i < j\\
&t_{ij} \in \{0, 1\},\ i < j.
\end{aligned}$$

The same is true for $v_{ij} = |y_i - y_j|$.

Sherali et al. (2003) also propose an alternative set of nonoverlapping constraints:

$$\begin{aligned}
&x_i - x_j \geq w_i + w_j - M_{ij}(1 - z_{ij}^h)\\
&y_i - y_j \geq h_i + h_j - M_{ij}(1 - z_{ij}^v)\\
&-(w_F - w_i^{\min} - w_j^{\min}) \leq x_i - x_j \leq w_F - w_i^{\min} - w_j^{\min}\\
&-(h_F - h_i^{\min} - h_j^{\min}) \leq y_i - y_j \leq h_F - h_i^{\min} - h_j^{\min}\\
&w_i^{\min} + w_j^{\min} \leq w_i + w_j \leq w_i^{\max} + w_j^{\max}\\
&h_i^{\min} + h_j^{\min} \leq h_i + h_j \leq h_i^{\max} + h_j^{\max}
\end{aligned}$$

$$z_{ij}^h + z_{ji}^h + z_{ij}^v + z_{ji}^v = 1$$
$$z_{ij}^{h(v)} \in \{0,1\}.$$

They construct a convex-hull representation of the above constraint set in a higher dimensional space. This convex hull can also be derived using the reformulation-linearization technique (RLT) of Sherali et al. (1998). Because of the size of this representation, they use it for just one pair of departments, the positively interacting (nonfixed) pair with the largest total area. Using this, together with constraints (B2) and (V2), symmetry-breaking constraints (see Section 5), and a new branching priority rule, Sherali et al. (2003) solve instances with up to 9 departments to global optimality. Meller et al. (2007) use inequalities (B2) and (V2) in the context of the sequence-pair representation formulation, plus symmetry-breaking constraints, and the same branching priority rule to solve instances with 11 departments within 24 hours.

Castillo and Westerlund (2005) use symmetry-avoidance constraints and tighten the nonoverlapping constraints via

$$\frac{1}{2}(w_i + w_j) - u_{ij} \leq w_F X_{ij},\ 1 \leq i < j \leq n \tag{114}$$
$$\frac{1}{2}(h_i + h_j) - v_{ij} \leq h_F(1 - X_{ij}),\ 1 \leq i < j \leq n. \tag{115}$$

Note that these are the same as (B2) and (V2).

7 Directions for future research

Facility layout continues to be the focus of much research, as evidenced for example in the bibliography of this review that includes more than 30 research articles published since 2010. We covered three classes of layout problems, namely row layout, unequal-areas layout, and multifloor layout, primarily from the perspective of mathematical optimization techniques, mostly MILO and SDO.

Within row layout problems, the single-row FLP is the one most studied in the literature, and major progress has been achieved in recent years since the first papers that modelled and solved this problem using SDO Anjos et al. (2005); Anjos and Vannelli (2008). The recent review in this journal by Keller and Buscher (2015) provides a detailed exposition of this progress. The double-row and multirow cases have also attracted attention recently, but have generally been less studied. The multirow FLP in particular sets a challenge to the research community in terms of providing new ideas for models and algorithms to compute global solutions.

Unequal-area layout is by far the most studied class of FLPs. Nevertheless only instances with up to 11 departments have been solved to global optimality using MILO formulations. Other mathematical optimization-based research has focused on two-stage heuristics that can provide good solutions for instances with up to 100 departments. Moreover, unlike for row FLPs, little work has been done with respect to applying conic optimization to this class of problems beyond the observation that the department area constraint can be relaxed in a conic form (see equation 49) that is handled efficiently by conic optimization software. Indeed, to the best of our knowledge, the only approach entirely based on conic optimization was given in Takouda et al. (2005) where it is applied to obtain global bounds for benchmark instances in the area of VLSI floorplanning.

Finally, multifloor layout has received the least attention in the literature. In fact there is no commonly agreed definition for the problem because most of the models in the literature are motivated by specific applications and therefore make assumptions about the structure of the problem that are not generally used. As a consequence it is difficult to compare the performance of different approaches. For this reason we proposed a general formulation for the MF-FLP in Section 4.2 that we hope will gain acceptance in the community and will motivate further research into this most challenging version of facility layout.

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