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Abstract. This paper provides an axiomatic foundation of the measurement of diversification in a one-period portfolio theory under the assumption that the investor has complete information about the joint distribution of asset returns. Four categories of portfolio diversification measures can be distinguished: the law of large numbers diversification measures, the correlation diversification measures, the market portfolio diversification measures and the risk contribution diversification measures. We offer the first step towards a rigorous theory of correlation diversification measures. We propose a set of nine desirable axioms for this class of diversification measures, and name the measures satisfying these axioms coherent diversification measures that we distinguish from the notion of coherent risk measures. We provide the decision-theoretic foundations of our axioms by studying their compatibility with investors' preference for diversification in two important decision theories under risk: the expected utility theory and Yaari's dual theory. We explore whether useful methods of measuring portfolio diversification satisfy our axioms. We also investigate whether or not our axioms have forms of representation.

Keywords: Portfolio theory, portfolio diversification, preference for diversification, correlation diversification, expected utility theory, dual theory.

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22 1 Introduction

23 Diversification is one of the major components of decision making in conditions of risk
24 and uncertainty, especially in portfolio theory (see [Markowitz, 1952](#); [Ross, 1976](#); [Sharpe,](#)
25 [1964](#)). It consists of investing in various assets. Its objective is to reduce risk, particularly
26 the likelihood and severity of portfolio loss, through *multilateral insurance* in which each
27 asset is insured by the other assets. Despite criticism after the 2007-2009 financial crisis
28 (see [Holton, 2009](#)), it is still an important risk management tool for many institutions
29 and regulators (see [Basel Committee on Banking Supervision, 2010, 2013](#); [Committee of](#)
30 [European Banking Supervisors, 2010](#); [Committee of European Insurance and Occupational](#)
31 [Pensions Supervisors, 2010a,b](#); [Committee on Risk Management and Capital Requirements,](#)
32 [2016](#); [European Insurance and Occupational Pensions Authority, 2014](#); [Ilmanen and Kizer,](#)
33 [2012](#); [Laas and Siegel, 2017](#); [Markowitz et al., 2009](#); [Sandstrom, 2011](#)). Its measurement
34 and management, outside the standard risk measurement frameworks (e.g. the theory of
35 monetary risk measures of [Artzner et al. \(1999\)](#) and [Föllmer and Weber \(2015\)](#)), remains
36 of fundamental importance in finance and insurance economics.

37 1.1 Existing Diversification Measures

38 Following [Markowitz \(1952\)](#)'s pioneering work on the mathematical formulation of diversi-
39 fication in portfolio theory, several measures of portfolio diversification have been proposed.
40 According to [Koumou \(2018\)](#), there are four categories of diversification measures: the
41 *law of large numbers* diversification measures, the *correlation* diversification measures, the
42 *market portfolio* diversification measures and the *risk contribution* diversification measures.

43 1.1.1 Law of Large Numbers Diversification Measures

44 This category includes measures designed to capture the effect of law of large numbers
45 diversification. This diversification strategy involves investing a small fraction of wealth in
46 each of a large number of assets. A specific example is *naive diversification* (or equal weight
47 portfolio), in which the same amount of wealth is invested in each available asset. Examples
48 of law of large numbers (and in particular, naive diversification) measures are the effective
49 number of constituents (see [Carli et al., 2014](#); [Deguest et al., 2013](#)) and [Bouchaud et al.'s](#)
50 [\(1997\)](#) class of measures which includes the Shannon and Gini-Simpson indexes ([Zhou et al.,](#)
51 [2013](#)). Other examples of naive diversification measures can be found in [Yu et al. \(2014\)](#)
52 and [Lhabitant \(2017\)](#).

53 1.1.2 Correlation Diversification Measures

54 This category includes measures designed to capture the effect of *correlation*¹ diversifica-
 55 tion. This diversification strategy is at the core of most of the decision theories including
 56 the expected utility theory and Yaari's (1987) dual theories, and is similar to the notion
 57 of *correlation aversion* (see Epstein and Tanny, 1980; Richard, 1975). Consequently, it can
 58 be viewed as the *rational* diversification principle for risk averse investors. It exploits in-
 59 terdependence between asset returns to reduce portfolio risk. The idea is that fewer assets
 60 are positively correlated, so the likelihood they do poorly at the same time in the same
 61 proportion is low and the protection offered by multilateral insurance, which is diversifi-
 62 cation, is better. Therefore, when there is correlation, it becomes dangerous to use law
 63 of large numbers diversification. *Correlation* diversification is recommended in Basel II
 64 (Committee of European Banking Supervisors, 2010) and Basel III (Basel Committee on
 65 Banking Supervision, 2010, 2013), and in Solvency II for calculating the solvency capital
 66 requirement (Committee of European Insurance and Occupational Pensions Supervisors,
 67 2010a,b). Examples of *correlation* diversification measures are Embrechts et al.'s (1999)
 68 class of measures, Tasche's (2006) class of measures, the diversification ratio of Chouiefaty
 69 and Coignard (2008), the diversification return of Booth and Fama (1992), the excess growth
 70 rate of Fernholz (2010), the return gap of Statman and Scheid (2005), the Goetzmann and
 71 Kumar's (2008) measure of diversification and the diversification delta of Vermorken et al.
 72 (2012).

