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# Production scheduling under uncertainty for an open pit mine using Lagrangian relaxation and branch-and-cut algorithm

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**Abstract:** The production scheduling of an open pit mine determines the optimal extraction sequencing that significantly impacts a mine's life as well as net present value. The optimization of production scheduling of large open pit mines with geological uncertainty is a computationally very intensive process. In this paper, an approximation algorithm is proposed to schedule an open pit mine, where instead of solving the whole problem at once, the production schedule is generated by sequentially solving sub-problems. The sub-gradient method is used to generate the upper bound solution of a Lagrangian relaxed sub-problem. If the upper bound relaxed algorithm is not a feasible solution, a mixed integer programming is applied on the generated upper-bound solution. The algorithm is validated by solving three problems and is compared to the linear relaxation of the original production scheduling problem. The results show that the proposed algorithm generates a solution very close to optimal with less than a 2% optimality gap.

An application at a copper mine with an orebody represented by 16532 mining blocks is presented using the proposed algorithm. Results show that all constraints are satisfied. The net present values are also calculated and results reveal that an 11% higher net present value (NPV) is generated when compared to the NPV generated when the same approach is applied to the deterministic model of the deposit where uncertainty is not accounted for. Furthermore, a comparative study with the conventional approach shows that the approach proposed here generates a schedule with at least 26% higher NPV. The comparative study also demonstrated that the proposed method generates a pit which is bigger in size (at least 10%) when compared with both the deterministic and conventional approaches.

**Key Words:** Open pit mine optimization, production scheduling, uncertain supply, Lagrangian relaxation, sub-gradient method, branch and cut algorithm.

# 1 Introduction

Open pit mine production scheduling is concerned with the problem of assigning an extraction sequence of mining blocks to different production periods. Mining blocks are represented by their weight, economic value (net profit from block), ore content (weight of block when economic value is positive), metal content, and more. The mine scheduling can be described as follows. A set of mining blocks are assigned to different production periods to maximize profit subject to certain constraints. These constraints include but are not limited to (a) each block can be assigned at most to one production period; (b) a set of blocks are overlying on each block which needs to be assigned the same production period or previous periods; and (c) the number of blocks assigned in a specific period should be such that the total amount of material, ore, and metal mined from the deposit are within the limit in specific production period.

The production scheduling of an open pit mine is a complex task, particularly for mines having a large number of mining blocks and a life of many years (Ramazan and Dimitrakopoulos 2004). Typically, open pit production scheduling is solved by mixed integer programming (Ramazan and Dimitrakopoulos 2004). However, finding the optimum production schedule is always a complex job due to the presence of large numbers of integer variables, large number of constraints, and the overall uncertainty associated with different parameters of mining blocks. Uncertainty arises mostly from geological uncertainty which states that the metal content of the blocks is not precisely known beforehand.

Neglecting the uncertainty of the block parameters and considering those values are known, different optimization methods have been proposed for solving the deterministic production scheduling. A Mixed Integer Linear Programming (MILP) model was proposed by Gershon (1983) to maximize the net present value of an actual mining operation. Alternative to that Dagdelen and Johnson (1986) proposed an exact method based on Lagrangian relaxation. Caccetta and Hill (2003) proposed a branch-and-cut algorithm for solving the production schedule of an open pit mine. Bley et al. (2010) proposed an algorithm using cutting plane algorithm to solve the mine production scheduling problem. Although, these algorithms provide an exact solution, they are restricted only to small size problems. To solve the real life mining problem, which consists of millions of mining blocks, these algorithms require huge computational time. To reduce the problem size of real mine scheduling and make it computationally manageable by exact methods, researchers proposed different alternative algorithms. Topal (2003) proposed an algorithm to solve large scale production scheduling by reducing the number variables by applying the critical path model (CPM). Ramazan (2007) developed the Fundamental Tree Algorithm (FTA) which reduces the problem size by aggregating mining blocks.

Deterministic approaches have been successfully applied for open pit optimisation since the late nineties (Johnson 1968; Dowd 1994; Tachefine and Soumis 1997), however, the underlying assumption of constant value of grade and quantity in a specific block over simplified the problem which leads to unrealistic assessment (Ravenscroft 1992; Dowd 1997; Dimitrakopoulos et al. 2002; Godoy and Dimitrakopoulos 2004; Goodfellow and Dimitrakopoulos 2013). Since the block grades and thus metal contents are calculated using a limited number of data sets, uncertainties are always associated with these estimated values (Boucher and Dimitrakopoulos 2009; Mustapha and Dimitrakopoulos 2010). In order to incorporate the uncertainty associated with block grades and metal content, multiple simulated orebody models are generated using geostatistical simulation algorithms (Govearts 1997; Boucher and Dimitrakopoulos 2012). The stochastic approach is therefore more realistic and researchers are giving it a large amount of attention. Ramazan and Dimitrakopoulos (2004) and Dimitrakopoulos and Ramazan (2009) proposed uncertainty-based approaches by maximizing net present value while minimizing the deviation from the target using the concept of geological risk discounting (GRD). Menabde et al. (2007) proposed an algorithm for mine production scheduling by maximizing the expected net present value over several geologically simulated orebody models while satisfying the production targets in an average sense. Boland et al. (2008) proposed a multi-stage stochastic programming approach for solving mining production scheduling.

These algorithms perform well when used to generate an open pit production schedule; however, they are restricted to small or medium size problems due to the computational issues. Godoy and Dimitrakopoulos (2004) proposed an alternative approach using simulated annealing algorithm to reduce computational time.

Leite and Dimitrakopoulos (2007), and Albor and Dimirakopoulos (2009) modified and tested in a large scale mine a scheduling optimization problem by using the series of pit shells generated by nested Lerchs and Grossmann algorithm (Lerchs and Grossmann 1965). Their algorithms have proven to be computationally faster with an increase of 15 to 28% in the project value as compared to the conventional approach (Leite and Dimitrakopoulos 2007; Albor and Dimirakopoulos 2009). Lamghari and Dimitrakopoulos (2012) proposed a meta-heuristic algorithm for addressing the computational issues associated with full scale stochastic integer program (SIP) and developed an algorithm using Tabu search. Ramazan and Dimitrakopoulos (2013) developed a two-stage SIP algorithm for full scale stochastic production scheduling incorporating stockpiling option and geological risk discounting.

In this paper, an alternative approach is proposed with uncertain economic values of mining blocks in order to reduce the computational time of mine production scheduling. The uncertainty of the economics of mining blocks is incorporated by using multiple geological simulated orebody models. A minimum cut algorithm (Picard 1976; Meagher et al. 2009) combined with Lagrangian relaxation (Tachefine and Soumis 1997; Asad and Dimitrakopoulos 2013) and brunch-and-cut (Caccetta and Hill 2003) are employed to generate a production schedule. Instead of solving the entire problem simultaneously, which is computationally intensive, the proposed algorithm solves the problem sequentially where blocks for the first period are first assigned and eliminated from the problem set and the remaining blocks in the problem set are used for the next period. The proposed approach is a two-step process: (i) first, the minimum graph cut algorithm with Lagrangian relaxation is applied and solved by sub-gradient method (Asad and Dimitrakopoulos 2013); and (ii) if the Lagrangian relaxed solution is infeasible, a mixed integer program is solved using the branch-and-cut algorithm on the violated solution to obtain a solution which respects all resource constraints over all simulations.

