CIRRELT

CIRRELT-2019-56

The Two-Echelon Production-Routing Problem

Cleder M. Schenekemberg
Cassius T. Scarpin
José E. Pécora Jr.
Thiago A. Guimarães
Leandro C. Coelho

November 2019

Document de travail également publié par la Faculté des sciences de l'administration de l'Université Laval, sous le numéro FSA-2019-018.

Bureau de Montréal
Université de Montréal
C.P. 6128, succ. Centre-Ville
Montréal (Québec) H3C 3J7
Tél. : 1-514-343-7575
Télécopie : 1-514-343-7121

Bureau de Québec
Université Laval,
2325, rue de la Terrasse
Pavillon Palasis-Prince, local 2415
Québec (Québec) G1V 0A6
Tél. : 1-418-656-2073
Télécopie : 1-418-656-2624

The Two-Echelon Production-Routing Problem

Cleder M. Schenekemberg[1,*], Cassius T. Scarpin[1], José E. Pécora Jr.[1,2], Thiago A. Guimarães[1,3], Leandro C. Coelho[1,4]

[1] Research Group of Technology Applied to Optimization (GTAO), Federal University of Paranã, Brazil

[2] Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation (CIRRELT)

[3] Federal Institute of Science and Technology of Paraná (IFPR), Curitiba, Brazil

[4] Department of Operations and Decision Systems, 2325 de la Terrasse, Université Laval, Québec, Canada G1V 0A6

Abstract. This paper introduces the Two-Echelon Production-Routing Problem. This problem is motivated from the petrochemical industry, enlarging the supply chain integration by taking into account production, inventory, and routing decisions in a two-echelon vendor-managed inventory system. We describe, model, and design a branch-and-cut (B&C) to solve the problem under different inventory policies. We also propose a novel exact algorithm, by employing parallel computing techniques, in order to combine local search procedures within a traditional B&C scheme. We evaluate the performance of our methods through extensive computational experiments, both by comparing the algorithms, the effectiveness of the different inventory policies, and the impact of these policies on the partial costs. We derive many managerial insights based on the results. We also validate our new exact algorithm by solving similar problems from the literature, such as the two-echelon multi-depot inventory-routing (2E-MDIRP) and the classical multi-vehicle production-routing problem (MV-PRP). Computational experiments show that our method is very competitive. Based on 512 experiments for the 2E-MDIRP, our algorithm was able to find 111 new best-known solutions (BKS), besides proving 412 optimal solutions, against 298 from the literature. For 336 experiments over small and medium size MV-PRP instances, we proved 242 optimal solutions, 11 more than the exact methods from the literature, besides providing 95 new BKS. Moreover, we were the first to tackle large MV-PRP instances exactly, and in this case, our algorithm provides all BKS for instances up to 50 customers, 20 periods and 5 vehicles, outperforming all meta/matheuristics procedures from the literature.

Keywords. Logistics, two-echelon production-routing, local search, branch-and-cut, parallel computing.

Acknowledgments: This paper was partly funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - grant 1554767, and by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant 2019-00094. We also thank the C3SL (Centro de Computação Científica e Software Livre) from Federal University of Paraná for providing parallel computing facilities.

---

* Corresponding author: cledercms@hotmail.com

Dépôt légal – Bibliothèque et Archives nationales du Québec
Bibliothèque et Archives Canada, 2019

1. Introduction

Production, inventory control, and routing decisions are key logistics functions. These problems have been widely studied individually as the lot sizing problem [23], inventory management [45], and vehicle routing problems (VRP) [31]. Many integrations of these problems have arisen in the last decades, such as the inventory-routing (IRP) [5, 19], the production-routing (PRP) [4], and with more rich features such as with heterogeneous fleet [16, 37] and multiple depots [11, 30].

Supply chains are now comprised of complex interactions between many facilities at different levels, including depots, production plants, distribution centers, and customers spread over large geographical areas. In these cases, if one is to optimize his operations from raw material acquisition to final product delivery, more than one echelon is present in the supply chain. In this context, a two-echelon distribution problem arises. This structure has been studied from the classical VRP standpoint [10, 22, 40], and extended to an IRP context by Guimarães et al. [29].

However, in industry, decisions concerning production and distribution are often taken separately, and most companies handle these two echelons sequentially [13]. This approach does not allow to achieve the benefits enabled by a closer integration and higher coordination between production and distribution, such as logistics costs reduction and service level improvement [25].

A practical example emerges in the petrochemical industry in South American countries. According to local regulations, commercial gasoline sold at gas stations requires a certain amount of ethanol on its blend, which defines its octane grade [44]. Ethanol is produced from sugarcane at ethanol suppliers, while refineries produce pure gasoline, and also blend these two components in order to yield commercial gasoline. Due the strict quality control required by the regulations, the refineries are for managing the whole system, encompassing pickups and inventory control of ethanol from the suppliers, production and inventory of pure and commercial gasoline, and deliveries of commercial gasoline to the final costumers. This structure defines a two-echelon supply chain with three layers (ethanol suppliers − refineries − gas stations), all controlled by the middle layer. The pickup decisions are taken on the first echelon, delivery decisions on the second one, and production decisions are taken between them.

When gas stations and the refineries are placed in a vertical chain, the owner company pays for the inventory cost at the plants and at the final customers. This structure allows to consider a vendor-managed inventory (VMI) system, where the plants control the customer's inventories. According to Govindan [28], VMI is mutually beneficial because customers do not need to spend resources controlling their inventory, while plants can achieve better logistics coordination, especially on the deliveries routes composition. Once a customer is visited and one must determine the quantity to be delivered, one of two policies are often applied. Under the maximum level (ML) policy, the plant is free to decide how much to deliver, as long as the inventory capacity is not exceeded. The order-up-to (OU) policy fills the customer's inventory capacity whenever a delivery occurs [6].

To the best of our knowledge, a problem with these features has not yet been considered in the literature. In order to fill this gap, this paper introduces the two-echelon production-routing problem (2E-PRP), in which a set of refineries manage a two-echelon supply chain under the VMI paradigm, integrating pickups, production, inventory, and delivery decisions simultaneously. The aim is to minimize routing (pickups of the ethanol and deliveries of commercial gasoline), production (pure and commercial gasoline) and inventories costs at refineries (ethanol, pure and commercial gasoline), and inventories costs at gas stations (commercial gasoline), avoiding stock-outs over a planning horizon.

The 2E-PRP combines the well-known PRP with the 2E-MDIRP in a richer supply chain structure. The PRP was initially considered by Chandra and Fisher [15] and a first rich set of instances was proposed by Boudia et al. [12], who solved the problem via GRASP and path relinking. Later, heuristic methods were proposed and assessed over this set by Boudia et al. [13] by an improved version of GRASP and path relinking, by Boudia and Prins [14] with a memetic algorithm, by Bard and Nananukul [9] who introduce a tabu search, by Armentano et al. [8] who combine tabu search and path relinking, by Adulyasak et al. [3] who designed a matheuristic based on adaptive large neighborhood search (ALNS), by Absi et al. [1] with a two-phase iterative method, by Solyalı and Süral [43] who developed a multi-phase heuristic based on mathematical programming techniques, by [41] who combines a multi-phase matheuristic with a set partitioning approach, by Chitsaz et al. [17] with a three-phase decomposition matheuristic, and more recently by Li et al. [36] who propose a two-phase iterative method with a repairing strategy and a fix-and-optimize procedure. In the meantime, Adulyasak et al. [2] proposed a B&C and introduced another benchmark instance set, with small and medium size ones. To the best of our knowledge, no exact method have been assessed for the set proposed by Boudia et al. [12], and no lower bounds (LB) are known for these instances. The 2E-MDIRP was introduced by Guimarães et al. [29] inspired by a two-echelon VMI system implementation of a real fuel distribution problem.

The scientific contributions of this work are:

1. we introduce explicit production decisions on a two-echelon IRP, defining a new and rich problem
2. we describe, model, design a B&C, solve, and compare different inventory policies with different cost structures
3. we also design a parallel exact algorithm based on branch-and-cut (B&C)
4. we assess the performance of our parallel algorithm by solving other problems from the literature, namely the 2E-MDIRP of Guimarães et al. [29], the MV-PRP of Boudia et al. [12] and Adulyasak

et al. [2]. The results show that our method is very competitive, proving optimality for 125 open instances (114 for 2E-MDIRP and 11 for MV-PRP), and finding new best known solutions for several ones

The remainder of the paper is organized as follows. In Section 2, we formally describe the 2E-PRP. A mixed-integer programming (MIP) formulation is proposed in Section 3, where we present sets of new and existing valid inequalities. The B&C algorithm is detailed in Section 4. In Section 5, we describe the parallel exact algorithm we propose to solve the 2E-PRP. In Section 6, we discuss the results of extensive computational experiments performed to assess the quality of the algorithms. We also derive business insights based on the results. Conclusions are presented in Section 7.

2. Problem description

The 2E-PRP is defined over an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where the vertex set $\mathcal{V}$ is formed by the union of the sets $\mathcal{F}$ of suppliers, $\mathcal{P}$ of plants, and $\mathcal{C}$ of customers, while $\mathcal{E}$ is the set of edges. A non-negative cost c_{uv} is associated with each edge $(u, v) \in \mathcal{E}$. Suppliers and plants are linked in the first echelon, defined by subgraph $G' = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}' = \mathcal{F} \cup \mathcal{P}$ and $\mathcal{E}' = \{(u, v) : u, v \in V', u \in \mathcal{F} \wedge v \in \mathcal{P}\}$. Plants and customers are linked in the second echelon, defined by subgraph $G'' = (\mathcal{V}'', \mathcal{E}'')$ with $\mathcal{V}'' = \mathcal{P} \cup \mathcal{C}$ and $\mathcal{E}'' = \{(u, v) : u, v \in \mathcal{V}'' \wedge u, v \oplus \mathcal{P}, u < v\}$, in which $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ and $\mathcal{E} = \mathcal{E}' \cup \mathcal{E}''$.

The planning horizon is defined over a set $\mathcal{T} = \{1, ..., p\}$ of periods. A fleet of $|\mathcal{K}|$ homogeneous vehicles of capacity Q is available at each plant j at each period t. Each vehicle $k \in \mathcal{K}$ is able to pick up ethanol (α) from a supplier, and/or to deliver commercial gasoline (γ) to the customers. As the fleet is non-compartmentalized, a plant cannot combine a pickup and a delivery in the same tour. Furthermore, a plant can pick up α from one supplier using one vehicle in a period. Likewise, split deliveries are not allowed, and a vehicle can perform at most one pick up and/or one delivery per period. Finally, all routes must start and finish in their origin plant.

In each period t, each plant needs to take the following decisions: the pick up of a certain amount of α from a supplier, the production of a certain amount of pure gasoline (β), the blending among both inputs in order to produce a certain amount of commercial gasoline (γ), and the delivery schedule of γ to the customers. The octane grade of commercial gasoline is modulated by φ, which measures the ratio of α in the blend, with $0 \leq \varphi \leq 1$. Thus, each unit of γ consists in a part φ from α, and $(1 - \varphi)$ from β.