73 1.1.3 Market Portfolio Diversification Measures

74 This category includes measures designed to capture the effect of market portfolio diver-
 75 sification. This diversification strategy was introduced by Sharpe (1964) and consists of
 76 holding a market portfolio or a market capitalization-weighted portfolio. It focuses on id-
 77 iosyncratic risk reduction, so it fails during systematic crashes like the 2007-2009 financial
 78 crisis. Examples of market portfolio diversification measures are *portfolio size* (see Evans
 79 and Archer, 1968), Sharpe (1972)'s measure and Barnea and Logue (1973)'s measures.

80 1.1.4 Risk Contribution Diversification Measures

81 The last category includes measures designed to capture the effect of *risk contribution*
 82 diversification. This diversification strategy, also known as risk parity, became popular after
 83 the 2007-2009 financial crisis (Maillard et al., 2010; Qian, 2006). It consists of allocating
 84 portfolio risk equally among its components. Examples of risk contribution diversification
 85 measures are the *effective number of correlated bets* (see Carli et al., 2014; Roncalli, 2014)
 86 and the *effective number of uncorrelated bets* of Meucci (2009) and Meucci et al. (2014).

¹The term *correlation* here refers here to any dependence measure including dissimilarity or similarity measures.

87 **1.2 Towards a Rigorous Theory of Correlation Diversification Measures**

88 We focus on the rich choice set of *correlation* diversification measures. A completely rigorous
 89 formulation of *correlation* diversification measurement is still lacking in the literature. None
 90 of the existing measures truly has theoretical foundations (axiomatic or decision-theoretic
 91 foundations). Although *correlation* diversification is at the core of most of the decision
 92 theories and is recommended by international regulatory agencies, no attention has been
 93 given to the conceptual problems involved in its measurement. Much of the academic liter-
 94 ature on the theoretical foundations of risk management has been focused on the study of
 95 risk measurement (see Artzner et al., 1999; Föllmer and Schied, 2002; Frittelli and Gianin,
 96 2002, 2005; Rockafellar et al., 2006). Unfortunately, even if *correlation* diversification is
 97 taken into account in the standard risk measurement frameworks through the properties of
 98 *convexity*, *sub-additivity*, *comonotonic additivity*, *homogeneity* and *non-additivity for inde-*
 99 *pendence*, these risk measures do not quantify the *correlation* diversification effect properly.
 100 The reason is that risk reduction is not equivalent to diversification. Diversification implies
 101 risk reduction, but the reverse is not true, because risk can also be reduced by concentra-
 102 tion. For example, in Artzner et al.'s (1999) and Föllmer and Weber's (2015) monetary risk
 103 measure theories, the possibility of reducing risks by concentration is taken into account
 104 through the property of *monotonicity*, and has the same importance as diversification. Con-
 105 sequently, standard risk measurement frameworks fail to adequately quantify and manage
 106 *correlation* diversification, except in the extreme case where all assets have the same risk.

107 The lack of rigorous theories of *correlation* diversification measures when the decision maker
 108 is risk averse does not favor (i) a rapid improvement in understanding the concept of
 109 diversification, (ii) a development of *coherent* measures, and (iii) a comparison of existing
 110 measures. The 2007-2008 crisis revealed that the concept of *correlation* diversification is
 111 misunderstood (Ilmanen and Kizer, 2012; Miccolis and Goodman, 2012; Statman, 2013).
 112 An example of the development of an inadequate *correlation* diversification measure is the
 113 *diversification delta* introduced in Vermorken et al. (2012) and revised in Flores et al. (2017).

114 Our paper is a first step towards a rigorous theory of *correlation* diversification measures.
 115 We provide an axiomatic foundation of the measurement of *correlation* diversification in a
 116 one-period portfolio theory under the assumption that the investor has complete information
 117 about the joint distribution of asset returns.

118 Specifically, in Section 3, we present and discuss a set of minimum desirable axioms that a
 119 measure of portfolio diversification must satisfy in order to be considered coherent.