The paper is organised as follows: Section 2 describes the proposed approach. Section 3 includes the validation of the proposed method by comparing results to the commercial solver CPLEX. A case study of the proposed method in a copper mine is presented in Section 4. Conclusions are presented in Section 5.

## 2 Method

The steps involved in the method proposed in this paper are presented hereafter.

### 2.1 Formulation

Mine production scheduling with geological uncertainty uses stochastic integer programs (SIPs) with time-indexed binary variables  $x_{i,t}$   $i \in N, t = \{1, \dots, T\}$ , where  $x_{i,t} = 1$  if a mining block is extracted at time  $t$  and  $x_{i,t} = 0$  otherwise. Unlike Ramazan and Dimitrakopoulos (2013), where the objective function is penalized for deviation from target by geological risk discounting factor, the formulation presented in this paper drops the penalty term; assuming that if the deviation is within the permissible limit no penalty is necessary. Therefore, some new constraints are added to limit the permissible deviation. This leads to the following formulation:

$$\max \frac{1}{S} \sum_{s=1}^S \sum_{t=1}^T \sum_{i=1}^N \frac{c_i^s}{(1+r)^t} x_{i,t} \quad (1)$$

s.t.

$$\sum_{t=1}^T x_{i,t} \leq 1 \quad i = 1, 2, \dots, N \quad (2)$$

$$x_{i,t} - \sum_{k=1}^t x_{j,k} \leq 0 \quad i = 1, 2, \dots, N, j \in \Gamma_i \quad (3)$$

$$\sum_{i=1}^N w_i x_{i,t} \leq W_t \quad t = 1, 2, \dots, T \quad (4)$$

$$\sum_{i=1}^N o_{i,s} w_i x_{i,t} \leq O_t + d_{t,s}^o \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (5)$$

$$\sum_{i=1}^N o_{i,s} w_i x_{i,t} \geq O_t - d_{t,s}^o \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (6)$$

$$\sum_{i=1}^N m_{i,s} w_i x_{i,t} \leq M_t + d_{t,s}^m \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (7)$$

$$\sum_{i=1}^N m_{i,s} w_i x_{i,t} \geq M_t - d_{t,s}^m \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (8)$$

$$d_{t,s}^o \leq y * O_t \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (9)$$

$$d_{t,s}^m \leq z * M_t \quad t = 1, 2, \dots, T, s = 1, 2, \dots, S \quad (10)$$

$$x_{i,t} = 0 \text{ or } 1 \quad t = 1, 2, \dots, T; \quad d_{t,s}^o, d_{t,s}^m \geq 0 \quad (11)$$

where,

$\Gamma_i$  is the set of overlying blocks of block  $i$

$c_i^s$  is the block economic value of block  $i$  from simulation  $s$

$N$  is the number of blocks in the deposit

$T$  is the number of production periods

$S$  is the number of simulated orebody models

$w_i$  is weight of blocks  $i$

$W_t$  is the maximum amount of material that can be excavated in period  $t$

$o_{i,s} = 1$  if block  $x_i$  is ore block in simulation  $s$  and 0 otherwise

$O_t$  is amount of ore precessed in period  $t$

$m_{i,s}$  is metal content in block  $i$  in simulation  $s$

$M_t$  is amount of metal obtained after processing in period  $t$

$d_{t,s}^o$  is deviation of ore production allowed in period  $t$  from simulation  $s$

$d_{t,s}^m$  is deviation of metal production allowed in period  $t$  from simulation  $s$

$y$  and  $z$  are maximum percentage of deviation from target ore and metal production allowed, respectively.

The formulation of the objective function (Eq. 1) tries to maximize the sum of discounted cash flow over all simulations by assigning block  $i$  to production period  $t$ .  $\frac{c_i^s}{(1+r)^t}$  is the discounted cash flow generated by mining block  $i$  to be mined in period  $t$  from simulation  $s$ , where  $r$  is discount rate.

The constraints in Eq. (2) indicate that each block is mined only once and are known as *reserve constraints*. The constraints of Eq. (3) are *precedence constraints* which ensure that block  $i$  cannot be extracted until and unless the set of overlying blocks  $\Gamma_i$  have already been extracted. The constraints in Eq. (4) ensure that the total production in period  $t$  can't be more than mining equipment capacity  $W_t$ . The constraints in Eq. (5) and Eq. (6) represent the bounds of total ore production in period  $t$ ; the constraints in Eq. (7) and Eq. (8) represent the bounds for metal production. The constraints in Eqs. (9) and (10) provide the maximum deviation allowed for the ore and metal production. Finally, Eq. (11) represents the decision variables: whether block  $i$  is going to be extracted in period  $t$  or not.

Solving the above formulation for large deposits is also computationally prohibitive due to the set of constraints in Eqs. (4)–(10). In this paper, an approach is followed to discretize the main problem to a number of small sub-problems. Instead of solving the above formulation for all time periods  $T$  simultaneously, the problem is solved sequentially: after solving for the first period, the blocks assigned (say  $n_1$ ) to the first period are eliminated from the orebody models, and the remaining blocks ( $N - n_1$ ) are considered for assigning in the next period. This process continues until all the blocks are extracted profitably from the deposit and assigned a specific production period. Figure 1 shows the schematic presentation for solving the formulation of Eq. (1) to (11) by solving sub-problems.