According to a pre-established contract with plants, each supplier i disposes of Φ_i units of α to serve all of plants for the whole planning horizon. Each plant has a minimum and a maximum inventory level for α, β, and γ, given by L_j^α and U_j^α, L_j^β and U_j^β, and L_j^γ and U_j^γ, respectively, and also incurs an inventory holding cost h_j^α, h_j^β, and h_j^γ, for each unit stocked per period. Regarding production activities for β and γ, a plant is able to produce up to P_j^β and P_j^γ in t, incurring in a unit variable cost c_j^β and c_j^γ, and also a fixed setup cost f_j^β and f_j^γ whenever a production batch is processed. Similarly, a customer l has a minimum and a maximum inventory level L_l^γ and U_l^γ, respectively, and incurs a holding cost h_l^γ. Its demand $d_l^{\gamma t}$ in each period t is known. In $t = 0$ it is also known the initial inventory levels at each plant j ($I_j^{\alpha 0}, I_j^{\beta 0}$, and $I_j^{\gamma 0}$) and at each customer l ($I_l^{\gamma 0}$).

Regarding the timing of activities, we assume that a plant is able to pick up α, to produce β, and to produce and deliver γ to the customers in the same period. Although the pick up of α and the production of β can be done simultaneously, whenever all activities (pick up of α, production of β, and production and delivery of γ) are performed in the same period, it is mandatory that it follows a sequence. As the production of γ is dependent of the availability of α and β, the blending process can only be scheduled after all pick up of α and production of β is finished. This is mandatory in order to allow the use of the total amount α and β obtained in t in the same period. Lastly and for the same reasons, deliveries can only be performed after all pickups and production decisions were taken.

The objective of the 2E-PRP is to minimize the total inventory, production, and transportation cost, determining, for each plant:

- when, how much and from which supplier to pickup α
- when and how much to produce β
- when and how much to blend α and β in order to yield γ
- how to manage the inventories of α, β, and γ (customers and plant)
- when to visit and how much γ to deliver to a customer
- how to combine customers deliveries into vehicle routes

3. Mathematical formulation for the 2E-PRP

We now introduce the mathematical formulation for the 2E-PRP. The following continuous variables are used:

- $r_{ij}^{\alpha kt}$: quantity of α picked up by plant j from supplier i in period t using vehicle k
- $q_j^{\beta t}$: quantity of β produced at plant j in period t

- $q_j^{\gamma t}$: quantity of γ blended at plant j in period t
- $q_{jl}^{\gamma kt}$: quantity of γ delivered by plant j to customer l in period t using vehicle k
- $I_j^{\alpha t}$: inventory level of α at plant j at the end of period t
- $I_j^{\beta t}$: inventory level of β at plant j at the end of period t
- $I_j^{\gamma t}$: inventory level of γ at plant j at the end of period t
- $I_l^{\gamma t}$: inventory level of γ at customer l at the end of period t

In addition, we also use integer and binary variables in our model as follows.

- $X_{ij}^{\alpha kt} = 1$ if vehicle k from plant j picks up α from supplier i in period t, 0 otherwise
- $X_j^{\beta t} = 1$ if β is produced at plant j in period t, 0 otherwise
- $X_j^{\gamma t} = 1$ if γ is produced at plant j in period t, 0 otherwise
- $Y_{jl}^{kt} = 1$ if vehicle k from plant j visits customer l in period t, 0 otherwise
- $Y_{jj}^{kt} = 1$ if vehicle k from plant j is used in period t, 0 otherwise
- $y_{uv}^{kjt} = 1$ if vehicle k from plant j travels between the customers u and v, in period t, 0 otherwise
- $y_{jl}^{kjt} \in \{0,1,2\}$. When $y_{jl}^{kjt} = 1$, vehicle k traverses edge (j,l) in period t. If $y_{jl}^{kjt} = 2$, a round trip is defined, 0 otherwise

Figure 1 shows a suitable representation of the 2E-PRP for a given plant in a given period. In order to simplify the example, indices i, k, and t are omitted from the variables. In the first echelon, the pick up of α feeds inventory I_j^α. The second echelon contains the delivery decisions of γ, which is provided by I_j^γ. Production decisions are taken between the two echelons. When a production batch of γ $\left(q_j^\gamma\right)$ is scheduled, a certain amount of α $\left(\varphi q_j^\gamma\right)$ and β $\left((1-\varphi)\, q_j^\gamma\right)$ are required and provided by their inventories, according to the octane grade φ. After the blending process finishes, q_j^γ is available at its inventory I_j^γ, and is ready to be delivered to the customers.

The 2E-PRP is formulated by (1)$-$(31):

$$\min \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{P}}\sum_{g\in\{\beta,\gamma\}} f_j^g X_j^{gt} + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{P}}\sum_{g\in\{\beta,\gamma\}} c_j^g q_j^{gt} + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{P}}\sum_{g\in\{\alpha,\beta,\gamma\}} h_j^g I_j^{gt} + \sum_{t\in\mathcal{T}}\sum_{l\in\mathcal{C}} h_l^\gamma I_l^{\gamma t} + \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{K}}\sum_{(i,j)\in\mathcal{E}'} 2c_{ij} X_{ij}^{\alpha kt} + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}}\sum_{(u,v)\in\mathcal{E}''} c_{uv} y_{uv}^{kjt} \quad (1)$$

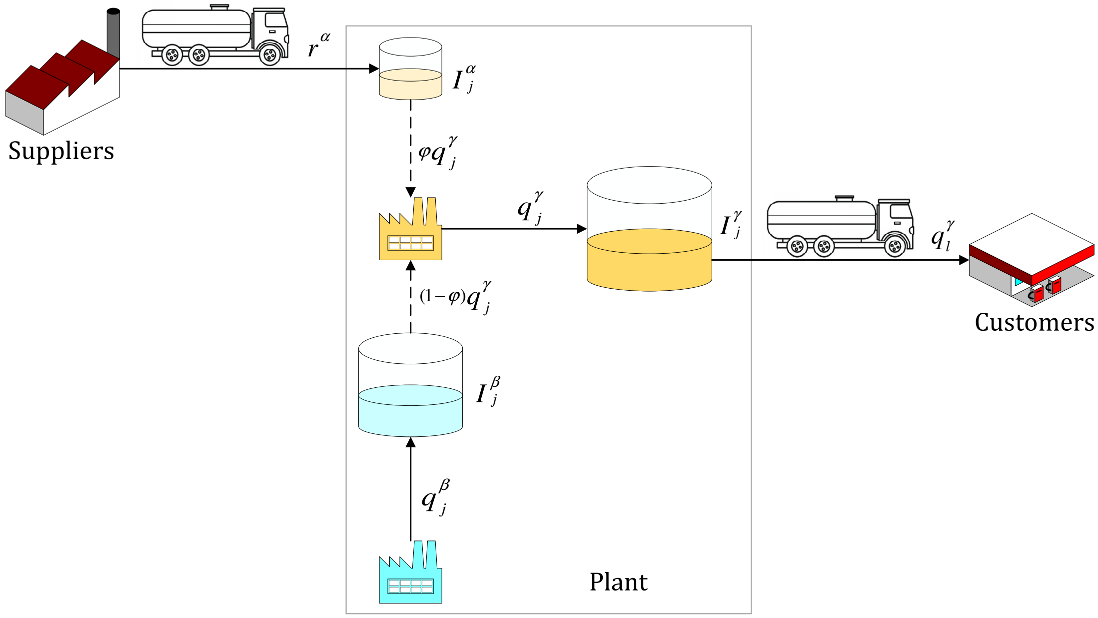

Figure 1: Graphical representation of the 2E-PRP

subject to

$$\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}} r_{ij}^{\alpha kt} \le \Phi_i \qquad i\in\mathcal{F} \tag{2}$$

$$I_j^{\alpha t} = I_j^{\alpha t-1} + \sum_{i\in\mathcal{F}}\sum_{k\in\mathcal{K}} r_{ij}^{\alpha kt} - \varphi q_j^{\gamma t} \qquad j\in\mathcal{P}, t\in\mathcal{T} \tag{3}$$

$$I_j^{\beta t} = I_j^{\beta t-1} + q_j^{\beta t} - (1-\varphi) q_j^{\gamma t} \qquad j\in\mathcal{P}, t\in\mathcal{T} \tag{4}$$

$$I_j^{\gamma t} = I_j^{\gamma t-1} + q_j^{\gamma t} - \sum_{l\in\mathcal{C}}\sum_{k\in\mathcal{K}} q_{jl}^{\gamma kt} \qquad j\in\mathcal{P}, t\in\mathcal{T} \tag{5}$$

$$L_j^g \le I_j^{gt} \le U_j^g \qquad j\in\mathcal{P}, t\in\mathcal{T}, g\in\{\alpha,\beta,\gamma\} \tag{6}$$

$$I_l^{\gamma t} = I_l^{\gamma t-1} + \sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}} q_{jl}^{\gamma kt} - d_l^{\gamma t} \qquad l\in\mathcal{C}, t\in\mathcal{T} \tag{7}$$

$$L_l^{\gamma} \le I_l^{\gamma t} \le U_l^{\gamma} \qquad l\in\mathcal{C}, t\in\mathcal{T} \tag{8}$$

$$q_j^{gt} \le P_j^g X_j^{gt} \qquad j\in\mathcal{P}, t\in\mathcal{T}, g\in\{\beta,\gamma\} \tag{9}$$

$$q_j^{gt} \le U_j^g - I_j^{gt-1} \qquad j\in\mathcal{P}, t\in\mathcal{T}, g\in\{\beta,\gamma\} \tag{10}$$

$$\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{F}} X_{ij}^{\alpha kt} \le 1 \qquad j\in\mathcal{P}, t\in\mathcal{T} \tag{11}$$

$$\sum_{i\in\mathcal{F}}\sum_{k\in\mathcal{K}} r_{ij}^{\alpha kt} \le U_j^{\alpha} - I_j^{\alpha t-1} \qquad j\in\mathcal{P}, t\in\mathcal{T} \tag{12}$$

$$\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}} q_{jl}^{\gamma kt} \le U_l^{\gamma} - I_l^{\gamma t-1} \qquad l\in\mathcal{C}, t\in\mathcal{T} \tag{13}$$

$$q_{jl}^{\gamma kt} \leq U_l^{\gamma} Y_{jl}^{kt} \qquad l \in \mathcal{C}, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{14}$$

$$r_{ij}^{\alpha kt} \leq Q X_{ij}^{\alpha kt} \qquad j \in \mathcal{P}, i \in \mathcal{F}, k \in \mathcal{K}, t \in \mathcal{T} \tag{15}$$

$$\sum_{l \in \mathcal{C}} q_{jl}^{\gamma kt} \leq Q Y_{jj}^{kt} \qquad j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{16}$$

$$\sum_{j \in \mathcal{P}} \sum_{k \in \mathcal{K}} Y_{jl}^{kt} \leq 1 \qquad l \in \mathcal{C}, t \in \mathcal{T} \tag{17}$$

$$\sum_{\substack{u \in \mathcal{V}'' \\ u < l}} y_{ul}^{kjt} + \sum_{\substack{u \in \mathcal{V}'' \\ l < u}} y_{lu}^{kjt} = 2 Y_{jl}^{kt} \qquad l \in \mathcal{V}'', j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{18}$$

$$\sum_{l \in S} \sum_{\substack{u \in S \\ l < u}} y_{lu}^{kjt} \leq \sum_{l \in S} Y_{jl}^{kt} - Y_{jm}^{kt} \qquad S \subseteq \mathcal{C}, |S| \geq 2, m \in S, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{19}$$