120 In Section 4, we provide decision-theoretic foundations of our axioms by studying their
 121 rationality with respect to the two important decision theories under risk. The first is the
 122 classical expected utility theory. The second is Yaari's (1987) dual theory. More specifically,
 123 for each framework, we examine the compatibility of our axioms with investors' preference

124 for diversification (PFD) by using the notion of PFD introduced by [Deikel \(1989\)](#) and
 125 extended later by [Chateauneuf and Tallon \(2002\)](#) and [Chateauneuf and Lakhnati \(2007\)](#).

126 We proceed as follows. First, using the notion of PFD, we identify the measure of portfolio
 127 diversification at the core of each theory. Next, we test the identified measure against our
 128 axioms. If the identified measure satisfies our axioms, we consider our axioms rationalized
 129 by the theory. In doing so, we show that our axioms are rationalized by (a) the expected
 130 utility theory if and only if one of the following conditions is satisfied: (i) risk is small in the
 131 sense of [Pratt \(1964\)](#) and absolute risk aversion is constant (see [Proposition 1](#)), or (ii) each
 132 distribution of asset returns belongs to the location-scale family and the certainty equivalent
 133 has a particular additive-separable form (see [Proposition 2](#)); and (b) [Yaari's \(1987\)](#) dual
 134 theory if and only if its probability distortion function is convex. These results strengthen
 135 the desirability, reasonableness and relevance of our axioms.

136 In [Section 5](#), we examine a list of some of the most frequently used methods for measuring
 137 *correlation* diversification in terms of the axioms. This list includes:

- 138 (i) portfolio variance, a risk measure following the mean-variance model, often used
 139 to capture the benefit of diversification ([Markowitz, 1952, 1959; Sharpe, 1964](#)) and
 140 formally analyzed in [Frahm and Wiechers \(2013\)](#) as a measure of portfolio diversi-
 141 fication;
- 142 (ii) the *diversification ratio* designed by [Choueifaty and Coignard \(2008\)](#), and used by
 143 the firm TOBAM² to manage billions in assets via its Anti-Benchmark[®] strategies
 144 in Equities and Fixed Income;
- 145 (iii) [Embrechts et al.'s \(1999\)](#) class of measures and its normalized version analyzed by
 146 [Tasche \(2006\)](#), which are widely used to quantify diversification in both the finance
 147 and insurance industries (see [Bignozzi et al., 2016; Dhaene et al., 2009; Embrechts
 148 et al., 2013, 2015; Tong et al., 2012; Wang et al., 2015](#)) and are recommended
 149 implicitly in some international regulatory frameworks (see [Basel Committee on
 150 Banking Supervision, 2010; Committee on Risk Management and Capital Require-
 151 ments, 2016](#)).

152 We show that portfolio variance satisfies our axioms, but under the very restrictive (if not
 153 impossible) conditions that assets have identical variances (see part (i) of [Proposition 4](#)).
 154 This result, rather than weakening our axioms, reveals the limits of portfolio variance as an
 155 adequate measure of diversification in the mean-variance model.

156 We also show that the *diversification ratio* satisfies our axioms (see part (ii) of [Proposi-
 157 tion 4](#)), and that [Embrechts et al.'s \(1999\)](#) (see part (iii) of [Proposition 4](#)) and [Tasche's
 158 \(2006\)](#) (see part (iv) of [Proposition 4](#)) classes of diversification measures satisfy our ax-
 159 ioms, but under the condition that the underlying risk measure is convex (or quasi-convex),

²<https://www.tobam.fr/>

160 homogeneous, translation invariant and *reverse lower comonotonic additive*, which means
 161 that if the benefit of diversification is exhausted, then risks have lower comonotonicity (see
 162 item (xi) on page 8). These findings constitute supplementary evidence for the desirability,
 163 reasonableness and relevance of our axioms. They also show that measures such as the
 164 *diversification ratio*, Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification
 165 measures can be justified by our axioms.

166 Our findings (parts (iii) and (iv) of Proposition 4) also establish the conditions under
 167 which a coherent risk measure in the sense of Artzner et al. (1999) induces a coherent
 168 diversification measure (see Corollary 1). This condition is the reverse lower comonotonic
 169 additive property. The expected shortfall (in the case of continuous distribution), which
 170 is chosen over Value-at-Risk in Basel III (see Basel Committee on Banking Supervision,
 171 2013), and the concave distortion risk measures (see Sereda et al., 2010) induce coherent
 172 diversification measures. Our findings (parts (iii) and (iv) of Proposition 4) also imply that
 173 the deviation risk measure (see Rockafellar et al., 2006) induces a coherent diversification
 174 measure, but the family of convex risk measures (see Follmer and Schied, 2002; Frittelli and
 175 Gianin, 2002, 2005) does not. Our findings (part ((iv) of Proposition 4 in particular) also
 176 support the findings of Flores et al. (2017) that the *diversification delta* of Vermorken et al.
 177 (2012) is an inadequate measure of portfolio diversification.