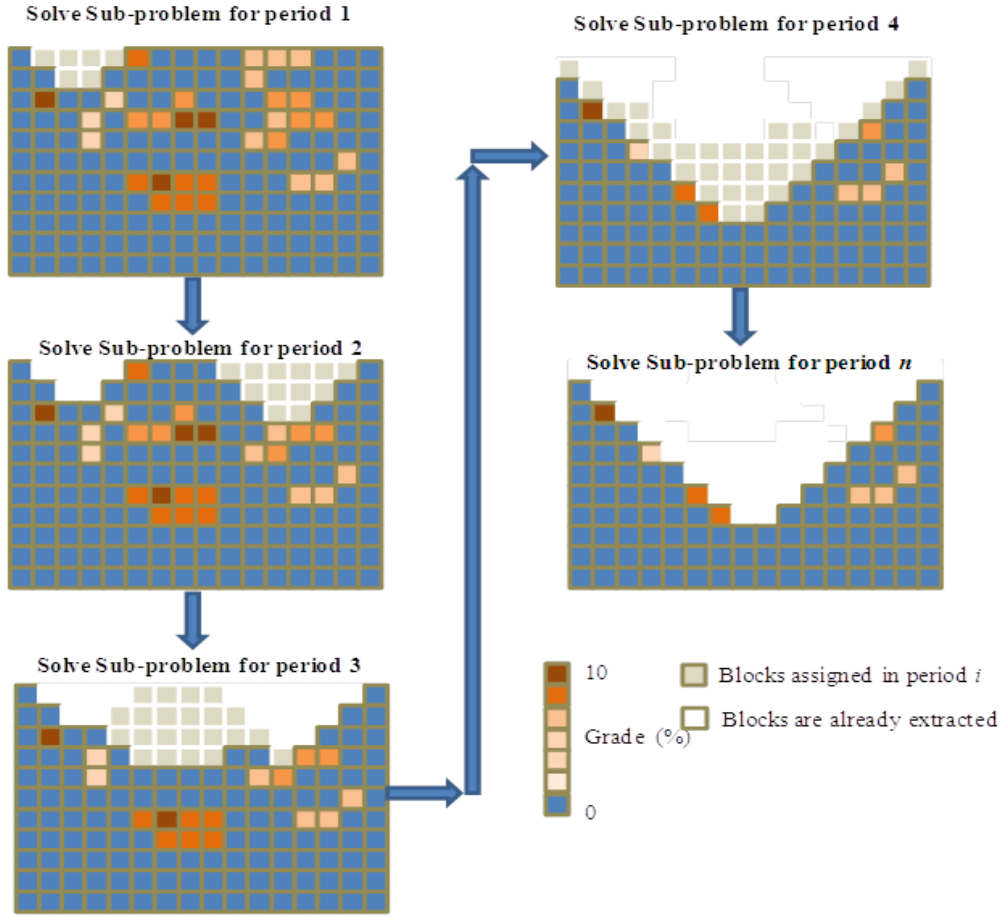


Figure 1: Schematic diagram of proposed approach

## 2.2 Solution procedure of the sub-problem

The sub-problem formulation of the open pit scheduling aims to assign the blocks which are going to be extracted by maximizing the present value of the cash flow subject to sets of constraints for the sub-problem. The actual formulation of Eq. (1) to (11) holds true for the sub-problem with time period  $T = 1$ . Although the sub-problem formulation significantly reduces the computational complexity of the original formulation, due to the presence of constraints Eq. (4) to (10), the sub-problem is also computationally complex. If the constraints of Eq. (4) to (10) are eliminated from the problem, the sub-problem can be solved efficiently using a network flow algorithm (Goldberg 1985; Goldberg and Tarjan 1988) due to the unimodularity of the remaining constraints.

In this paper, the Lagrangian relaxation algorithm is used to solve the sub-problem by bringing these constraints into the objective function by multiplying Lagrangian parameters. However, due to the presence of the linear variables  $d_s^0$  and  $d_s^m$  in the Lagrangian relaxed problem, the integer property of the decision variables will be lost, thus difficult to solve using the network flow algorithm. Therefore, the value of  $d_s^0$  and  $d_s^m$  in the relaxed problem are replaced by their upper bound values  $yO$  and  $zM$ , respectively. The Lagrangian relaxed problem can now be represented as follows:

$$\begin{aligned} \Phi(\lambda) = \max \sum_{s=1}^S \sum_{i=1}^N c_i^s x_i + \eta \left( W - \sum_{i=1}^N w_i x_i \right) + \sum_{s=1}^S \gamma_s \left( O + yO - \sum_{i=1}^N o_{i,s} w_i x_i \right) + \sum_{s=1}^S \beta_s \left( \sum_{i=1}^N o_{i,s} w_i x_i - O + yO \right) \\ + \sum_{s=1}^S \alpha_s \left( M + zM - \sum_{i=1}^N m_{i,s} w_i x_i \right) + \sum_{s=1}^S \delta_s \left( \sum_{i=1}^N m_{i,s} w_i x_i - M + zM \right) \quad (12) \end{aligned}$$

s.t.

$$x_i - x_j \leq 0 \quad i = 1, 2, \dots, N, j \in \Gamma_i \quad (13)$$

$$x_{i_s} = 0 \text{ or } 1 \quad i = 1, 2, \dots, N \quad (14)$$

where,  $\boldsymbol{\lambda} = \{\eta, \gamma_s, \beta_s, \alpha_s, \delta_s, s = 1, 2, \dots, S\}$  and  $\eta \geq 0, \gamma_s \geq 0, \beta_s \geq 0, \alpha_s \geq 0, \delta_s \geq 0, s = 1, 2, \dots, S$  are Lagrangian multipliers. Since, the problem is only being solved for the first period; the subscript  $t$  from all variables has been omitted. Now, the present formulation can be seen as maximum graph closure or minimum graph cut problem of graph  $G(N, A)$  for fixed values of Lagrangian parameters  $\boldsymbol{\lambda} = \{\eta, \gamma_s, \beta_s, \alpha_s, \delta_s, s = 1, 2, \dots, S\}$  and that can be efficiently solved by minimum cut algorithm. We have followed the same approach for solving the maximum graph closure problem for multiple simulations  $S$  as proposed in Meagher et al. (2009) and Asad and Dimitrakopoulos (2013). For a given  $\boldsymbol{\lambda} \geq 0; \boldsymbol{\lambda} = \{\eta, \lambda_s, \beta_s, \alpha_s, \delta_s, s = 1, 2, \dots, S\}$ , the optimum value of  $\Phi(\boldsymbol{\lambda})$  for the optimization problem defined by Eqs. (12)–(14) is an upper bound on the value for any feasible solution of the problem of Eqs. (1)–(11) for  $T = 1$ . Therefore, if  $Z$  is the optimal solution to the initial problem, then  $Z \leq \Phi(\boldsymbol{\lambda})$ .

The best upper bound corresponds to the multiplier  $\boldsymbol{\lambda}^*$  such that  $\boldsymbol{\lambda}^* \in \arg \min\{\Phi(\boldsymbol{\lambda}), \boldsymbol{\lambda} \geq 0\}$ . Equation (12) is piecewise linear convex function and this can be solved by the sub-gradient method. In the sub-gradient method, a random initialization of  $\boldsymbol{\lambda}_0 \geq 0; \boldsymbol{\lambda} = \{\eta, \lambda_s, \beta_s, \alpha_s, \delta_s, s = 1, 2, \dots, S\}$  is made and each  $\lambda, \lambda \in \boldsymbol{\lambda}$  is systematically updated in successive iterations  $p$  (Tachefine and Soumis 1997):

$$\lambda_{p+1} = \max(0, \lambda_p - k_p l_p) \text{ for all } \boldsymbol{\lambda} \in (\eta, \lambda_d, \beta_s, \alpha_s, \delta_s, s = 1, 2, \dots, S) \quad (15)$$

where, 
$$k_p = \frac{\rho[\Phi(\boldsymbol{\lambda}) - Z]}{\|\boldsymbol{l}_p\|^2} \text{ with } 0 < \rho < 2, \quad (16)$$

$$l_p = \begin{cases} \left( W - \sum_{i=1}^N w_i x_i(\boldsymbol{\lambda}_p) \right) & \text{for } \lambda = \mu \\ \left( O + yO - \sum_{i=1}^N o_{i,s} w_i x_i(\boldsymbol{\lambda}_p) \right) & \text{for } \lambda = \{\gamma_1, \dots, \gamma_S\} \\ \left( \sum_{i=1}^N o_{i,s} w_i x_i(\boldsymbol{\lambda}_p) - O + yO \right) & \text{for } \lambda = \{\beta_1, \dots, \beta_S\} \\ \left( M + zM - \sum_{i=1}^N m_{i,s} w_i x_i(\boldsymbol{\lambda}_p) \right) & \text{for } \lambda = \{\alpha_1, \dots, \alpha_S\} \\ \left( \sum_{i=1}^N m_{i,s} w_i x_i(\boldsymbol{\lambda}_p) - M + zM \right) & \text{for } \lambda = \{\delta_1, \dots, \delta_S\} \end{cases} \quad (17)$$

$x_j(\boldsymbol{\lambda}_p)$  is the  $j$ th component of the solution vector in the problem defined by Eqs. (12)–(14) at the  $p$ th iteration,  $\|\cdot\|$  is an operator to calculate Euclidian norm, and  $Z$  the optimal solution value of the objective function corresponding to a feasible solution. Since the optimal solution  $Z$  is not known in advance, it is replaced by the best-known value of the objective function corresponding to a feasible solution. After solving the formulation of Eqs. (12)–(14) using the sub-gradient algorithm,  $\Phi(\boldsymbol{\lambda}^*)$  serves as an upper-bound solution for the initial problems and will be denoted by  $Z_{UB}$ .