$$r_{ij}^{\alpha kt} \geq 0 \qquad j \in \mathcal{P}, i \in \mathcal{F}, k \in \mathcal{K}, t \in \mathcal{T} \tag{20}$$

$$I_j^{\alpha t}, I_j^{\beta t}, I_j^{\gamma t} \geq 0 \qquad j \in \mathcal{P}, t \in \mathcal{T} \tag{21}$$

$$I_l^{\gamma t} \geq 0 \qquad l \in \mathcal{C}, t \in \mathcal{T} \tag{22}$$

$$0 \leq q_j^{\beta t} \leq \min\left\{P_j^{\beta}, U_j^{\beta}\right\} \qquad j \in \mathcal{P}, t \in \mathcal{T} \tag{23}$$

$$0 \leq q_j^{\gamma t} \leq \min\left\{P_j^{\gamma}, U_j^{\gamma}\right\} \qquad j \in \mathcal{P}, t \in \mathcal{T} \tag{24}$$

$$0 \leq q_{jl}^{\gamma kt} \leq \min\left\{Q, U_l^{\gamma}\right\} \qquad l \in \mathcal{C}, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{25}$$

$$X_{ij}^{\alpha kt} \in \{0, 1\} \qquad j \in \mathcal{P}, i \in \mathcal{F}, k \in \mathcal{K}, t \in \mathcal{T} \tag{26}$$

$$X_j^{\beta t}, X_j^{\gamma t} \in \{0, 1\} \qquad j \in \mathcal{P}, t \in \mathcal{T} \tag{27}$$

$$Y_{jl}^{kt} \in \{0, 1\} \qquad l \in \mathcal{C}, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{28}$$

$$Y_{jj}^{kt} \in \{0, 1\} \qquad j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{29}$$

$$y_{jl}^{kjt} \in \{0, 1, 2\} \qquad l \in \mathcal{C}, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T} \tag{30}$$

$$y_{uv}^{kjt} \in \{0, 1\} \qquad u, v \in \mathcal{C}, u < v, j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T}. \tag{31}$$

The objective function (1) minimizes the associated costs, given by six terms: fixed and variable production costs, inventory costs at the plants and at the customers, pickup transportation costs, and delivery transportation costs. Constraints (2) limit the availability for α at the suppliers, according to the contract with plants. Constraints (3)−(6) balance the flow and impose inventory bounds at the plants, while (7)−(8) do the same for γ at the customers. Constraints (9)−(10) guarantee that the production decisions at the plants do not exceed their production and inventory capacities for β and γ. Constraints (11) allow the plants to perform at most one pick up per period. The ML policy is formulated for α at the plants by (12) and for γ at the customers by (13). Constraints (14) link the quantity delivered with the visit to the customer. Constraints (15)−(16) ensure that the total amount picked up and the total delivered quantities do not exceed the vehicle capacity, while constraints (17) avoid split deliveries. Linking and subtour elimination conditions are imposed by (18)−(19). The variables domain is defined by constraints (20)−(31).

The OU policy for α at the plants is formulated by (32), while the OU policy for γ at customers is imposed by (33).

$$\sum_{i\in\mathcal{F}}\sum_{k\in\mathcal{K}} r_{ij}^{\alpha kt} \geq U_j^{\alpha} \sum_{i\in\mathcal{F}}\sum_{k\in\mathcal{K}} X_{ij}^{\alpha kt} - I_j^{\alpha t-1} \qquad j\in\mathcal{P}, t\in\mathcal{T}. \tag{32}$$

$$\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}} q_{jl}^{\gamma kt} \geq U_l^{\gamma} \sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}} Y_{jl}^{kt} - I_l^{\gamma t-1} \qquad l\in\mathcal{C}, t\in\mathcal{T}. \tag{33}$$

We also present a set of well-known valid inequalities (VI) from the IRP. Archetti et al. [6] introduced several VI for the single-vehicle IRP, which are extended to the multi-vehicle IRP (MIRP) by Coelho and Laporte [21], and to the multi-depot MIRP by Bertazzi et al. [11].

$$y_{jl}^{kjt} \leq 2Y_{jl}^{kt} \qquad l\in\mathcal{C}, j\in\mathcal{P}, k\in\mathcal{K}, t\in\mathcal{T} \tag{34}$$

$$y_{ul}^{kjt} \leq Y_{jl}^{kt} \qquad u,l\in\mathcal{C}, u<l, j\in\mathcal{P}, k\in\mathcal{K}, t\in\mathcal{T} \tag{35}$$

$$y_{lu}^{kjt} \leq Y_{jl}^{kt} \qquad u,l\in\mathcal{C}, l<u, j\in\mathcal{P}, k\in\mathcal{K}, t\in\mathcal{T} \tag{36}$$

$$Y_{jl}^{kt} \leq Y_{jj}^{kt} \qquad l\in\mathcal{C}, j\in\mathcal{P}, k\in\mathcal{K}, t\in\mathcal{T} \tag{37}$$

$$Y_{jj}^{kt} \leq Y_{jj}^{k-1,t} \qquad j\in\mathcal{P}, k\in\mathcal{K}\setminus\{1\}, t\in\mathcal{T} \tag{38}$$

$$Y_{jl}^{kt} \leq \sum_{\substack{u\in C\\ u<l}} Y_{ju}^{k-1,t} \qquad l\in\mathcal{C}, j\in\mathcal{P}, k\in\mathcal{K}\setminus\{1\}, t\in\mathcal{T}. \tag{39}$$

Inequalities (34) link the variable used to perform a direct delivery to customer l with its corresponding visit variable. Analogously, inequalities (35) do the same when customer l is preceded, and (36) when the customer is succeeded by another customer u. Inequalities (37) impose that vehicle k must depart from plant j if customer l is being served by this vehicle. Inequalities (38) ensure that vehicle k cannot depart from plant j if vehicle $k-1$ is not already in use, while inequalities (39) break symmetry by ensuring that customers with smaller indexes are served by vehicles with smaller indexes.

We also consider inequalities (40)−(42) proposed by Coelho and Laporte [21] for the MIRP and adapted by Guimarães et al. [29] for the two-echelon multi-depot IRP (2E-MDIRP), which compute the minimum number of deliveries on the interval $[t_1, t_2]$ for customer l to avoid a stock-out.

$$\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}}\sum_{t=t_1}^{t_2} Y_{jl}^{kt} \geq \left\lceil \frac{\sum_{t=t_1}^{t_2} d_l^{\gamma t} - U_l^{\gamma}}{min\left\{Q, U_l^{\gamma}\right\}} \right\rceil \qquad l\in\mathcal{C}, t_1, t_2\in\mathcal{T}, t_2>t_1 \tag{40}$$

$$\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}}\sum_{t=t_1}^{t_2} Y_{jl}^{kt} \geq \frac{\sum_{t=t_1}^{t_2} d_l^{\gamma t} - I_l^{\gamma t_1-1}}{min\left\{Q, U_l^{\gamma}\right\}} \qquad l\in\mathcal{C}, t_1, t_2\in\mathcal{T}, t_2>t_1 \tag{41}$$

$$\sum_{j\in\mathcal{P}}\sum_{k\in\mathcal{K}}\sum_{t=t_1}^{t_2} Y_{jl}^{kt} \geq \frac{\sum_{t=t_1}^{t_2} d_l^{\gamma t} - I_l^{\gamma t_1-1}}{\sum_{t=t_1}^{t_2} d_l^{\gamma t}} \qquad l\in\mathcal{C}, t_1, t_2\in\mathcal{T}, t_2>t_1. \tag{42}$$

Desaulniers et al. [26] note that an optimal solution of the IRP can be associated with the first-in, first-out rule, and introduce a suitable notation as follows. Let $I_l^{\gamma 0,s} = max\left\{0, I_l^{\gamma 0} - \sum_{t=1}^{s} d_l^{\gamma t}\right\}$ be the residual initial inventory for the customer l at the end of period $s \in \mathcal{T}$. The portion of the demand not met by the initial inventory is known as the residual demand, and is formally given by:

$$\bar{d}_l^{\gamma s} = \begin{cases} \max\left\{0, d_l^{\gamma 1} - I_l^{\gamma 0}\right\} & \text{if } s = 1 \\ \max\left\{0, d_l^{\gamma s} - I_l^{\gamma 0,s-1}\right\} & \text{otherwise} \end{cases} \quad \forall s \in \mathcal{T}. \tag{43}$$

Furthermore, the same authors define the following set P_{lt}^{+}, representing all periods in which a sub-delivery of γ for customer l in period t can be consumed in the period t, or to be kept in inventory to meet future demands.

$$P_{lt}^{+} = \left\{t|\bar{d}_l^{\gamma t} > 0\right\} \cup \left\{s > t|\bar{d}_l^{\gamma s} > 0 \text{ and } \sum_{t'=t}^{s-1} d_l^{\gamma t'} < U_l^{\gamma}\right\} \cup \left\{p+1|\sum_{t'=t}^{p} d_l^{\gamma t'} < U_l^{\gamma}\right\}. \tag{44}$$

Finally, the set of periods for which a delivery can be performed to satisfy the demand of customer l in $s \in \mathcal{T}$ is defined by $P_{ls}^{-} = \left\{t \in \mathcal{T} | s \in P_{lt}^{+}\right\}$. Based on that, Desaulniers et al. [26] introduce the following valid inequalities for the IRP, which we employ to the 2E-PRP context, as follows:

$$\sum_{j \in \mathcal{P}} \sum_{k \in \mathcal{K}} \sum_{t \in P_{ls}^{-}} Y_{jl}^{kt} \geq 1, \qquad l \in \mathcal{C}, s \in \mathcal{T}, \text{ with } P_{ls}^{-} \neq \emptyset. \tag{45}$$

Using a remaining quantity to restrict the range of continuous variables $I_l^{\gamma t}$ and $q_{jl}^{\gamma kt}$, Lefever et al. [35] propose the following inequalities to the IRP, which are automatically extended to the 2E-PRP.

$$I_l^{\gamma t} \geq I_l^{\gamma 0,t}, \qquad l \in \mathcal{C}, t \in \mathcal{T} \tag{46}$$

$$q_{jl}^{\gamma kt} \leq U_l^{\gamma} - I_l^{\gamma 0,t}, \qquad j \in \mathcal{P}, l \in \mathcal{C}, k \in \mathcal{K}, t \in \mathcal{T}. \tag{47}$$

Based on the aforesaid definitions, Lefever [34] formulates a set of valid inequalities to the IRP with transshipment, in order to establish the minimum number of delivery routes along $\mathcal{T}$. We also adapt these inequalities to the 2E-PRP, as:

$$\sum_{j \in \mathcal{P}} \sum_{k \in \mathcal{K}} \sum_{t=1}^{s} Y_{jj}^{kt} \geq \left\lceil \frac{\sum_{l \in \mathcal{C}} \sum_{t=1}^{s} \bar{d}_l^{\gamma t}}{Q} \right\rceil, \qquad s \in \mathcal{T}. \tag{48}$$

Finally, we also propose inequalities (49) which strengthen the minimum number of delivery routes performed by plant j in period t, according to the total amount of γ planned to be delivered and the vehicle capacity.

$$\frac{\sum_{k\in\mathcal{K}}\sum_{l\in\mathcal{C}} q_{jl}^{\gamma kt}}{Q} \leq \sum_{k\in\mathcal{K}} Y_{jj}^{kt} \qquad j\in\mathcal{P}, t\in\mathcal{T}. \tag{49}$$

4. Branch-and-cut algorithm

The model presented in Section 3 can be fully generated only for very small instances size. For realistic size problems, the number of subtour elimination constraints (SEC) (19) increases exponentially and their full enumeration is impracticable. To overcome this limitation, we present a branch-and-cut scheme where these constraints are dynamically generated and added along the search process, whenever subtours are found at the current solution. Our branch-and-cut works as follows.