178 Finally, in Section 6, we investigate the structure of representation of our axioms. We
 179 show that our axioms imply a family of representations, but this family is not unique. We
 180 provide some examples and one counterexample of this family of representations to support
 181 our argument.

182 Section 7 concludes the paper. Proofs are given in the appendix. Throughout the paper,
 183 vectors and matrices have bold style.

184 1.3 Related Literature

185 As mentioned above, the literature has focused exclusively on the design of *correlation*
 186 diversification measures. Some studies have presented and discussed desirable axioms to
 187 support their proposed measures. For example, Choueifaty et al. (2013) introduce the
 188 axiom of *duplication invariance* to support the *diversification ratio*. Evans and Archer
 189 (1968), Rudin and Morgan (2006) and Vermorken et al. (2012) present the *monotonicity*
 190 *in portfolio size* axiom to support portfolio size, the *portfolio diversification index* and the
 191 *diversification delta*, respectively. In addition to the axioms of *duplication invariance* and
 192 *monotonicity in portfolio size*, Carmichael et al. (2015) discuss the axioms of *degeneracy in*
 193 *portfolio size* and *degeneracy relative to dissimilarity* to support Rao's *Quadratic Entropy*.
 194 Meucci et al. (2014, Example 1, pp 4) discuss the axiom of *market homogeneity* to support
 195 the *effective number of bets*. Our research contributes to this literature by generalizing,
 196 completing and rationalizing this list of axioms to obtain a coherent axiomatic system.

197 In a recent contribution, [De Giorgi and Mahmoud \(2016b\)](#) develop an axiomatic structure
 198 for a diversification measure that is designed to capture the effect of naive diversification.
 199 By contrast, our axiomatic system is relevant for measures based on the notion of *correla-*
 200 *tion* diversification, assuming that the risk averse decision maker has complete information
 201 about the joint distribution of asset returns. Our work complements [De Giorgi and Mah-](#)
 202 [moud \(2016b\)](#) but it differs on an important point: their axiomatic system has a unique
 203 representative form and ours does not.

204 Finally, our work complements that of [Artzner et al. \(1999\)](#) on risk measurement. Whereas
 205 [Artzner et al. \(1999\)](#) provide a coherent axiomatic system of risk measures, our work pro-
 206 vides a coherent axiomatic system of *correlation* diversification measures. Our work differs
 207 from that of [Artzner et al. \(1999\)](#) in two other important respects. First and foremost, we
 208 study, in [Section 4](#), the rationality of our axiomatic system with respect to the expected
 209 utility ([Propositions 1](#) and [2](#)) and [Yaari's \(1987\)](#) dual ([Propositions 3](#)) theories. Second,
 210 our axiomatic system does not imply a unique family of representations.

211 2 Preliminaries

212 We consider a one-period model, so time diversification is impossible. We assume that the
 213 investor is risk averse and the investment opportunity set is a universe $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}_N}$ of N
 214 assets (risky or not), where A_i denotes asset i of \mathcal{A} , $\mathcal{I}_N = \{1, \dots, N\}$ is an index set and N
 215 is a strictly positive integer ($N \geq 1$). We also assume that short sales are restricted and
 216 we denote by $\mathbb{W} = \{\mathbf{w} = (w_1, \dots, w_N)^\top \in \mathbb{R}_+^N : \sum_{i=1}^N w_i = 1\}$ the set of long-only portfolios
 217 associated with \mathcal{A} , where w_i is the weight of asset i in portfolio \mathbf{w} , \top is a transpose operator
 218 and \mathbb{R}_+ is the set of positive real numbers. Our findings remain valid when short sales in
 219 the sense of [Lintner \(1965\)](#) are allowed, i.e. when the set of long/short portfolios is defined
 220 as $\mathbb{W}^- = \{\mathbf{w} = (w_1, \dots, w_N)^\top \in \mathbb{R}^N : \sum_{i=1}^N |w_i| = 1\}$ with \mathbb{R} the set of real numbers and $|\cdot|$ the
 221 absolute value operator. A single-asset i portfolio is denoted $\delta_i = (\delta_{i1}, \dots, \delta_{iN})^\top$, where δ_{ij}
 222 is the Kronecker delta i.e. $\delta_{ii} = 1$ for each $i \in \mathcal{I}_N$ and $\delta_{ij} = 0$ for $i \neq j$, $i, j \in \mathcal{I}_N$. A portfolio
 223 that holds at least two assets is considered a diversified portfolio, while a portfolio that
 224 maximizes or minimizes a portfolio diversification measure is a well-diversified portfolio.