The Lagrangean relaxation generates a good upper-bound solution for the problem; however the solution is not guaranteed to be feasible. The solution  $Z_{UB}$  obtained from Lagrangean relaxation may be infeasible where some or all constraints of Eqs. (4)–(8) are violated. If the upper-bound solution is not feasible, it is necessary to eliminate some blocks from the solution set to make it feasible. To respect constraints, some blocks that belong to  $Z_{UB}$  from the solution vector  $\{\boldsymbol{x}_{UB}; x_j(\boldsymbol{\lambda}^*) \in \boldsymbol{x}_{UB}, j = 1, 2, \dots, N_1; N_1 \leq N\}$  need to be eliminated. To select the set of blocks which can be considered for elimination, a solution  $Z_{LB}$  is generated with the solution vector  $\{\boldsymbol{x}_{LB}; x_k \in \boldsymbol{x}_{LB}, k = 1, 2, \dots, N_2, N_2 \leq N_1 \leq N\}$  such that the amounts of mining, processing, and metal quantities within the solution are less than the target mining, processing, and metal quantities.

### 2.3 Generating the lower bound solution

To generate the lower bound solution  $Z_{LB}$ , the same procedure as presented in Meagher et al. (2008) is followed. In this approach, instead of using the whole objective function of Eq. (12), the first term is used, i.e.  $\sum_{s=1}^S \sum_{i=1}^N c_i^s x_i$ , and parameterized using only the single parameter  $\lambda$ ,  $1 \geq \lambda \geq 0$  to  $\Psi(\lambda) = \max \sum_{s=1}^S \sum_{i=1}^N d_i^s x_i$ , where  $d_i^s = \lambda * c_i^s$  if  $c_i^s > 0$  else  $d_i^s = c_i^s$ . The multiplier  $\lambda$  is a parameterization factor bounded within (0 1]. When the value of  $\lambda$  is 1, the  $\Psi(\lambda)$  will generate the ultimate pit. The algorithm starts with a small value of  $\lambda$  and goes through the loop to update the  $\lambda$  value by  $\lambda_{p+1} = \lambda_p + \nabla \lambda_p$  as long as the constraints of Eqs. (4), (5), and (7) are valid for period  $T = 1$ .

For the specific  $\lambda$  value, this problem became a simple graph cut problem with multiple simulations  $S$ , and can be solved by minimum cut algorithm (Meagher et al. 2009; Asad and Dimitrakopoulos 2013). After solving this problem, a solution will be obtained where none of the upper bound mining, processing, and metal quantity constraints will be violated. This solution along with the solution generated in Section 2.2 will be used to generate a feasible solution.

### 2.4 Imposing constraints on the upper-bound solution

To add new solution elements with  $\mathbf{x}_{LB}$  or eliminate solution elements from  $\mathbf{x}_{UB}$ , a new vector set  $\{\mathbf{x}_{new}; x_v \in \mathbf{x}_{new}, v = 1, 2, \dots, N_{new}, N_{new} = N_1 - N_2\}$  is identified which belongs to the solution vector  $\mathbf{x}_{UB}$  but does not belong to  $\mathbf{x}_{LB}$ , i.e.  $\mathbf{x}_{new} \subseteq \mathbf{x}_{UB}; \mathbf{x}_{new} \cup \mathbf{x}_{LB} = \mathbf{x}_{UB}; \mathbf{x}_{new} \cap \mathbf{x}_{LB} = \emptyset$ .

A mixed integer programming problem is formulated using binary variables  $x_v, v \in N_{new}$ , where  $x_v = 1$  if the  $v$ th element of new vector  $\mathbf{x}_{new}$  is added to the solution vector  $\mathbf{x}_{LB}$ , or  $x_{i,t} = 0$  otherwise. The formulation can be written as

$$\max \frac{1}{S} \sum_{s=1}^S \sum_{v=1}^{N_{new}} c_v^s x_v \quad (18)$$

$$x_v - x_u \leq 0 \quad v = 1, 2, \dots, N_{new}, u \in \Gamma_v \quad (19)$$

$$\sum_{v=1}^{N_{new}} w_i x_v \leq W - W_{LB} \quad (20)$$

$$\sum_{v=1}^{N_{new}} o_{v,s} w_v x_v \leq O + d_s^o - O_{s,LB} \quad s = 1, 2, \dots, S \quad (21)$$

$$\sum_{v=1}^{N_{new}} o_{v,s} w_v x_v \geq O - d_s^o - O_{s,LB} \quad s = 1, 2, \dots, S \quad (22)$$

$$\sum_{v=1}^{N_{new}} m_{v,s} w_v x_v \leq M + d_s^m - M_{s,LB} \quad s = 1, 2, \dots, S \quad (23)$$

$$\sum_{v=1}^{N_{new}} m_{v,s} w_v x_v \geq M - d_s^m - M_{s,LB} \quad s = 1, 2, \dots, S \quad (24)$$

$$d_s^o \leq y * O \quad s = 1, 2, \dots, S \quad (25)$$

$$d_s^m \leq z * M \quad s = 1, 2, \dots, S \quad (26)$$

$$x_v = 0 \text{ or } 1 \quad d_s^m, d_s^o \geq 0 \quad (27)$$

where,  $W_{LB} = \sum_{k=1}^{N_2} w_k$ ,  $O_{s,LB} = \sum_{k=1}^{N_2} o_{k,s} w_k$ ,  $M_{s,LB} = \sum_{k=1}^{N_2} m_{k,s} w_k$ , and  $x_k = 1; x_k \in \mathbf{x}_{UB}$ , and  $k = 1, 2, \dots, N_2, N_2 \leq N_1 \leq N$ . Since,  $N_{new} \ll N$ , the computational time of this formulation is substantially less. The branch-and-cut algorithm is used to solve this problem. The solution vector of the optimization formulation of Eqs. (18)–(27) is then added with the lower bound solution vector  $\mathbf{x}_{LB}$  to get the solution for the sub-problem of the first period. The final solution vector of the first period is then eliminated from the

original problem set  $N$ . The resultant set is used for solving the sub-problem for the next time period. The process is repeated until  $t = T$ .

### 3 Validation of the method

To validate the proposed method, three real life data sets from mining companies are considered. The production scheduling problems for all three data sets are solved using the proposed approach as well as the commercial solver CPLEX. The original formulation defined by Eqs. (1)–(11) is solved using the commercial solver CPLEX. As some of these test problems need large computational times to solve optimally using the CPLEX solver, all three problems are solved after linear relaxation in order to obtain an upper bound solution. The results of CPLEX are compared with the results obtained from the proposed algorithm. For all three test problems, the mining blocks are of  $20 \text{ m} \times 20 \text{ m} \times 10 \text{ m}$  size. The numerical tests are completed on an Intel(R) dual core machine 2.33 GHz with 2.0 GB of RAM. Table 1 presents the characteristic of all three test problems, the optimality gap, and the time required to solve the problem using the proposed algorithm and CPLEX solver.