At the root node, all constraints (except SEC) and valid inequalities are generated. Whenever a node is optimized by a MIP solver, a search for violated SEC is performed. To this end, we have used the CVRPSEP package of Lysgaard et al. [38]. When subtours are identified, their corresponding SEC are added at the search tree. This process is repeated until a feasible or dominated solution is reached, or until there are no more cuts to be added. At this point, a new fractional variable is chosen for branching, and a new subproblem is generated. Thus, the model is reoptimized in a new node. We provide the pseudocode of our branch-and-cut in **Algorithm 1**.

5. Parallelized Algorithm

The NP-hard aspect of integrated production and distribution problems leads to the development of heuristics [1, 2, 3, 7, 14, 13, 17, 43], while exact algorithms are usually limited to solve only small instances [2, 7]. In this section, we introduce a novel scheme that combines local search (LS) procedures with B&C, in a parallelized processing framework, which we call LS-B&C. At the beginning, an initial solution is provided to the LS front and to the B&C front and both start to run, continuously and independently, feeding each other whenever a new best solution is found. The LS-B&C stops when an optimal solution is proved or when a time limit is reached.

Algorithm 1 Pseudocode of the proposed B&C algorithm

```
At the root node, generate (1)−(31), except (19), and all valid inequalities (34)−(42)∪(45)−(49).
Solve the LP relaxation of the node.
Termination check:
if there are no more nodes to evaluate then
    Stop.
else
    Select one node from the B&C tree.
end if
while the solution of the current LP relaxation contains subtours do
    Add violated subtour elimination constraints.
    Solve the LP relaxation of the node.
end while
if the solution of the current LP relaxation is integer then
    Go to the termination check.
else
    Branching: branch on one of the fractional variables.
    Go to the termination check.
end if
```

5.1. Initial solution procedure

On the initial solution procedure (ISP) all deliveries are scheduled based on direct links between plants and customers, by removing y variables from the model presented in the Section 3, while minimizing pickups, inventory holding, and production costs. After we identify a delivery plan, the visiting sequence is determined exactly by the B&C proposed by Padberg and Rinaldi [39]. The ISP model is formulated by:

$$\min \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{P}} \sum_{g \in \{\beta,\gamma\}} f_j^g X_j^{gt} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{P}} \sum_{g \in \{\beta,\gamma\}} c_j^g q_j^{gt} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{P}} \sum_{g \in \{\alpha,\beta,\gamma\}} h_j^g I_j^{gt} + \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{C}} h_l^{\gamma} I_l^{\gamma t} + \sum_{t \in \mathcal{T}} \sum_{k \in \mathcal{K}} \sum_{(i,j) \in \mathcal{E}'} 2c_{ij} X_{ij}^{\alpha kt} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{P}} \sum_{k \in \mathcal{K}} \sum_{l \in \mathcal{C}} c_{jl} Y_{jl}^{kt} \tag{50}$$

subject to (2)−(17), (20)−(29) and to:

$$Y_{jj}^{kt} \leq \sum_{l \in \mathcal{C}} Y_{jl}^{kt} \qquad j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T}. \tag{51}$$

The objective function (50) minimizes the production costs (fixed and variable), inventory holding, pickup and approximated delivery transportation costs, calculated as a direct link cost c_{jl} between customer l and plant j. Constraints (51) ensure that each vehicle from a plant performs a delivery route if at least one customer is served in this period. Valid inequalities (37)−(42) and (45)−(49) also apply and the OU policy for α and γ is handled through constraints (32) and (33), respectively.

5.2. Local search mechanism

The LS front reduces the search space by decomposing the problem into several smaller MIP subproblems, which are solved exactly. As the B&C front controls the stopping criteria, our LS-B&C stops when a solution, either from LS or from B&C, is proven to be optimal. At the same time, the advantages of the parallel approach allows the LS-B&C to handle very large instances, which could not be done by previous exact algorithms.

The LS front runs in two stages. The first one works by fixing strategic variables at the original model, in order to yield ad hoc neighborhoods (MIP subproblems), which are then solved by a MIP solver. It is inspired by Larrain et al. [33], which proposed a variable MIP neighborhood descent (VMND) algorithm to solve an IRP variant from banking industry. The VMND is sequential and alternates between B&C and LS phase from time to time, while the depth of the neighborhood explored in the LS is dynamically adjusted, according to the performance of each phase. This idea has been successfully employed in other contexts by Darvish and Coelho [25] Darvish et al. [24], Larrain et al. [32]. Our approach differs from these authors because the B&C and LS run independently and continuously inside a parallel framework, and each new best solution found is immediately shared by both fronts. Each ad hoc neighborhood $\mathcal{N}_i$ is defined as the solutions that can be reached by applying an operator i to a current solution. Furthermore, a chosen neighborhood is explored up to a certain time limit or when its optimal solution is proven. The depth of the neighborhood is dependent on the operators, which are described now:

- ***Random single period (RSP):*** Randomly selects a period t' and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $u, v \in \mathcal{E}''$, and $t \neq t'$, where $\overline{Y}_{jl}^{kt}$ and $\overline{y}_{uv}^{kjt}$ are the values of the variables in the current solution.
- ***Random double period (RDP):*** Randomly selects two periods $t'_1 \neq t'_2$, and sets $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $t \neq t'_1$ and $t \neq t'_2$.
- ***Random plant (RP):*** Randomly selects a plant j', and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $j \neq j'$.
- ***Random vehicle (RV):*** Randomly selects a vehicle k', and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $k \neq k'$.

- ***Random plant-period (RP-P):*** Randomly selects a plant j' and a period t', and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $j \neq j', t \neq t'$.

- ***Random vehicle-period (RV-P):*** Randomly selects a vehicle k' and a period t', and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $k \neq k', t \neq t'$.

- ***Random plant-vehicle (RP-V):*** Randomly selects a plant j' and a vehicle k', and sets variables $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, and $y_{uv}^{kjt} = \overline{y}_{uv}^{kjt}$, with $j \neq j', k \neq k'$.

- ***Largest deliveries (LD):*** Sets $Y_{jl}^{kt} = 1$ for one third of the largest deliveries scheduled. This operator is based on our observation that smaller deliveries are easily consolidated into existing routes or new ones.

- ***Random deliveries (RD):*** Randomly selects Θ visit variables and sets $Y_{jl}^{kt} = \overline{Y}_{jl}^{kt}$, where Θ follows a discrete uniform distribution in the interval $\mathcal{U}[1/3 \times |Y_{jl}^{kt}|; 2/3 \times |Y_{jl}^{kt}|]$, where $|Y_{jl}^{kt}| = |\mathcal{P}||\mathcal{C}||\mathcal{K}||\mathcal{T}|$.

- ***All operator (ALL):*** It sequentially applies all valid operators in the order presented. In this case, the time limit is uniformly shared between all operators.

Table 1 shows the neighborhood size, according to the number of variables fixed by each operator chosen. We highlight that the continuous processing allows to setup a wide range of neighborhoods to be explored by the local search, switching between intensification and diversification, based on the deterministic and random characteristic of the operators.

Table 1: Neighborhood definitions

Operator	Fixed Variables	Number of Fixed Variables	Neighborhood Size	Not Applicable
RSP	Y_{jl}^{kt} and y_{uv}^{kjt}	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|(\|\mathcal{T}\|-1) + \|\mathcal{E}''\|\|\mathcal{K}\|(\|\mathcal{T}\|-1)$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	
RDP	Y_{jl}^{kt} and y_{uv}^{kjt}	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|(\|\mathcal{T}\|-2) + \|\mathcal{E}''\|\|\mathcal{K}\|(\|\mathcal{T}\|-2)$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	
RP	Y_{jl}^{kt} and y_{uv}^{kjt}	$(\|\mathcal{P}\|-1)\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + (\|\mathcal{E}''\| - \|\mathcal{P}\|\|\mathcal{C}\|)\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{P}\| = 1$
RV	Y_{jl}^{kt} and y_{uv}^{kjt}	$\|\mathcal{P}\|\|\mathcal{C}\|(\|\mathcal{K}\|-1)\|\mathcal{T}\| + \|\mathcal{E}''\|(\|\mathcal{K}\|-1)\|\mathcal{T}\|$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{K}\| = 1$
RP-P	Y_{jl}^{kt} and y_{uv}^{kjt}	$(\|\mathcal{P}\|-1)\|\mathcal{C}\|\|\mathcal{K}\|(\|\mathcal{T}\|-1) + (\|\mathcal{E}''\| - \|\mathcal{P}\|\|\mathcal{C}\|)\|\mathcal{K}\|(\|\mathcal{T}\|-1)$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{P}\| = 1$
RV-P	Y_{jl}^{kt} and y_{uv}^{kjt}	$\|\mathcal{P}\|\|\mathcal{C}\|(\|\mathcal{K}\|-1)(\|\mathcal{T}\|-1) + \|\mathcal{E}''\|(\|\mathcal{K}\|-1)(\|\mathcal{T}\|-1)$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{K}\| = 1$
RP-V	Y_{jl}^{kt} and y_{uv}^{kjt}	$(\|\mathcal{P}\|-1)\|\mathcal{C}\|(\|\mathcal{K}\|-1)\|\mathcal{T}\| + (\|\mathcal{E}''\| - \|\mathcal{P}\|\|\mathcal{C}\|)(\|\mathcal{K}\|-1)\|\mathcal{T}\|$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\| + \|\mathcal{E}''\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{P}\| = 1$ or $\|\mathcal{K}\| = 1$
LD	Y_{jl}^{kt}	$1/3 * \|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\|$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\|$	
RD	Y_{jl}^{kt}	$[1/3 * \|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\|; 2/3 * \|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\|]$	$\|\mathcal{P}\|\|\mathcal{C}\|\|\mathcal{K}\|\|\mathcal{T}\|$	

5.3. General improvement procedure

The second stage of LS front improves the current solution by applying a general improvement procedure (GIP). It was proposed by Schenekemberg et al. [42], and it can perform removals, insertions, and customers swaps only on established routes, avoiding customers from being served by new ones. In this sense, for all route variables $Y_{jj}^{kt} = 1$, let $\mathcal{A}_{jkt}$ be the set of customers $l \in C$ served by vehicle k, departing from plant j in period t. The authors also introduce a subset of edges, adjacent to plant j, given by $\mathcal{E}_{jkt} \subseteq \mathcal{E}''$, with $\mathcal{E}_{jkt} = \left\{(u,v) : u,v \in \{j\} \cup \mathcal{A}_{jkt} \cup \mathcal{A}_{jkt}^c\right\}$, and $\mathcal{A}_{jkt}^c = \mathcal{C} \backslash \mathcal{A}_{jkt}$. Formally, GIP sets free $q_{jl}^{\gamma kt}$, Y_{jl}^{kt} and y_{uv}^{kjt} to be optimized if and only if their associated routes exist, where $(u,v) \in \mathcal{E}_{jkt}$ and $|\mathcal{A}_{jkt}| > 0$. Otherwise, $q_{jl}^{\gamma kt}$, Y_{jl}^{kt} and y_{uv}^{kjt} are set to zero when $\mathcal{A}_{jkt} = \emptyset$. We highlight that all other decision variables of the 2E-PRP model are also free to be optimized.