225 $R_i \in \mathcal{R}$ denotes the future return of asset i , where $\mathcal{R} = \mathbb{L}^\infty(\Omega, \mathcal{E}, P)$ is the vector space
 226 of bounded real-valued random variables on a probability space (Ω, \mathcal{E}, P) , where Ω is the
 227 set of states of nature, \mathcal{E} is the σ - algebra of events, and P is a σ - additive probability
 228 measure on (Ω, \mathcal{E}) . We assume that the investor has complete information about the joint
 229 distribution of $\mathbf{R} = (R_1, \dots, R_N)^\top$. The expected value of R_i is $\mu_i = E(R_i)$, its variance
 230 $\sigma_i^2 = \text{Var}(R_i)$, its cumulative function $F_{R_i}(r_i)$, and its decumulative (survival) function
 231 $\bar{F}_{R_i}(r_i) = 1 - F_{R_i}(r_i)$, where $E(\cdot)$ and $\text{Var}(\cdot)$ are the operators of expectation and variance,
 232 respectively. The covariance between R_i and R_j is σ_{ij} and the covariance matrix is $\Sigma =$
 233 $(\sigma_{ij})_{i,j=1}^N$. The Pearson's correlation between R_i and R_j is $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ and the correlation

234 matrix is $\boldsymbol{\rho} = (\rho_{ij})_{i,j=1}^N$. The vector of asset volatility is denoted $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)^\top$. The
 235 future return of portfolio \mathbf{w} is $R(\mathbf{w}) = \mathbf{w}^\top \mathbf{R}$. Its expected value is $\mu(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\mu}$ and its
 236 variance $\sigma^2(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$. The cumulative and decumulative functions of \mathbf{R} are $F_{\mathbf{R}}(\mathbf{r})$ and
 237 $\bar{F}_{\mathbf{R}}(\mathbf{r})$, respectively. When necessary, the subscript i will be replaced by A_i , \mathbb{W} will be
 238 denoted \mathbb{W}^N or $\mathbb{W}_{\mathcal{A}}^N$ and \mathbf{w} will be denoted $\mathbf{w}_{\mathcal{A}}$.

239 Let ϱ denote a risk measure on \mathcal{R} .³ More formally, ϱ is a mapping defined from \mathcal{R} into \mathbb{R} ,
 240 which can have the following desirable properties:

- 241 (i) *Monotonicity*: for all $X, Y \in \mathcal{R}$, if $X \leq Y$, then $\varrho(X) \geq \varrho(Y)$;
- 242 (ii) *Sub-additivity*: for all $X, Y \in \mathcal{R}$, $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$;
- 243 (iii) *Convexity*: for all $X, Y \in \mathcal{R}$, $\lambda \in [0, 1]$, $\varrho(\lambda X + (1 - \lambda)Y) \leq \lambda \varrho(X) + (1 - \lambda) \varrho(Y)$;
- 244 (iv) *Quasi-convexity*: for all $X, Y \in \mathcal{R}$, $\lambda \in [0, 1]$, $\varrho(\lambda X + (1 - \lambda)Y) \leq \max(\varrho(X), \varrho(Y))$;
- 245 (v) *Comonotonic additivity*: for comonotonic $X, Y \in \mathcal{R}$, $\varrho(X + Y) = \varrho(X) + \varrho(Y)$;
- 246 (vi) *Non-additivity for independence*: for independent $X, Y \in \mathcal{R}$, $\varrho(X + Y) \neq \varrho(X) +$
 247 $\varrho(Y)$;
- 248 (vii) *Translation invariance*: for all $a \in \mathbb{R}$, $X \in \mathcal{R}$ and $\eta \geq 0$, $\varrho(X + a) = \varrho(X) - \eta a$;
- 249 (viii) *Positive homogeneity*: for all $\kappa \in \mathbb{R}$, $X \in \mathcal{R}$ and $b \geq 0$, $\varrho(bX) = b^\kappa \varrho(X)$;
- 250 (ix) *Law invariance*: if X, Y are identically distributed, denoted by $X = Y$, then $\varrho(X) \leq$
 251 $\varrho(Y)$.
- 252 (x) *Positivity*: for all nonconstant X , $\varrho(X) > 0$ and for all constant X , $\varrho(X) = 0$.
- 253 (xi) *Reverse lower comonotonic additivity*: for $X_i \in \mathcal{R}$, $i = 1, \dots, N$, $\varrho(\sum_{i=1}^N X_i) = \sum_{i=1}^N \varrho(X_i)$
 254 implies that the sequence X_1, \dots, X_N is lower comonotonic.