Table 1: Test problems for validating the proposed approach

Problem number	Number of blocks ( $N$ )	Number of periods ( $T$ )	Optimality gap (%)	Computational time in minutes	
				Our proposed approach	CPLEX solver
Problem 1	2914	2	1.85	8.1	18.2
Problem 2	5132	3	1.88	13.8	93.2
Problem 3	16532	7	2.76	37.2	1377

The gap between the proposed approach and the solution obtained from CPLEX is calculated from the following equation:

$$gap = \frac{(O_1 - O_2)}{O_1} * 100 \quad (28)$$

where  $O_1$  is value of the objective function of Eq. (1) using CPLEX and  $O_2$  is the value of the objective of Eq. (1) using the proposed approach. The value of the gap is presented in percentage. Results reveal that the optimality gap is very reasonable, within 3%. However, the computational times are substantially less in the proposed approach when compared to the CPLEX solver, particularly when the problem size increases. Table 1 shows that computational time in the proposed method increases linearly; while, it increases exponentially when the number of integer variables and number of production periods increase when using CPLEX. These test problems demonstrate that the proposed approach reduces the computational time significantly while maintaining a small gap. The proposed approach appears to be a good choice for solving the real-world mine scheduling problems where the numbers of integer variables are significantly large.

## 4 Case study

### 4.1 The data set

A case study of stochastic production scheduling using the proposed approach is presented for a copper deposit. The drillholes are on a pseudo-regular grid of  $50 \text{ m} \times 50 \text{ m}$ , covering an approximately rectangular area of  $1600 \times 900 \text{ m}^2$ . A total number of 185 drillholes are available from the deposit. The borehole data is composited over 10-m down-hole lengths. A wireframe of the mineralized domain is created using the geological information of the deposit. Geological uncertainty modelling is obtained by performing geo-statistical modelling within the wireframe. The direct block simulation technique was used to generate a simulation orebody model of the deposit (Godoy 2003; Boucher and Dimitrakopoulos, 2012). A total number of 20 simulated models are generated to incorporate geological uncertainty.

The economic block values ( $c_i^s$ ) of the individual blocks are calculated using the following equation:

$$c_i^s = \begin{cases} NR_{ii}^s - MC - PC, & \text{if } NR_{ii}^s > PC \\ -MC, & \text{otherwise} \end{cases} \quad (29)$$

where,  $NR_{ii}^s = w_i * g_i^s * R * (MT - SC)$ ,  $MC$  is the mining cost,  $PC$  is the processing cost,  $w_i$  is weight of block  $i$ ,  $g_i^s$  is grade of block  $i$  from simulation  $s$ ,  $R$  is recovery,  $MT$  is metal price,  $SC$  is selling cost. Table 2 presents the economic parameters used for calculating the value of  $c_i^s$ .

Table 2: Economic parameters

Copper price (US\$/lb)	1.9
Selling cost (US\$/lb)	0.4
Mining cost (\$/tonne)	1.0
Processing cost (\$/tonne)	9.0
Processing recovery	0.9

## 4.2 Results

In this case study, a total number of 16,532 blocks are scheduled over 7 years. The target amounts of ore and metal are to be extracted in each production periods are 7.1 Mt and 4.8 kt, respectively. The maximum amount of deviations allowed from target ore and metal production is 13%. The maximum amount of total production of ore and waste is 25 Mt. The economic values of all 16,532 blocks are calculated using Eq. (29), the financial data shown in Table 2, and the simulated grade values of all 20 simulations. For solving the Lagrangian relaxed problem, the push re-label minimum cut algorithm is applied for its computational efficiency (Goldberg and Tarjan, 1988). A directed graph is constructed where ore blocks are connected with the source node, and waste blocks are connected to the sink node, as described in Asad and Dimitrakopoulos (2013). To maintain slope constraints, an infinite capacity arc is formed for underlying blocks to overlying blocks. The infinity capacity arcs are directed from an underlying block to nine overlying neighbour blocks. To get the upper bound solution of sub-problems, the sub-gradient algorithm is performed with a maximum of 50 iterations, as discussed in Section 2.2. The lower-bound solution is generated by applying the algorithm presented in Section 2.3. Both the upper- and lower-bound solutions are then used to generate the solution of the sub-problem by applying the algorithm proposed in Section 2.4 and solving it by the branch and cut algorithm. A total number of 7 sub-problems are solved for seven production periods. All sub-problems are then combined to produce the production schedule. Figure 2 shows two sections of the mine's schedule using the method proposed herein.

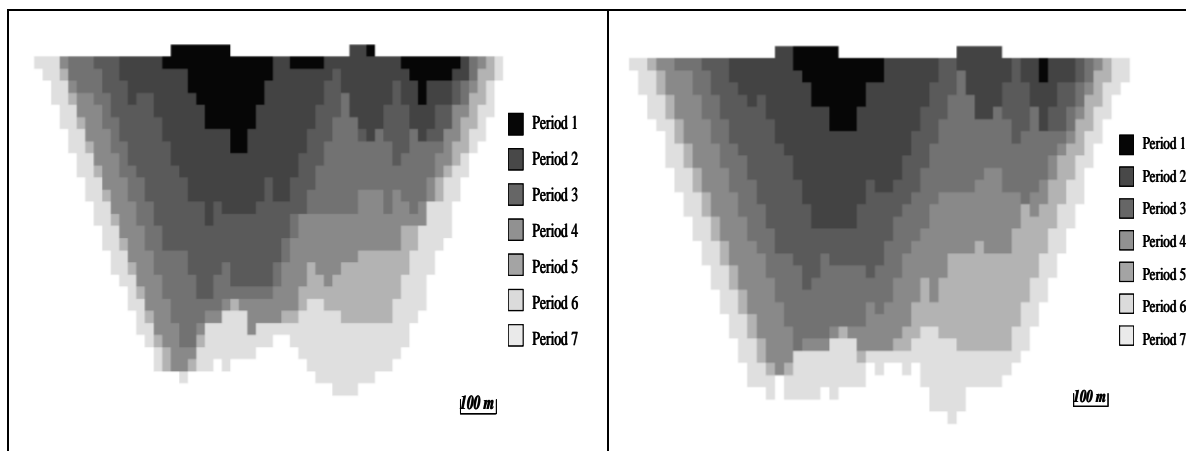


Figure 2: Two different E-W sections of the production schedule

After generating a schedule, the risk profiles of ore and metal production were calculated using the 20 simulated orebody models, and presented in Figure 3. It can be observed from Figure 3(a) that the expected ore production is more or less constant over production periods. It can also be seen from Figure 3 that the minimum and maximum values of ore in all production periods are falling within the allowable deviation. The same observation was also true for metal production as can be seen in Figure 3(b). The cumulative metal quantity and its risk profiles are also presented in Figure 4. The net present value (NPV) was calculated for economic analysis of the deposit using the proposed uncertainty-based scheduling algorithm. The minimum, maximum, and expected NPV were calculated over 20 simulated models and presented in Figure 5. The expected NPV generated from the mine was 285 M\$, with a minimum and maximum value of 249 M\$ and 322 M\$, respectively. It was also demonstrated from this figure that the variation of NPV generated from this deposit is significantly less in the initial production period then in the later periods. The computation time required to complete the schedule of the entire deposit needs a total of 37.2 minutes.