Although GIP explores a small search space, it can still be difficult to find good solutions in an acceptable time. To handle this limitation, we limit the number of movements on each existing route, by adding the following constraints to the GIP model:

$$\sum_{l \in \mathcal{A}_{jkt}} \left(1 - Y_{jl}^{kt}\right) + \sum_{l \in \mathcal{A}_{jkt}^c} Y_{jl}^{kt} \leq \mathcal{B} \qquad j \in \mathcal{P}, k \in \mathcal{K}, t \in \mathcal{T}, |\mathcal{A}_{jkt}| > 0. \qquad (52)$$

Constraints (52) are inspired by the local branching of Fischetti and Lodi [27]. The positive integer parameter $\mathcal{B}$ counts the number of binary variables switching their value with respect to each existing route from a solution $\bar{s}$, either from 1 to 0 or from 0 to 1. The set of solutions satisfying (52) define the $\mathcal{B}$-OPT neighborhood $\mathcal{N}(\bar{s}, \mathcal{B})$ of $\bar{s}$. The parameter $\mathcal{B}$ must be properly chosen, since when $\mathcal{B}$ is too small the probability of finding solutions better than $\bar{s}$ is low. Otherwise, a big value for $\mathcal{B}$ does not allow through exploration of the neighborhood in a suitable time. We also note that all valid inequalities and OU constraints are applicable to GIP.

5.4. LS-B&C general framework

Our algorithm starts simultaneously in both fronts (B&C and LS), after ISP provides an initial solution Z_{Ini}. When a new best solution Z_{Best} is found by B&C (LS), it is feed to LS (B&C). The LS front runs sequentially in two steps: the first one diversifies the search process by randomly choosing and applying an operator on Z_{LS}, allowing LS to explore different neighborhoods. The second one intensifies the search by applying GIP on the current solution. The B&C front controls the stopping criteria, either when the time limit is reached, or when an optimal solution is proven.

6. Computational experiments

All algorithms were coded in C++, executed on a grid of Intel(R) Xeon(R) processors at 2.60GHz, running in CentOS Linux operating system. All MIPs were solved by Gurobi 8.1.0. We have generated a set of instances for the 2E-PRP, by adapting the benchmarks proposed by Archetti et al. [6] for the IRP. We have considered the ML and OU policies for the α and γ, yielding four different combinations (ML-ML, ML-OU, OU-ML, and OU-OU). After a tuning phase, we have used six threads for the B&C algorithm of Section 4. For the LS-B&C experiments, we split in four dedicated threads for the B&C and two threads for the LS front.

We have also assessed our algorithms by comparing their performance against two similar problems from the literature. We have investigated our methods by solving the instances of the 2E-MDIRP, recently introduced by Guimarães et al. [29]. We also considered multi-vehicle production-routing problem (MV-PRP), introduced by Chandra and Fisher [15]. The set of large- to very large-size instances of the MV-PRP proposed by [12] has three groups of 30 instances each, being B1 with 50 customers and 5 vehicles, B2 with 100 customers and 9 vehicles, and B3 with 200 customers and 13 vehicles, and all of them with 20 periods. These sets have been widely studied by the literature, exclusively through heuristic and matheuristics methods. We highlight that we are the first to solve these instances exactly. Finally, we also considered a set of instances proposed and solved (exactly and heuristically) by Adulyasak et al. [2].

Regarding the processing time, B&C and LS-B&C ran up to 7200s for the 2E-PRP and 2E-MDIRP. Despite the difference on computational features (hardware, solver version, and programming language), our tests were performed with the same time limit of Guimarães et al. [29] for the 2E-MDIRP. For the Adulyasak et al. [2] instances, we ran up to 14400s, one third of the processing time of the authors. Finally, the experiments on the Boudia et al. [13] instances, we also considered 14400s. We justify this choice by the fact that exact methods on similar size instances [12] have been tested for up to 86400s [20], so we use only a sixth of that time.

After preliminary experiments, we set the time limit for each LS operator to 300s, while GIP ran for up to 900s for the 2E-PRP, 2E-MDIRP, and MV-PRP [2]. For the [12] instances, each operator ran for up to 900s and 1200 for the GIP. In all cases, the parameter $\mathcal{B}$ of constraints (52) was defined as $\left\lceil \frac{|\mathcal{C}|}{2} \right\rceil$, except for the B2 and B3 groups for [12] instances, where $\mathcal{B} = \left\lceil \frac{|\mathcal{C}|}{10} \right\rceil$ for B2 and $\mathcal{B} = \left\lceil \frac{|\mathcal{C}|}{20} \right\rceil$ for B3.

All instances generated for the 2E-PRP and detailed results for all problems considered are available from `https://www.leandro-coelho.com/two-echelon-production-routing-problem/`.

6.1. Results for the 2E-PRP

We have generated a set of instances for the 2E-PRP adapted from the single plant IRP instances of Archetti et al. [6]. We considered instances with 1 and 3 vehicles, and 3 and 6 periods, low and high inventory cost at customers. The supply chain structure defines four additional combinations: one supplier-one plant, two suppliers-two plants, two suppliers-three plants, and three suppliers-two plants. We also consider 5, 10, 25, and 50 customers. We kept all information about the customers (coordinates, demand, initial inventory,

inventory cost and capacity), and the coordinates of the first plant. To configure a two-echelon supply chain with multiples suppliers and plants we generated the parameters as follows.

- Supplier coordinates follow a discrete uniform distribution in the interval $X_i \sim \mathcal{U}[500, 1000]$ and $Y_i \sim \mathcal{U}[0, 1000]$
- Plants coordinates: when $|P| > 1$, X_j and Y_j follow a discrete uniform distribution in the interval $\mathcal{U}[0, 500]$
- $U_j^\gamma = \left\lceil \frac{1}{|\mathcal{P}|} \left(\mathcal{U}[1.1, 1.2] \left[\frac{p(\sum_{l\in\mathcal{C}} U_l^\gamma) - \sum_{l\in\mathcal{C}} I_l^{\gamma 0}}{p} \right] \right) \right\rceil$
- $U_j^\alpha = \lceil \varphi U_j^\gamma \rceil$
- $U_j^\beta = \lceil (1-\varphi) U_j^\gamma \rceil$
- $P_j^\beta = \left\lceil \Theta U_j^\beta \right\rceil$ and $P_j^\gamma = \lceil \Theta U_j^\gamma \rceil$, where Θ is randomly chosen among $\{0.5; 0.75\}$
- $Q = \left\lceil \frac{|\mathcal{P}| \max\{U_j^\gamma\}}{|\mathcal{K}|} \right\rceil, j \in \mathcal{P}$
- $I_j^{\alpha 0} = I_j^{\beta 0} = 0$
- $I_j^{\gamma 0} = \lceil \mathcal{U}[0.2, 0.3] U_j^\gamma \rceil$
- $\Phi_i = \left\lceil \Theta \frac{\varphi}{|\mathcal{F}|} \left(p(\sum_{l\in\mathcal{C}} U_l^\gamma) - \sum_{l\in\mathcal{C}} I_l^{\gamma 0} - \sum_{j\in\mathcal{P}} I_j^{\gamma 0} \right) \right\rceil$, where Θ is randomly chosen among $\{0.25; 0.5; 0.75\}$
- $h_j^\alpha = h_j^\beta = h_j^\gamma = h_j$
- $c_j^\beta = \Theta h_j^\beta$, where Θ is randomly chosen among $\{0.75; 1.0; 1.25\}$
- $c_j^\gamma = \Theta h_j^\gamma$, where Θ is randomly chosen among $\{0.75; 1.0; 1.25\}$
- $f_j^\beta = 0.4 \left(\frac{1}{|\mathcal{F}|} \sum_{i\in\mathcal{F}} c_{ij} \right)$
- $f_j^\gamma = 0.6 \left(\frac{1}{|\mathcal{F}|} \sum_{i\in\mathcal{F}} c_{ij} \right)$

According to the groups of vehicles, supply chain structure, customers, periods, and inventory cost, we generated 128 instances ($2 \times 4 \times 4 \times 2 \times 2$), and each one of them was solved by the four inventory policies, totaling 512 tests on each algorithm.

We start our analysis by comparing the results obtained by B&C and LS-B&C for each inventory policy. Tables 2−5 report the results for the ML-ML, ML-OU, OU-ML and OU-OU, respectively. Each row presents the average results for 16 instances, according to the supply chain structure, periods, and inventory cost. Columns $|\mathcal{K}|$ and $|\mathcal{C}|$ show the number of vehicles and customers. Column SF reports the number upper bounds (UB) obtained by the B&C, OPT is the number of optimal solutions proven, $\overline{LB}$, is the average lower bounds for the 16 instances, while $\overline{UB^*}$ is the average UB only for instances for which B&C finds a solution. Finally, column $\overline{T(s)}$ shows the average processing time. As the ISP provides a feasible initial solution for the LS-B&C, its correspondent value for SF is 16 in all rows. We note that LS-B&C outperforms the B&C in UB and LB for all policies in a equivalent processing time, proving two more optimal solutions for the ML-OU and one more for the OU-ML. Moreover, for the whole experiment over 512 tests, LS-B&C obtained 97 exclusive BKS, against only 14 by B&C.

Table 2: Results for the 2E-PRP under the ML-ML policy

$\|\mathcal{K}\|$	$\|\mathcal{C}\|$	B&C					LS-B&C			
		SF	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$
1	5	16	16	6549.5	6549.5	306.2	16	6549.5	6549.5	159.4
	10	16	14	9191.1	9280.7	1138.5	14	9192.7	9279.6	1107.1
	25	16	13	9010.9	9182.9	2152.7	13	9005.8	9159.8	2339.6
	50	16	9	13189.8	13740.9	3521.0	9	13167.7	13716.2	3645.3
3	5	16	15	6910.3	6943.1	1127.7	15	6922.8	6943.1	1135.8
	10	16	13	9513.6	9689.4	1527.7	13	9520.0	9685.2	1614.1
	25	16	10	9009.7	9476.7	3121.1	10	9008.1	9443.4	3133.2
	50	11	3	12879.9	11056.2	6344.8	3	12899.6	10903.9	6338.1
Avg				**9531.9**	**9489.9**	**2405.0**		**9533.3**	**9460.1**	**2434.1**

*Where B&C finds a solution.

In order to derive managerial insights, we also compare the inventory policies regarding to the average result, considering the BKS for each instance. Table 6 presents the average UB for the ML-ML policy, and also the average increase in total cost by imposing the OU policy for α, for γ, and for both. The results are consistent with the findings of Archetti et al. [6] for the basic IRP, Coelho et al. [18] for the MIRP, Coelho and Laporte [20] for IRP with transshipment, and Guimarães et al. [29] for the 2E-MDIRP.