255 A random vector \mathbf{X} is comonotonic if and only if there are non-decreasing functions f_i , $i \in \mathcal{I}_N$
 256 and a random variable X such that $\mathbf{X} \stackrel{d}{=} (f_1(X), \dots, f_N(X))$, where $\stackrel{d}{=}$ stands for “equally
 257 distributed” (see [Dhaene et al., 2008](#)). Intuitively, the comonotonicity corresponds to an
 258 extreme form of positive dependency. All returns are driven linearly or nonlinearly by a
 259 unique factor, but positively. For more details about the concept of comonotonicity and its
 260 applications in finance, we refer readers to [Dhaene et al. \(2002b\)](#) and [Dhaene et al. \(2002a\)](#).

261 In a real world environment, asset returns are usually not comonotonic, but can be comono-
 262 tonic in the tails, as observed during the 2007-2009 financial crisis. The concept of upper
 263 comonotonicity was introduced and investigated in [Cheung \(2009\)](#). A random vector \mathbf{X} is
 264 upper comonotonic if and only if \mathbf{X} exhibits a comonotonicity behavior in the upper tail. The
 265 lower comonotonicity is the opposite: a random vector \mathbf{X} is lower comonotonic if and only
 266 if its opposite $-\mathbf{X}$ is upper comonotonic. We refer readers to [Nam et al. \(2011\)](#), [Dong et al.](#)
 267 [\(2010\)](#) and [Hua and Joe \(2012\)](#) for more details on the concept of upper comonotonicity.

268 The property of monotonicity is a natural requirement for a reasonable risk measure. The
 269 properties of sub-additivity, convexity, quasi-convexity, comonotonic additivity and non-

³Note that $\varrho(\cdot) = \text{Var}(\cdot)$ in the case of the variance risk measure.

270 additivity for independence capture the diversification effect. The property of sub-additivity
 271 states that “a merger does not create extra risk” (Artzner et al., 1999). The properties
 272 of convexity and quasi-convexity imply that diversification should not increase risk. The
 273 property of comonotonic additivity implies that there is no benefit in terms of risk reduction
 274 to diversify across comonotonic risks. The property of non-additivity for independence rules
 275 out the possibility that the pooling of independent risks does not have a diversification
 276 effect. The property of translation invariance states that risk can be reduced by adding
 277 cash, except in the case where $\eta = 0$. The property of law invariance states that a risk
 278 measure $\varrho(X)$ depends only on the distribution of X i.e. $\varrho(X) = \varrho(F_X)$. The property of
 279 positive homogeneity states that a linear increase of the return by a positive factor leads
 280 to a non-linear increase in risk, except in the case where $\kappa = 1$. The property of positivity
 281 captures the idea that $\varrho(\cdot)$ measures the degree of uncertainty in X . The property of reverse
 282 lower comonotonic additivity requires that if $\varrho(\sum_{i=1}^N X_i)$ is additive, then risks X_1, \dots, X_N
 283 are lower comonotonic. In other words, if the benefit of diversification is exhausted, then
 284 risks are lower comonotonic.

285 $\varrho(\cdot)$ is called a coherent risk measure (see Artzner et al., 1999) if it satisfies the properties of
 286 monotonicity, sub-additivity, translation invariance and positive homogeneity (with $\kappa = 1$).
 287 $\varrho(\cdot)$ is called a convex risk measure (see Follmer and Schied, 2002; Frittelli and Gianin, 2002,
 288 2005), if it satisfies the properties of monotonicity, translation invariance and convexity. $\varrho(\cdot)$
 289 is called a deviation risk measure (see Rockafellar et al., 2006), if it satisfies the properties
 290 of sub-additivity, translation invariance ($\eta = 0$), positive homogeneity (with $\kappa = 1$) and
 291 positivity. For more details about these properties and for other desirable properties, readers
 292 are referred to Pedersen and Satchell (1998), Artzner et al. (1999), Song and Yan (2006),
 293 Song and Yan (2009), Sereda et al. (2010), Follmer and Schied (2010) and Wei et al. (2015).

294 **3 Axioms**

295 We present and discuss a set of minimum desirable axioms that obtain a measure of portfolio
 296 diversification that can be considered coherent. The next section provides the decision-
 297 theoretic foundations of these axioms.

Let us first introduce the definition of a portfolio diversification measure. How can we
 define a diversification measure of a portfolio \mathbf{w} ? Although there is no unique definition of
 diversification in portfolio theory, the diversification interest variable and benefit, which is
 to say the distribution of portfolio weight \mathbf{w} and risk or uncertainty reduction, respectively,
 are unique. Let Φ be a continuous measure of portfolio diversification. In the case where
 the investor is risk averse and has complete information about the joint distribution of
 asset returns \mathbf{R} , it is natural to represent Φ as a mapping from \mathbb{W} into \mathbb{R} conditional to \mathbf{R}

explicitly or implicitly; formally

$$\Phi: \mathbb{W} \rightarrow \mathbb{R} \quad (1)$$

$$\mathbf{w} \mapsto \Phi(\mathbf{w}|\mathbf{R}). \quad (2)$$

298 The form of the function $\Phi(\cdot|\mathbf{R})$ depends on some properties of the portfolio diversification
299 measure.