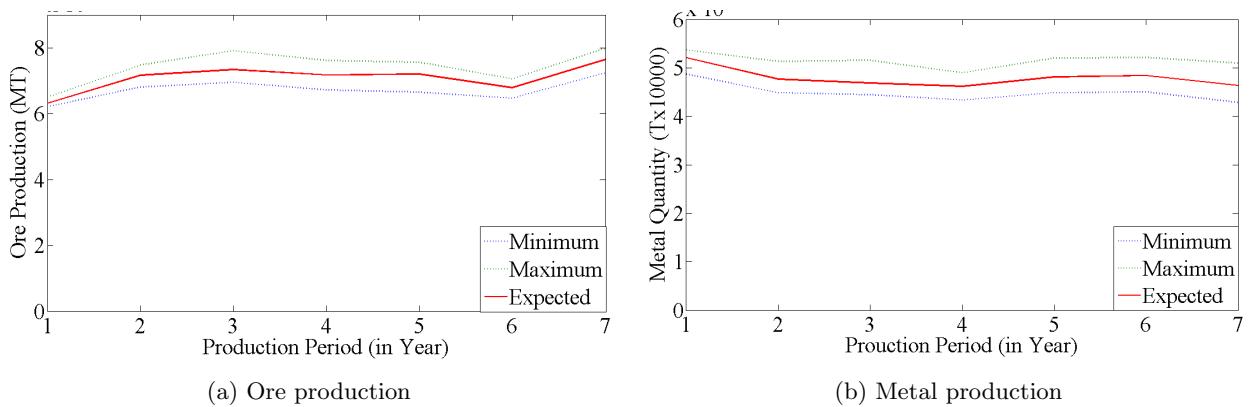


Figure 3: Ore and metal production for the mine production schedule, generated using the proposed algorithm

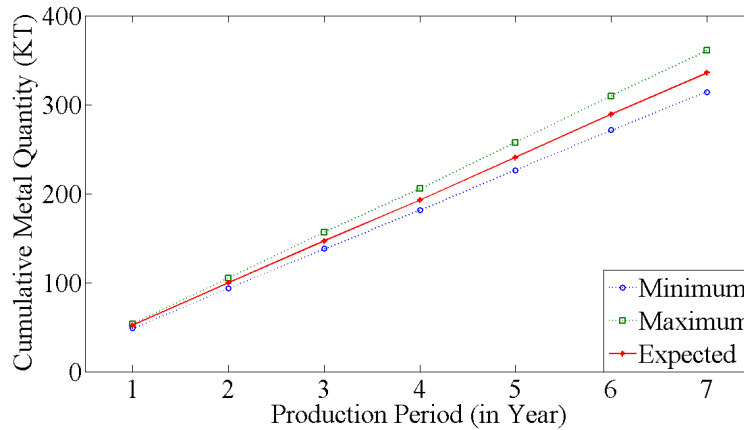


Figure 4: Cumulative metal productions from the 7-year schedule of the deposit

### 4.3 Comparative studies

#### 4.3.1 Comparison with the deterministic equivalent model

First, a comparative study was performed between the stochastic and deterministic model using the same approach proposed in this paper. The main aim of this comparison is to see how the results of stochastic formulation of production scheduling differ from the deterministic formulation when the same solution approach is applied. The deterministic model was generated using a single orebody model that presents the ore grade and metal content of the deposit, and using the same algorithm proposed in this paper with number

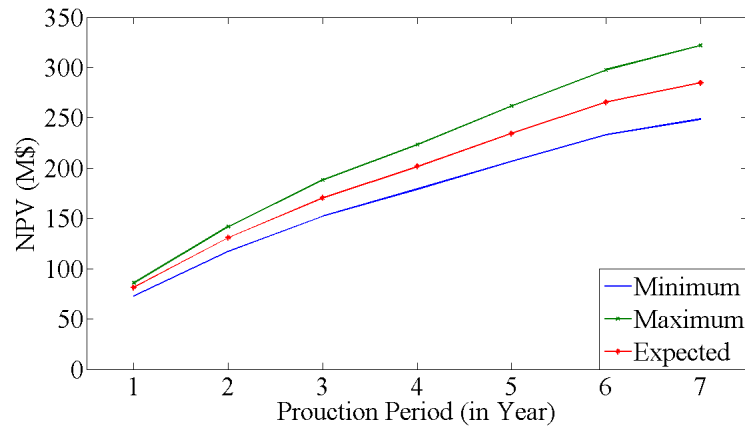


Figure 5: Cumulative NPV generated using the proposed approach

of simulation  $S$  equal to 1. The estimated ore body model was generated by averaging 20 simulated orebody models. The economic parameters and slope angle were kept the same in all cases. After generating the production scheduling of the estimated orebody model, the generated schedule is used to calculate the risk profile of the schedule considering all 20 simulated orebody models.

Ore and metal production from each production period were calculated from each of the simulated orebody models. Maximum deviation from the target ore and metal production were calculated for both stochastic and deterministic schedules. Figure 6 shows the maximum deviations from targeted ore production. In Figure 6, the constant line represents the maximum allowable deviation used in this paper, i.e. 13% of target ore production. The results show that deviation is within the allowable limit in all production periods for the stochastic schedule. By comparison, in the deterministic schedule, deviation is more than the allowable deviation at least in three production periods. Although, the stochastic schedule is able to keep the deviation within the limit, the deviation is not smoothed over the production period; the reason being that in the proposed stochastic approach the scheduling is done sequentially, therefore, the decision to schedule of a block which has already been made can't be changed in a later period.

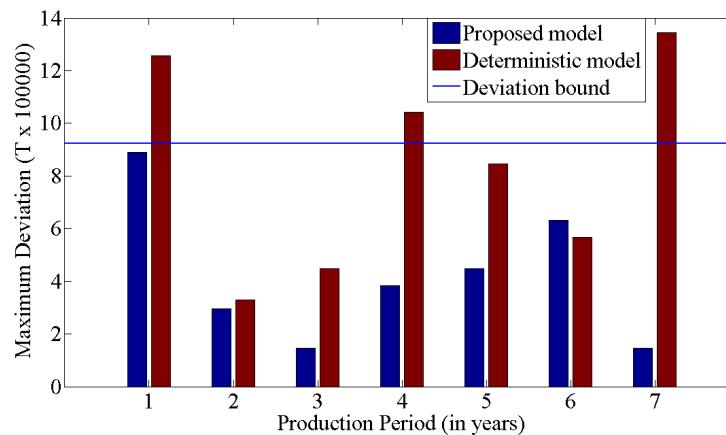


Figure 6: Maximum deviation of ore production over all production periods

Figure 7 represents the maximum deviations from metal production targets for both the stochastic and deterministic schedules with respect to the 20 simulated orebody models. Similar to the previous figure, the deviation of our approach is within the permissible limit from the target metal production; however with the deterministic schedule, it was observed that in three production periods the deviations have crossed the permissible limit. It was also observed from this figure that the deviation was lower at the initial period and increases in the latter periods.