Table 7 details the average cost of each component for $|\mathcal{T}| = 3$, and its percentage deviation by imposing the OU inventory policy to α at the plants, to γ at the customers, and both. When inventory costs are low and OU is imposed to α, the major effects are limited to the inventory at the plants. Despite its low representation on the total cost, the average inventory cost at the plants increased by nearly 20%. On the other hand, when the OU policy is imposed to the customers, all costs increased (except for the inventory at plants), especially the most representative ones, such as delivery cost (by 11%), and pickup cost (around by 5%). Since the deliveries to the customers take place at the end of the supply chain, the OU policy creates a cascade effect on the other activities. When the inventory cost changes from low to high, the pickups from

Table 3: Results for the 2E-PRP under the ML-OU policy

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{C}\rvert$	B&C					LS-B&C			
		SF	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$
1	5	16	16	7038.3	7038.3	97.2	16	7038.3	7038.3	144.1
	10	16	14	9665.7	9722.9	1392.6	14	9673.1	9722.9	1465.6
	25	16	12	9605.7	9896.5	2238.8	12	9640.8	9874.6	2276.4
	50	15	9	14290.0	14043.7	3489.0	9	14288.4	14018.1	3466.2
3	5	16	15	7328.3	7347.6	738.1	16	7347.6	7347.6	749
	10	16	13	9995.2	10168.0	1759.5	13	9996.3	10147.4	1891.5
	25	16	10	9696.1	10207.5	3574.6	10	9692.5	10196.6	3622.6
	50	10	3	13964.8	10810.4	6412.1	5	13979.3	10805.1	6342.2
Avg				**10198.0**	**9904.4**	**2462.7**		**10207.0**	**9893.8**	**2494.7**

*Where B&C finds a solution.

Table 4: Results for the 2E-PRP under the OU-ML policy

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{C}\rvert$	B&C					LS-B&C			
		SF	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$
1	5	16	16	6583.1	6583.1	273	16	6583.1	6583.1	342.9
	10	16	14	9227.6	9315.4	1095.8	14	9249.2	9314.3	1103.1
	25	16	14	9091.6	9238.4	1951.5	14	9092.3	9210.0	2314.8
	50	16	10	13212.5	13798.4	3224.1	9	13225.3	13738.4	3690.4
3	5	16	15	6929.5	6947.9	1087.7	16	6947.9	6947.9	947.1
	10	16	13	9547.1	9713.3	1464.3	13	9557.1	9711.7	1489.0
	25	16	10	9044.9	9595.4	3061.7	10	9060.3	9510.7	3075.5
	50	10	3	12910.9	9858.8	6203.3	3	12927.0	9777.4	6245.1
Avg				**9568.4**	**9381.3**	**2295.2**		**9580.2**	**9349.2**	**2401.0**

*Where B&C finds a solution.

Table 5: Results for the 2E-PRP under the OU-OU policy

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{C}\rvert$	B&C					LS-B&C			
		SF	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$	**OPT**	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$
1	5	16	16	7042.8	7042.8	71.5	16	7042.8	7042.8	79.6
	10	16	14	9703.0	9777.6	1329.9	14	9714.5	9771.8	1283.7
	25	16	12	9717.3	9945.5	2374	12	9697.2	9941.1	2390.1
	50	15	8	14353.6	14151.7	3755.4	9	14372.2	14051.6	3712.9
3	5	16	16	7388.3	7388.3	722.1	16	7388.3	7388.3	802.3
	10	16	13	10068.9	10224.9	1607.7	13	10080.5	10221.5	1566.6
	25	16	10	9755.7	10357.7	3659.4	10	9725.7	10300.9	3632.1
	50	12	4	14061.5	12896.4	6210.1	4	14044.0	12354.2	6322.5
Avg				**10261.4**	**10223.1**	**2466.3**		**10258.1**	**10134.0**	**2473.7**

*Where B&C finds a solution.

Table 6: Comparison of solution cost between the inventory policies for the 2E-PRP

$\lvert\mathcal{K}\rvert$	$\lvert\mathcal{C}\rvert$	**ML-ML**	**Δ% OU(α)**	**Δ% OU(γ)**	**Δ% OU(α and γ)**
1	5	6549.6	0.5	7.5	7.5
	10	9279.6	0.4	4.8	5.3
	25	9159.8	0.5	7.8	8.5
	50	13711.1	0.2	8.6	8.9
3	5	6943.1	0.1	5.8	6.4
	10	9684.8	0.3	4.8	5.5
	25	9441.1	0.7	8.0	9.1
	50	14147.6	0.3	8.1	8.4
Avg		**9864.6**	**0.4**	**6.9**	**7.5**

the suppliers are more frequent and the number of production setups is lower.

Table 7: Comparison between the inventory policies for the 2E-PRP on each cost component, $|\mathcal{T}| = 3$.

Group	Cost component	ML-ML	OU-ML	ML-OU	OU-OU	Δ% OU(α)	Δ% OU(γ)	Δ% OU(α and γ)
Low	Inventory at Plant	21.1	25.3	18.3	20.5	19.9	-13.4	-2.8
	Fixed Production	962.1	980.0	1028.3	1024.4	1.9	6.9	6.5
	Variable Production	47.5	48.6	56.4	56.3	2.3	18.7	18.5
	Pickups	1284.8	1287.9	1348.0	1354.3	0.2	4.9	5.4
	Deliveries	2323.6	2320.9	2579.1	2616.7	-0.1	11.0	12.6
	Inventory at Customer	46.7	47.3	76.6	76.3	1.3	64.1	63.4
	Total	**4685.6**	**4710.0**	**5106.7**	**5148.6**	**0.5**	**9.0**	**9.9**
High	Inventory at Plant	201.9	236.3	129.0	167.1	17.0	-36.1	-17.2
	Fixed Production	1160.7	1151.2	1205.5	1205.5	-0.8	3.9	3.9
	Variable Production	549.6	556.3	621.7	622.5	1.2	13.1	13.3
	Pickups	1379.6	1379.8	1520.1	1542.6	0.0	10.2	11.8
	Deliveries	2471.4	2491.8	2862.4	2872.9	0.8	15.8	16.3
	Inventory at Customer	469.4	462.0	752.6	754.5	-1.6	60.3	60.7
	Total	**6232.7**	**6277.3**	**7091.3**	**7165.1**	**0.7**	**13.8**	**15.0**

Table 8 presents the results considering $|\mathcal{T}| = 6$. We note that with a longer planning horizon, the cascade effect by imposing the OU policy is mitigated and stays concentrated on the extremes of the supply chain, i.e., the major change occurs at the inventory levels. The impact is even more significant on the inventory at customers when the inventory cost changes from low to high. In general, when the planning horizon is larger, the effects are dissipated among the periods, and the impact on the total cost caused by this change is smaller, as detailed in the Table 9.

Given the superior performance of our LS-B&C, and also its flexibility to deal with similar problems, all following analysis are preformed with respect to its results.

6.2. Results for the 2E-MDIRP

Guimarães et al. [29] introduced the 2E-MDIRP and proposed a B&C algorithm. The authors also designed a matheuristic based on ALNS to solve large instances. We considered all 128 instances, each one solved by the same four inventory policy considered for the 2E-PRP, totaling 512 tests.

Table 10 compares the B&C and ALNS proposed by [29], with respect to our LS-B&C. Columns SF and OPT have the same previous definition, while BKS and BKS-Excl reports the number of instances where the algorithm reached the BKS, and did it solely among the compared methods, respectively. Regarding the

Table 8: Comparison between the inventory policies for the 2E-PRP on each cost component, $|\mathcal{T}| = 6$.

Group	Cost component	ML-ML	OU-ML	ML-OU	OU-OU	Δ% OU(α)	Δ% OU(γ)	Δ% OU(α and γ)
Low	Inventory at Plant	71.4	78.0	80.9	84.9	9.2	13.3	18.9
	Fixed Production	3146.2	3163.7	3204.5	3196.5	0.6	1.9	1.6
	Variable Production	225.9	225.9	228.7	227.2	0.0	1.2	0.6
	Pickups	3718.3	3697.9	3783.9	3789.8	-0.6	1.8	1.9
	Deliveries	5687.7	5701.4	6047.9	6056.5	0.2	6.3	6.5
	Inventory at Customer	117.9	114.9	170.5	170.9	-2.5	44.6	45.0
	Total	**12967.4**	**12981.8**	**13516.3**	**13525.8**	**0.1**	**4.2**	**4.3**
High	Inventory at Plant	447.8	496.9	409.8	451.4	11.0	-8.5	0.8
	Fixed Production	2868.4	2872.5	2904.0	2960.0	0.1	1.2	3.2
	Variable Production	2058.1	2060.1	2092.7	2105.1	0.1	1.7	2.3
	Pickups	3474.9	3456.8	3509.6	3477.8	-0.5	1.0	0.1
	Deliveries	5754.2	5743.7	5888.3	5891.9	-0.2	2.3	2.4
	Inventory at Customer	973.2	1001.7	1746.5	1746.7	2.9	79.5	79.5
	Total	**15576.5**	**15631.6**	**16550.8**	**16633.0**	**0.4**	**6.3**	**6.8**

Table 9: Percentage effect on the average total cost when the inventory cost changes from low to high.

Policy	$|\mathcal{T}| = 3$	$|\mathcal{T}| = 6$
ML-ML	33.0	20.1
OU-ML	33.3	20.4
ML-OU	38.9	22.5
OU-OU	39.2	23.0

comparison with B&C, we note that our LS-B&C was able to prove 412 optimal solutions, against 298 of the previous one. Moreover, the results show that LS-B&C also outperforms the matheuristic, finding 111 exclusive BKS when ALNS gets only 22. Overall, LS-B&C obtained 490 BKS out of 512.

Table 10: Comparative statistics for the 2E-MDIRP

Policy	#	B&C [29]				ALNS [29]		LS-B&C		
		SF	**OPT**	**BKS**	**BKS-Excl**	**BKS**	**BKS-Excl**	**OPT**	**BKS**	**BKS-Excl**
ML-ML	128	103	80	83	0	100	3	104	125	24
ML-OU	128	98	69	70	0	98	1	101	114	28
OU-ML	128	101	80	81	0	97	14	106	127	27
OU-OU	128	95	69	69	0	92	4	101	124	32
Total	**512**	**397**	**298**	**303**	**0**	**387**	**22**	**412**	**490**	**111**

The average results presented in the Table 11 show that LS-B&C is also very superior in terms of LB, when compared to the B&C. Besides the difference on computational features, we highlight that the processing time of the LS-B&C is nearly half that of the B&C.

Table 11: Comparative results for the 2E-MDIRP

Policy	B&C [29]			ALNS [29]		LS-B&C			
	$\overline{LB}$	$\overline{UB^*}$	$\overline{T(s)}$	$\overline{UB}$	$\overline{T(s)}$	$\overline{LB}$	$\overline{UB^*}$	$\overline{UB}$	$\overline{T(s)}$
ML-ML	4897.1	4508.5	2877.6	5355.7	458.6	5156.8	4432.5	5334.5	1536.7
ML-OU	5179.7	4823.6	3438.9	5847.4	701.2	5539.9	4625.7	5847.8	1783.4
OU-ML	5011.5	4578.0	2879.6	5577.5	426.8	5357.7	4534.1	5544.5	1527.1
OU-OU	5277.8	4881.6	3438.2	6063.9	715.0	5725.7	4659.3	6043.2	1828.3
Avg	**5091.5**	**4697.9**	**3158.6**	**5711.1**	**575.4**	**5445.0**	**4562.9**	**5692.5**	**1668.9**

*Where B&C finds solution.