300 In the case where the investor has no information about \mathbf{R} , Φ can also be represented as a
301 function of \mathbf{w} i.e. $\Phi(\mathbf{w})$, and used to capture the effect of naive diversification.

Let now introduce our set of desirable axioms. There is no loss of generality in assuming that the well-diversified portfolio of $\Phi(\mathbf{w}|\mathbf{R})$, denoted \mathbf{w}^* , is obtained by maximization i.e.

$$\mathbf{w}^* \in \arg \operatorname{Max}_{\mathbf{w} \in \mathbb{W}} \Phi(\mathbf{w}|\mathbf{R}).$$

302 Therefore, given a measure Φ , we say that “portfolio \mathbf{w}_1 is more diversified than portfolio
303 \mathbf{w}_2 ” if $\Phi(\mathbf{w}_1) > \Phi(\mathbf{w}_2)$.

304 Our first axiom formalizes investors’ preference for diversification over \mathbb{W} . This axiom was
305 first formulated in [Carmichael et al. \(2015\)](#) and is expressed in

306 CONCAVITY (C). For each \mathbf{w}_1 and $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0, 1]$ and $\mathbf{R} \in \mathcal{R}^N$,

$$307 \quad \Phi(\alpha \mathbf{w}_1 + (1 - \alpha) \mathbf{w}_2|\mathbf{R}) \geq \alpha \Phi(\mathbf{w}_1|\mathbf{R}) + (1 - \alpha) \Phi(\mathbf{w}_2|\mathbf{R}) \quad (3)$$

308 and strict inequality for at least one α .

309 **Concavity** implies that holding different assets increases total diversification. It also ensures
310 that the diversification is always beneficial and can be decomposed across asset classes.

311 **Concavity** can be replaced by a less restrictive axiom.

312 QUASI-CONCAVITY (QC). For each \mathbf{w}_1 and $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0, 1]$ and $\mathbf{R} \in \mathcal{R}^N$,

$$313 \quad \Phi(\alpha \mathbf{w}_1 + (1 - \alpha) \mathbf{w}_2|\mathbf{R}) \geq \min(\Phi(\mathbf{w}_1|\mathbf{R}), \Phi(\mathbf{w}_2|\mathbf{R})) \quad (4)$$

314 and strict inequality for at least one α .

315 Our next axiom is complementary to **Concavity**. It is favored by [Carmichael et al. \(2015\)](#)
316 and is expressed in

317 SIZE DEGENERACY (SD). There is a constant (for a normalization) $\underline{\Phi} \in \mathbb{R}$ such that for
318 each $\mathbf{R} \in \mathcal{R}^N$,

$$319 \quad \Phi(\delta_i|\mathbf{R}) = \underline{\Phi} \text{ for each } i \in \mathcal{I}_N. \quad (5)$$

320 It states that all single-asset portfolios have the same degree of diversification and are
 321 the least diversified portfolio. **Concavity** and **Size Degeneracy** taken together imply that
 322 diversification is always better than full concentration or specialization; formally for each
 323 $i \in \mathcal{I}_N$, $\Phi(\delta_i|\mathbf{R}) \leq \Phi(\mathbf{w}|\mathbf{R})$. **Size Degeneracy** is clearly necessary to prevent portfolio
 324 concentration to remain undetected.

325 Our next axiom formalizes the behavior of $\Phi(\mathbf{w}|\mathbf{R})$ when \mathbf{R} is homogeneous in the sense
 326 of perfect similarity. It is expressed in

327 RISK DEGENERACY (RD). Let $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}_N}$ be a universe of N assets such that $A_i =$
 328 A , for each $i \in \mathcal{I}_N$. Then, for each $\mathbf{w} \in \mathbb{W}$

$$329 \quad \Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}. \quad (6)$$

330 **Risk Degeneracy** ensures that there is no benefit to diversifying across perfectly similar
 331 assets. Such diversification is equivalent to full concentration. **Risk Degeneracy** is also
 332 necessary to keep portfolio concentration from going undetected. **Risk Degeneracy** gener-
 333 alizes Carmichael et al.'s (2015) axiom of *degeneracy relative to dissimilarity*, which states,
 334 “a portfolio formed solely with perfect similar assets must have the lowest diversification
 335 degree.”