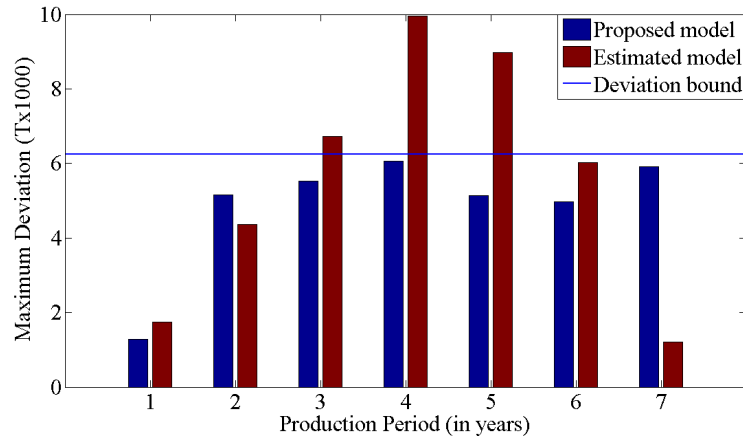


Figure 7: Maximum deviation of metal production over all production periods

When comparing the pit size of the stochastic model with deterministic model of the proposed approach, it was observed that the size of the pit is 7% bigger in the stochastic model than the deterministic model. The amount of metal is also higher in the stochastic model as compared to the deterministic model.

The NPVs generated using the stochastic schedule and deterministic schedule are presented in Figure 8. The figure indicates that the stochastic schedule generates a greater expected NPV of 285 M\$, as compared to its deterministic counterpart (254 M\$). The proposed stochastic scheduling approach produces 11% more net present value as compared to the deterministic approach. This difference is due to the incorporation of uncertainty in the proposed stochastic schedule model.

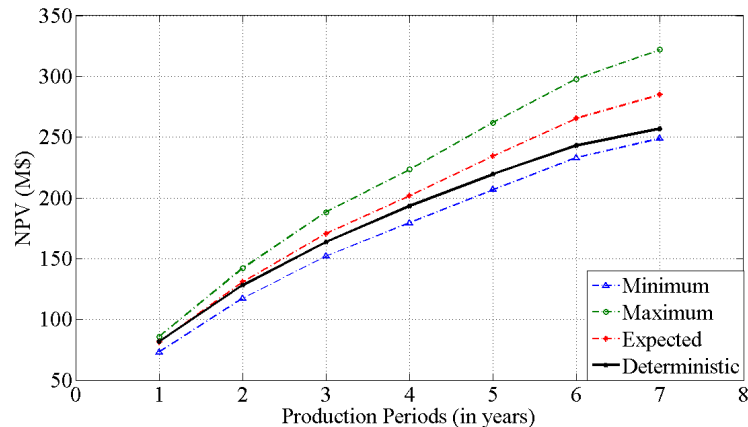


Figure 8: Comparison of NPV generated using different algorithms

#### 4.3.2 Comparison with the conventional approach

To compare the proposed stochastic production scheduling present herein, the schedule was generated conventionally using Whittle software (Whittle 1998, 1999). The results of a conventionally generated schedule were taken from Leite and Dimitrakopoulos (2007) for comparison purposes. For valid comparison, the schedule of the proposed approach was generated for the same data set and constraint bounds used in Leite and Dimitrakopoulos (2007). The technical and financial parameters used here are also the same as Leite and Dimitrakopoulos (2007). In Leite and Dimitrakopoulos (2007), ore (economic block value positive) and waste (economic block value negative) target constraints are considered, thus here the same constraints are imposed in the proposed method. The ore and waste target for this comparative study are 7.5 million tonnes (MT) and 20.5 million tonnes (MT) per year, respectively. The maximum allowable deviations of ore and waste from the target productions are 0.5 MT and 2 MT, respectively.

Figure 9 shows the schedule generated by the proposed method and the conventional approach along a specific east-west section. It is observed from the schedule that the pit size of the conventional method is relatively smaller than the proposed approach. The number of blocks within the pit of the proposed approach (16058) is 11% more than the conventional approach (14480).

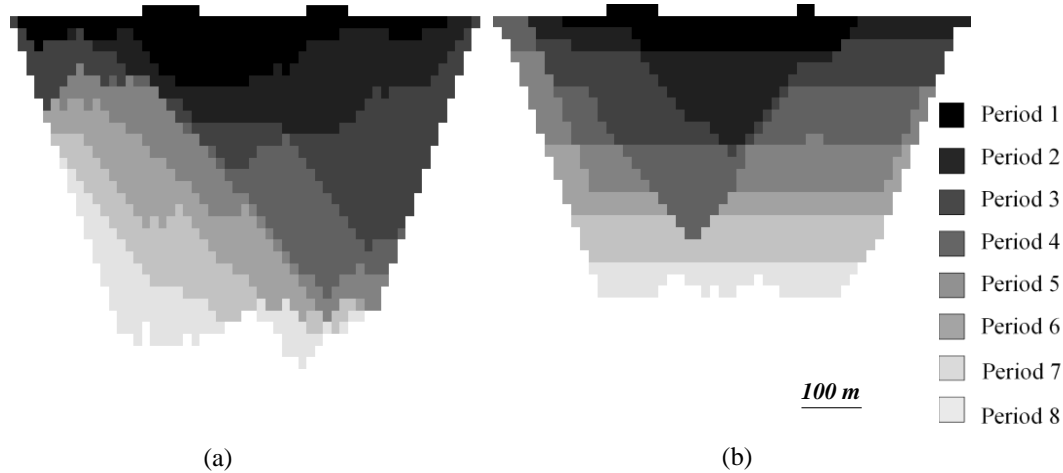


Figure 9: The production schedule of (a) the proposed method and (b) the conventional method using the same constraints and data sets as in Leite and Dimitrakopoulos (2007)

The risk analysis for ore and waste quantity was carried out for the schedule generated both from the proposed algorithm and conventional schedule using the simulated orebody models. Figure 10 presents the risk profiles of ore and waste tonnages for both methods. It is observed from the figure that the proposed algorithm respects the ore target constraints over the life of the mine, however, the conventional method deviates largely over the production periods. If it is assumed that the simulated orebody models are representing the uncertainty and heterogeneity of the actual deposit then it is observed from the results that the conventional schedule has a high risk of deviating from ore production targets over the different production periods.