6.3. Results for MV-PRP

Since the 2E-PRP is a generalization of the MV-PRP for a two-echelon supply chain, we also consider two sets of instances for this problem. A very close variant of PRP is tackled by Adulyasak et al. [2], who propose 168 instances, with $n = 10$ to 50 customers with 3 periods, $n = 10$ to 40 with 6 periods, and $n = 10$ to 30 with 9 periods. When $n \leq 25$, there are 2 vehicles, and 3 to 4 vehicles when $25 < n \leq 50$. The authors considered the ML and OU policies, and proposed a B&C and an ALNS matheuristic, called Op-ALNS, exploring parallel optimization from commercial solvers, with an eight core processor computer. In our case, LS-B&C combines two different algorithms in a separated framework, managing itself the optimization process.

Table 12 reports the average results for the MV-PRP under the ML policy. The number of customers is shown in the column $|\mathcal{C}|$, while # shows the number of instances on each row. In all cases, B&C was able to find a feasible solution. Columns $\overline{UB}$ bring the average UB obtained, and $\overline{GAP}(\%)$ is the average of gaps with respect to the LB. $\overline{T(s)}$ is the average processing time. We note that our LS-B&C was able to prove 134 optimal solutions, against 125 from [2]. Besides that, our method yields smaller gaps in much shorter processing times. The overall results corroborates the positive effect of the parallelized approach. Results for the OU policy are presented on Table 13, and an equivalent performance is reported. We highlight that LS-B&C proved two more optimal solutions than the B&C from [2].

Table 12: Comparative results for MV-PRP under the ML policy

$\|\mathcal{C}\|$	#	**Eight Core B&C** [2]				**Op-ALNS** [2]	**LS-B&C**			
		OPT	$\overline{UB}$	$\overline{GAP}(\%)$	$\overline{T(s)}$	$\overline{UB}$	**OPT**	$\overline{UB}$	$\overline{GAP}(\%)$	$\overline{T(s)}$
10	24	24	80437.9	0.0	63.9	81039.8	24	80437.9	0.0	13.2
15	24	20	113757.4	0.4	7734.1	114740.5	20	113749.8	0.2	2540.0
20	24	22	128911.1	0.1	4083.8	129861.5	24	128897.9	0.0	1030.0
25	24	19	152953.4	0.3	10535.4	153894.0	22	152883.5	0.2	1983.9
30	24	14	171515.8	1.2	20621.5	172498.3	16	171044.7	0.6	5787.3
35	16	10	139836.4	0.7	20562.5	141059.9	13	139696.3	0.3	4799.7
40	16	6	175077.3	1.3	29370.3	175763.7	7	174791.3	0.8	9859.8
45	8	5	134771.9	1.1	27798.3	135454.8	2	134570.6	0.9	12652.5
50	8	5	122846.6	0.5	21318.1	123882.6	6	122812.3	0.3	6404.2
Avg			**135567.5**	**0.5**	**15787.5**	**136466.1**		**135431.6**	**0.3**	**5007.8**

An overall performance is presented in Table 14. Our LS-B&C proved 241 optimal solutions, 11 more than the B&C from [2]. Furthermore, LS-B&C obtained 95 exclusive BKS, against only 10 from the B&C, while Op-ALNS was not competitive. In general, our algorithm reached the BKS in 325 cases, B&C in 241 cases and Op-ALNS in only 25 cases.

Finally, we also analyze the performance of our LS-B&C by solving the set of instances proposed by Boudia et al. [12]. These instances were also solved heuristically by Boudia et al. [13] (GRASP and path relinking), by Boudia and Prins [14] (memetic algorithm), by Bard and Nananukul [9] (tabu seach), by Armentano et al. [8] (tabu search and path relinking), by Adulyasak et al. [3] (Op-ALNS), by Absi et al. [1] (two-phase iterative method), by Solyalı and Süral [43] (multi-phase heuristic based on mathematical programming techniques), [41] (multi-phase set partitioning approach), Chitsaz et al. [17] (three-phase decomposition

Table 13: Comparative results for MV-PRP under the OU policy

$\|\mathcal{C}\|$	#	Eight Core B&C [2]				Op-ALNS [2]	LS-B&C			
		OPT	$\overline{UB}$	$\overline{GAP}$(%)	$\overline{T(s)}$	$\overline{UB}$	**OPT**	$\overline{UB}$	$\overline{GAP}$(%)	$\overline{T(s)}$
10	24	24	106038.5	0.0	14.2	106389.7	24	106038.5	0.0	65.1
15	24	23	155475.2	0.0	3610.9	156570.8	22	155475.2	0.4	2198.7
20	24	23	169639.0	0.1	3361.2	170368.3	23	169641.5	0.3	1982.8
25	24	20	215111.0	0.3	13515.8	216681.5	18	215074.2	0.7	6468.1
30	24	9	230555.9	1.4	31419.7	232122.2	12	230013.0	1.6	10128.8
35	16	5	226566.7	2.5	32834.3	227194.2	7	225262.2	1.5	10744.6
40	16	2	298812.1	2.3	42443.1	300725.6	2	297738.8	3.5	13447.6
45	8	0	363779.3	3.8	43225.2	364520.3	0	361468.1	4.0	14400.0
50	8	0	351695.1	4.4	43227.4	351815.9	0	348515.8	4.5	14400.0
Avg			**235297.0**	**1.1**	**23739.1**	**236265.0**		**234358.6**	**1.3**	**8204.0**

Table 14: Overall results for MV-PRP

Policy	#	Eight Core B&C [2]			Op-ALNS [2]		LS-B&C		
		OPT	**BKS**	**BKS-Excl**	**BKS**	**BKS-Excl**	**OPT**	**BKS**	**BKS-Excl**
ML	168	125	130	5	8	0	134	162	38
OU	168	106	111	5	17	0	108	163	57
Total	**336**	**231**	**241**	**10**	**25**	**0**	**242**	**325**	**95**

matheuristic), and more recently by Li et al. [36] (two-phase iterative method with a repairing strategy and a fix-and-optimize procedure).

Table 15 presents the detailed results for group B1, with 50 customers, 5 vehicles and 20 periods. The first column shows the instance number, followed by the best UB from the literature and its reference. Columns UB, LB, Gap % and T(s) present the complete results of our LS-B&C. The last column brings the deviation between our results with respect to the best UB, computed by $\left(\frac{UB - BKS}{BKS}\right) \times 100$. We highlight that our LS-B&C finds better solutions for all of 30 instances, holding all BKS in this group, besides providing LB for the first time for these instances.

For set B2, considered as very large instances with 100 customers, we were able to solve all of them and to provide the first LB. As mentioned before, we were the first to address these instances exactly, and at this point, we did not observe any improvement in terms of UB. Results are shown in Table 16. Regarding the set B3, with 200 customers, we considered up to 500 GB of RAM, but it was not enough to solve the root node, and no LB is reported.

7. Conclusions

In this work, we have introduced the 2E-PRP, an integration between the PRP and the two-echelon IRP, inspired from a real-case of VMI in the petrochemical industry. We have designed a B&C and an exact parallel algorithm, combining MIP-based local searches with B&C, which we call LS-B&C. Computational experiments shown that LS-B&C outperforms B&C in terms of the number of optimal solutions proven, and the quality of bounds (UB and LB), without compromising the processing time. Managerial analysis have shown that OU policy increase the total cost by following the same patterns from the literature; inventory cost at the customers increased by around 70%, compared to the ML policy. Transportation costs have also increased by around 7% when the OU policy is applied to the customers.

The LS-B&C proposed in this study is flexible enough to deal with other problems from the literature. Experiments over 512 cases (128 instances with 4 different inventory policies) on the 2E-MDIRP have shown that our LS-B&C outperforms the B&C from the literature, by proving 412 optimal solutions, including 114 new ones. In general, LS-B&C was also competitive against an ALNS-based matheuristic algorithm, finding 490 BKS, being 111 exclusive ones.

On classical PRP instances, our algorithm provides better results than the previous methods from the literature. On 336 MV-PRP instances (168 for ML and 168 for OU), LS-B&C obtained 9 more optimal solutions than a state-of-the-art B&C from literature for ML policy, also finding 22% of new exclusive BKS. For the OU policy, our method obtained two additional optimal solutions and detained 34% of exclusive

Table 15: Results for instances B1 of Boudia et al. [12]

Inst	BKS	Reference	UB	LB	Gap %	T(s)	Δ%UB
1	339391	Russell [41]	**337499**	258352.4	23.5	14400	-0.56
2	355490	Russell [41]	**338961**	264283.2	22.0	14400	-4.65
3	343274	Russell [41]	**336000**	254851.0	24.2	14400	-2.12
4	337265	Russell [41]	**330113**	248792.4	24.6	14400	-2.12
5	342209	Russell [41]	**333413**	259757.4	22.1	14400	-2.57
6	346471	Russell [41]	**342026**	260482.7	23.8	14400	-1.28
7	346178	Russell [41]	**337323**	258141.3	23.5	14400	-2.56
8	343483	Russell [41]	**339326**	256474.2	24.4	14400	-1.21
9	343477	Russell [41]	**340908**	264268.3	22.5	14400	-0.75
10	331606	Russell [41]	**330174**	250015.1	24.3	14400	-0.43
11	331702	Russell [41]	**329559**	250243.8	24.1	14400	-0.65
12	337257	Solyalı and Süral [43]	**327727**	250601.4	23.5	14400	-2.83
13	336392	Russell [41]	**332399**	258037.9	22.4	14400	-1.19
14	344009	Russell [41]	**332552**	253715.9	23.7	14400	-3.33
15	344811	Russell [41]	**337187**	254364.4	24.6	14400	-2.21
16	338888	Solyalı and Süral [43]	**334062**	257236.8	23.0	14400	-1.42
17	345273	Russell [41]	**341415**	255154.4	25.3	14400	-1.12
18	349145	Russell [41]	**349030**	254780.1	27.0	14400	-0.03
19	344140	Russell [41]	**334833**	246369.5	26.4	14400	-2.70
20	335590	Russell [41]	**326780**	248632.3	23.9	14400	-2.63
21	330425	Russell [41]	**324828**	256769.5	21.0	14400	-1.69
22	327559	Russell [41]	**322572**	257887.9	20.1	14400	-1.52
23	339646	Russell [41]	**332317**	253960.3	23.6	14400	-2.16
24	335753	Russell [41]	**334477**	250997.7	25.0	14400	-0.38
25	345980	Russell [41]	**339938**	256081.5	24.7	14400	-1.75
26	344861	Russell [41]	**331634**	263941.7	20.4	14400	-3.84
27	333767	Russell [41]	**324688**	253070.3	22.1	14400	-2.72
28	334890	Russell [41]	**326480**	256618.7	21.4	14400	-2.51
29	361210	Solyalı and Süral [43]	**345272**	263189.6	23.8	14400	-4.41
30	334361	Russell [41]	**333128**	252905.2	24.1	14400	-0.37
Avg	**340816.8**		**334220.7**	**255665.9**	**23.5**	**14400**	**-1.90**

Table 16: Results for instances B2 of Boudia et al. [12]

Inst	BKS	Reference	UB	LB	Gap %	T(s)	Δ%UB
1	627481	Solyalı and Süral [43]	673665	379246.0	43.7	14400	7.36
2	629654	Russell [41]	696327	365296.8	47.5	14400	10.59
3	626168	Adulyasak et al. [3]	696474	361480.3	48.1	14400	11.23
4	617529	Li et al. [36]	697277	369090.3	47.1	14400	12.91
5	638752	Li et al. [36]	703347	353076	49.8	14400	10.11
6	624242	Li et al. [36]	695470	344000.2	50.5	14400	11.41
7	612937	Russell [41]	668993	362252.7	45.9	14400	9.15
8	613980	Li et al. [36]	679533	360120.7	47.0	14400	10.68
9	629671	Solyalı and Süral [43]	690379	365830	47.0	14400	9.64
10	628162	Russell [41]	689501	342802.7	50.3	14400	9.76
11	632208	Li et al. [36]	696839	365539.9	47.5	14400	10.22
12	639171	Solyalı and Süral [43]	703374	348985	50.4	14400	10.04
13	627361	Russell [41]	688045	350884.6	49.0	14400	9.67
14	626362	Russell [41]	681686	360691.2	47.1	14400	8.83
15	628841	Russell [41]	682164	358071.7	47.5	14400	8.48
16	634063	Li et al. [36]	699943	386378.6	44.8	14400	10.39
17	640918	Russell [41]	711193	355322.1	50.0	14400	10.96
18	624512	Russell [41]	694429	358031.4	48.4	14400	11.20
19	638363	Solyalı and Süral [43]	703687	342533.9	51.3	14400	10.23
20	641094	Russell [41]	713022	362014.6	49.2	14400	11.22
21	632475	Russell [41]	704632	341909.3	51.5	14400	11.41
22	637021	Russell [41]	699609	367799.3	47.4	14400	9.83
23	618008	Li et al. [36]	683665	346167.6	49.4	14400	10.62
24	636148	Russell [41]	698056	342546.1	50.9	14400	9.73
25	631508	Russell [41]	695604	350477.2	49.6	14400	10.15
26	624516	Solyalı and Süral [43]	683735	350479.7	48.7	14400	9.48
27	617961	Absi et al. [1]	671018	370703.6	44.8	14400	8.59
28	620148	Solyalı and Süral [43]	672444	346725.3	48.4	14400	8.43
29	629760	Russell [41]	697149	360864.7	48.2	14400	10.70
30	622453	Solyalı and Süral [43]	693710	359029.2	48.2	14400	11.45
Avg	**628382.2**		**692165.7**	**357611.7**	**48.3**	**14400**	**10.15**

BKS. Finally, we were the first to solve exactly instances of Boudia et al. [12], and we generated LB for all instances of groups B1 and B2. We highlight that our LS-B&C attained all BKS up to 50 customer (B1 group), by comparing with heuristic, metaheuristics, and matheuristic previous approach.

References

[1] Absi, N., Archetti, C., Dauzère-Pérèz, S., Feillet, D., 2015. A two-phase iterative heuristic approach for the production routing problem. Transportation Science 49, 784–795.

[2] Adulyasak, Y., Cordeau, J.F., Jans, R., 2014a. Formulations and branch-and-cut algorithms for multi-vehicle production and inventory routing problems. INFORMS Journal on Computing 26, 103–120.

[3] Adulyasak, Y., Cordeau, J.F., Jans, R., 2014b. Optimization-based adaptive large neighborhood search for the production routing problem. Transportation Science 48, 20–45.

[4] Adulyasak, Y., Cordeau, J.F., Jans, R., 2015. The production routing problem: A review of formulations and solution algorithms. Computers & Operations Research 55, 141–152.

[5] Andersson, H., Hoff, A., Christiansen, M., Hasle, G., Løkketangen, A., 2010. Industrial aspects and literature survey: Combined inventory management and routing. Computers & Operations Research 37, 1515–1536.

[6] Archetti, C., Bertazzi, L., Laporte, G., Speranza, M.G., 2007. A branch-and-cut algorithm for a vendor-managed inventory-routing problem. Transportation Science 41, 382–391.

[7] Archetti, C., Bertazzi, L., Paletta, G., Speranza, M.G., 2011. Analysis of the maximum level policy in a production-distribution system. Computers & Operations Research 38, 1731 – 1746.

[8] Armentano, V.A., Shiguemoto, A.L., Løkketangen, A., 2011. Tabu search with path relinking for an integrated production–distribution problem. Computers & Operations Research 38, 1199 – 1209.

[9] Bard, J.F., Nananukul, N., 2008. The integrated production-inventory-distribution-routing problem. Journal of Scheduling 12, 257.

[10] Belgin, O., Karaoglan, I., Altiparmak, F., 2018. Two-echelon vehicle routing problem with simultaneous pickup and delivery: mathematical model and heuristic approach. Computers & Industrial Engineering 115, 1 – 16.

[11] Bertazzi, L., Coelho, L.C., Maio, A.D., Laganà, D., 2019. A matheuristic algorithm for the multi-depot inventory routing problem. Transportation Research Part E: Logistics and Transportation Review 122, 524–544.

[12] Boudia, M., Louly, M., Prins, C., 2005. Combined optimization of production and distribution, in: Proc. Internat. Conf. Indust. Engrg. Systems Management, IESM, Marrakesh, Marrocco, pp. 3–4.

[13] Boudia, M., Louly, M., Prins, C., 2007. A reactive GRASP and path relinking for a combined production–distribution problem. Computers & Operations Research 34, 3402 – 3419.

[14] Boudia, M., Prins, C., 2009. A memetic algorithm with dynamic population management for an integrated production–distribution problem. European Journal of Operational Research 195, 703 – 715.

[15] Chandra, P., Fisher, M.L., 1994. Coordination of production and distribution planning. European Journal of Operational Research 72, 503 – 517.

[16] Cheng, C., Yang, P., Qi, M., Rousseau, L.M., 2017. Modeling a green inventory routing problem with a heterogeneous fleet. Transportation Research Part E: Logistics and Transportation Review 97, 97 – 112.

[17] Chitsaz, M., Cordeau, J.F., Jans, R., 2019. A unified decomposition matheuristic for assembly, production, and inventory routing. INFORMS Journal on Computing 31, 134–152.

[18] Coelho, L.C., Cordeau, J.F., Laporte, G., 2012. Consistency in multi-vehicle inventory-routing. Transportation Research Part C: Emerging Technologies 24, 270–287.

[19] Coelho, L.C., Cordeau, J.F., Laporte, G., 2014. Thirty years of inventory routing. Transportation Science 48, 1–19.

[20] Coelho, L.C., Laporte, G., 2013. The exact solution of several classes of inventory-routing problems. Computers & Operations Research 40, 558–565.

[21] Coelho, L.C., Laporte, G., 2014. Improved solutions for inventory-routing problems through valid inequalities and input ordering. International Journal of Production Economics 155, 391–397.

[22] Crainic, T.G., Perboli, G., Mancini, S., Tadei, R., 2010. Two-echelon vehicle routing problem: A satellite location analysis. Procedia - Social and Behavioral Sciences 2, 5944 – 5955. The Sixth International Conference on City Logistics.

[23] Cárdenas-Barrón, L.E., Chung, K.J., Treviño-Garza, G., 2014. Celebrating a century of the economic order quantity model in honor of ford whitman harris. International Journal of Production Economics 155, 16–38.

[24] Darvish, M., Archetti, C., Coelho, L.C., 2019. Trade-offs between environmental and economic performance in production and inventory-routing problems. International Journal of Production Economics 217, 269 – 280.

[25] Darvish, M., Coelho, L.C., 2018. Sequential versus integrated optimization: Production, location, inventory control, and distribution. European Journal of Operational Research 268, 203 – 214.

[26] Desaulniers, G., Rakke, J.G., Coelho, L.C., 2016. A branch-price-and-cut algorithm for the inventory-routing problem. Transportation Science 50, 1060–1076.

[27] Fischetti, M., Lodi, A., 2003. Local branching. Mathematical Programming 98, 23–47.

[28] Govindan, K., 2013. Vendor-managed inventory: a review based on dimensions. International Journal of Production Research 51, 3808–3835.

[29] Guimarães, T.A., Coelho, L.C., Schenekemberg, C.M., Scarpin, C.T., 2019. The two-echelon multi-depot inventory-routing problem. Computers & Operations Research 101, 220–233.

[30] Lahyani, R., Coelho, L.C., Renaud, J., 2018. Alternative formulations and improved bounds for the multi-depot fleet size and mix vehicle routing problem. OR Spectrum 40, 125–157.

[31] Laporte, G., 2009. Fifty years of vehicle routing. Transportation Science 43, 408–416.

[32] Larrain, H., Coelho, L.C., Archetti, C., Speranza, M.G., 2019. Exact solution methods for the multi-period vehicle routing problem with due dates. Computers & Operations Research 110, 148 – 158.

[33] Larrain, H., Coelho, L.C., Cataldo, C., 2017. A variable MIP neighborhood descent algorithm for managing inventory and distribution of cash in automated teller machines. Computers & Operations Research 85, 22 – 31.

[34] Lefever, W., 2018. Stochastic and robust optimization algorithms for the inventory-routing problem and its extensions. Ph.D. thesis. Ghent University.

[35] Lefever, W., Aghezzaf, E.H., Hadj-Hamou, K., Penz, B., 2018. Analysis of an improved branch-and-cut formulation for the inventory-routing problem with transshipment. Computers & Operations Research 98, 137–148.

[36] Li, Y., Chu, F., Chu, C., Zhu, Z., 2019. An efficient three-level heuristic for the large-scaled multi-product production routing problem with outsourcing. European Journal of Operational Research 272, 914 – 927.

[37] Low, C., Chang, C.M., Li, R.K., Huang, C.L., 2014. Coordination of production scheduling and delivery problems with heterogeneous fleet. International Journal of Production Economics 153, 139 – 148.

[38] Lysgaard, J., Letchford, A.N., Eglese, R.W., 2004. A new branch-and-cut algorithm for the capacitated vehicle routing problem. Mathematical Programming 100, 423–445.

[39] Padberg, M., Rinaldi, G., 1991. A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. SIAM Review 33, 60–100.

[40] Perboli, G., Tadei, R., Vigo, D., 2011. The two-echelon capacitated vehicle routing problem: Models and math-based heuristics. Transportation Science 45, 364–380.

[41] Russell, R.A., 2017. Mathematical programming heuristics for the production routing problem. International Journal of Production Economics 193, 40 – 49.

[42] Schenekemberg, C.M., Scarpin, C.T., Pecora Jr., J.E., Guimarães, T.A., Coelho, L.C., 2019. The two-echelon inventory-routing problem with fleet management. Technical Report CIRRELT-2019-36. Montréal.

[43] Solyalı, O., Süral, H., 2017. A multi-phase heuristic for the production routing problem. Computers & Operations Research 87, 114 – 124.

[44] Sorda, G., Banse, M., Kemfert, C., 2010. An overview of biofuel policies across the world. Energy Policy 38, 6977–6988.

[45] Williams, B.D., T., T., 2008. A review of inventory management research in major logistics journals: themes and future directions. The International Journal of Logistics Management 19, 212–232.