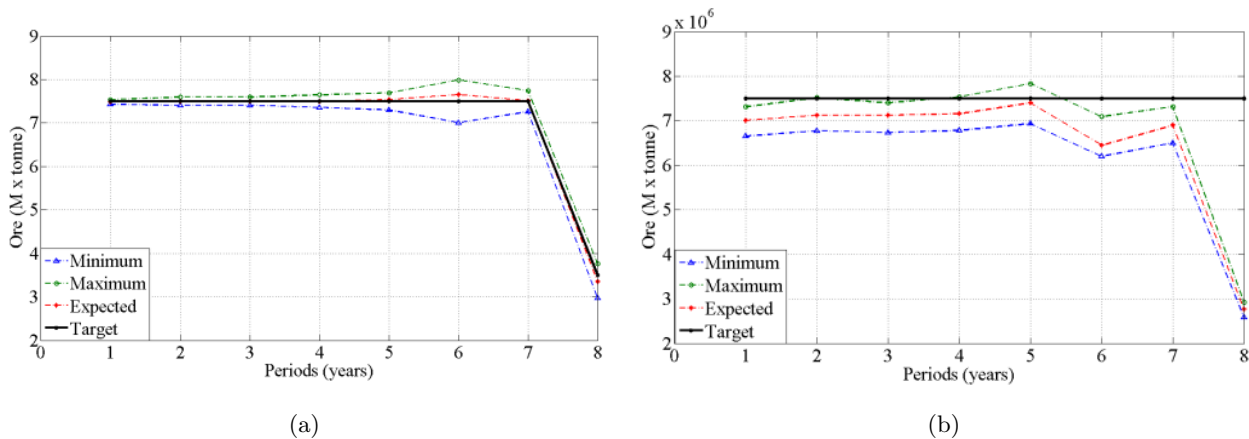


Figure 10: Ore production risk profile of the proposed method (a) and the conventional method (b) using the same constraints and data sets as in Leite and Dimitrakopoulos (2007)

Figure 11 represents the waste risk profile for the proposed approach and the conventional approach. It is observed from the figure that both algorithms respect the waste target over the life of the mine. It is noted that after 5th year, the waste target was relaxed and no restriction of the deviation of waste target was imposed.

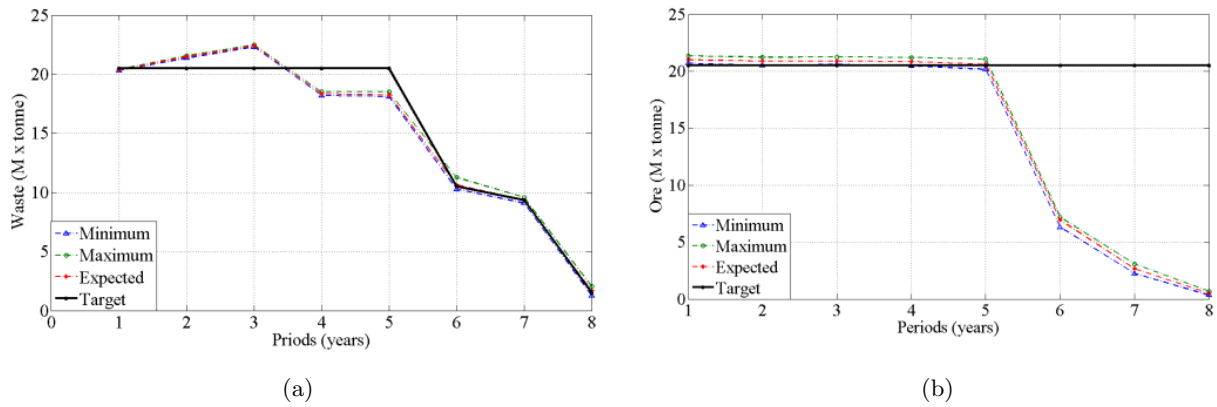


Figure 11: Waste production risk profile of the proposed method (a) and the conventional method (b) using the same constraints and data sets as in Leite and Dimitrakopoulos (2007)

It can be concluded from the risk profiles of ore and waste that the proposed algorithm respects both constraints thus generates a feasible solution; whereas the conventional method fails to respect the ore production target within deviation.

When comparing the proposed method with the conventional one in terms of financial benefit, the cumulative net present value was calculated and is presented in Figure 12. It is observed from the figure that the proposed algorithm generates approximately 26% more NPV as compared to the conventional method.

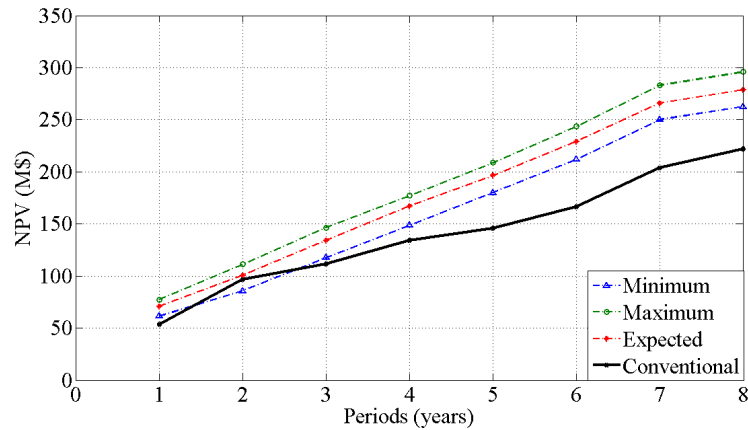


Figure 12: Cumulative net present value comparison between the proposed method and the conventional method

It is noted that the conventional approach overestimates ore tonnages (Leite and Dimitrakopoulos 2007) therefore the life of the mine is still eight years even though the number of blocks within the pit are significantly less than the proposed approach. Although both algorithms schedule the deposit in eight years, the proposed algorithm generates more revenue due to different sequences of extraction by assigning high probable ore blocks in the initial production periods. It is noteworthy to mention that the other stochastic approaches (Godoy 2004; Ramazan and Dimitrakopoulos 2013) also demonstrated the same type of observation when compared with the conventional approach. Moreover, the proposed algorithm applied Lagrangian relaxation which provides a good upper bound solution and the branch-and-cut algorithm which ensures the minimum deviation from the target.

## 5 Conclusions

In this paper, an efficient approach was proposed using the Lagrangian relaxation and branch-and-cut algorithm to solve the mine production scheduling. The algorithm proceeds sequentially and at each sequence the sub-problems are solved using Lagrangian relaxation. The feasibility of the Lagrangian relaxed solution was respected by the branch-and-cut algorithm. The computational time of the proposed method is significantly less when compared with the optimal scheduling algorithm where all decisions were made simultaneously.

The proposed method was first verified with three test data sets and compared with the linear relaxation of the original integer programming problem. The comparative results demonstrate that the proposed approach is not only computationally faster but also that the optimality gap is reasonably low.

Lastly, the method was applied at a copper mine. Results show that the proposed method can solve the larger size mine scheduling efficiently. The proposed method shows that the schedule generated is obeying the resource constraints within the range of deviation set. Although the algorithm is not optimal over the entire deposit since the decisions were made sequentially, it can be very useful for generating a solution within a reasonable amount of time.

Moreover, the proposed model uses a multiple simulated orebody for optimizing production schedules in open pit mines, therefore, it is accounting for uncertainty in the model. This method incorporates the uncertainty in the material supply from the deposit which isn't possible with traditional scheduling methods. When comparing the proposed method between multiple simulated orebody models capturing the uncertainty in the material supply and a single estimated orebody model with fixed material supply; there is at least 11% more NPV generated when uncertainty in the material supply is considered.

The proposed method was also compared with the conventional practical approach. The results reveal that the proposed algorithm respects all constraints over the life of the mine whereas the conventional method fails to respect the constraints. The comparative results also demonstrate that the proposed algorithm generates 26% more NPV than the conventional method and with 11% bigger pit size and 12% more metal production.

Although the example presented herein is based on uncertain block values due to metal uncertainty, the same method can be implemented to incorporate other uncertainties in demand (commodity price, exchange rate) as well as mining costs, processing costs, metal recovery, or any other input used to calculate the economic value of a mining block. Moreover, period wise geological risk discount can also be incorporated. The main advantage of the proposed algorithm is that it is computationally very fast, thus it is feasible to integrate multiple uncertainties in the optimization process on a routine basis.

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