336 Our next axiom is complementary to **Risk Degeneracy** and is expressed in

337 REVERSE RISK DEGENERACY (RRD). Consider the equation $\Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}$ on \mathcal{R} for each
 338 $\mathbf{w} \in \mathbb{W}$ such that $\mathbf{w} \neq \delta_i$, $i \in \mathcal{I}_N$ and without loss of generality assume that $w_i > 0$, for each $i \in$
 339 \mathcal{I}_N . Assume that a solution exists and is \mathbf{R}^* . Then \mathbf{R}^* must be lower comonotonic. Note
 340 that \mathbf{R}^* can be different from \mathbf{R} .

341 **Reverse Risk Degeneracy** is also necessary to prevent undetected portfolio concentration.
 342 It states that when $\Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}$ with \mathbf{w} a diversified portfolio, $\mathbf{R}_{\mathcal{I}_+} = (R_i)_{i \in \mathcal{I}_+}$ or its
 343 transformation is necessarily lower comonotonic, where $\mathcal{I}_+ = \{i | w_i > 0\}$. The following
 344 example is provided to get more of a sense of the importance of **Reverse Risk Degeneracy**.

345 EXAMPLE 1 (EMBRECHTS ET AL.'S (2009) CLASS OF DIVERSIFICATION MEASURES).

346 Consider Embrechts et al.'s (2009) class of diversification measures (see item (itememb)
 347 in Section 5) when the risk measure $\varrho(\cdot)$ is additive for independence i.e for independent
 348 $X, Y \in \mathcal{R}$, $\varrho(X + Y) = \varrho(X) + \varrho(Y)$. This is the case when $\varrho(\cdot)$ is the *mixed Esscher*
 349 *premium* or the *mixed exponential premium* analyzed in Goovaerts et al. (2004). In that
 350 case, according to Embrechts et al.'s (2009) class of diversification measures, any portfolio
 351 with assets with independent returns and single-asset portfolios would have the same degree
 352 of diversification, which is counterintuitive. **Reverse Risk Degeneracy** rules out this sub-class
 353 of Embrechts et al.'s (2009) and Tasche's (2006) diversification measures.

354 Our next axiom is the formalization of the property of *duplication invariance* of [Choueifaty](#)
 355 [et al. \(2013\)](#) analyzed in [Carmichael et al. \(2015\)](#). It is expressed in

DUPLICATION INVARIANCE (DI). Let $\mathcal{A}^+ = \{A_i^+\}_{i \in \mathcal{I}_{N+1}}$ be a universe of assets such that $A_i^+ = A_i$, for each $i \in \mathcal{I}_N$ and $A_{N+1}^+ = A_k$, for $k \in \mathcal{I}_N$. Then

$$\Phi(\mathbf{w}_{\mathcal{A}}^* | \mathbf{R}_{\mathcal{A}}) = \Phi(\mathbf{w}_{\mathcal{A}^+}^* | \mathbf{R}_{\mathcal{A}^+}) \quad (7)$$

$$\mathbf{w}_{A_i}^* = \mathbf{w}_{A_i^+}^* \text{ for each } i \neq k, i \in \mathcal{I}_N \quad (8)$$

$$\mathbf{w}_{A_k}^* = \mathbf{w}_{A_k^+}^* + \mathbf{w}_{A_{N+1}^+}^*. \quad (9)$$

356 The reasonableness and relevance of **Duplication Invariance** is evident. It allows us to
 357 avoid risk concentration by ensuring that the optimal diversified portfolio is not biased
 358 towards multiple representative assets. It is necessary to prevent portfolio concentration
 359 from going undetected. The following example is provided to demonstrate the importance
 360 of **Duplication Invariance**.

361 EXAMPLE 2 (CASE $N = 2$). Consider the case where $N = 2$. In that case $\mathcal{A} = \{A_1, A_2\}$ and
 362 $\mathcal{A}^+ = \{A_1, A_2, A_3^+\}$ such that $A_3^+ = A_1$. **Duplication Invariance** states that the degree of
 363 diversification of \mathcal{A} and \mathcal{A}^+ must be equal and optimally the weight of A_2 in \mathcal{A} must be
 364 equal to the sum of the weights of A_2 and A_3^+ in \mathcal{A}^+ .

365 Our next axiom formalizes the relationship between diversification and portfolio size. It is
 366 expressed in

367 SIZE MONOTONICITY (M). Let $\mathcal{A}^{++} = \{A_i^{++}\}_{i \in \mathcal{I}_{N+1}}$ be a universe of assets such that $A_i^{++} =$
 368 A_i , for each $i \in \mathcal{I}_N$ and $A_{N+1}^{++} \neq A_i$, for each $i \in \mathcal{I}_N$. Then

$$\Phi(\mathbf{w}_{\mathcal{A}^{++}}^* | \mathbf{R}_{\mathcal{A}^{++}}) \geq \Phi(\mathbf{w}_{\mathcal{A}}^* | \mathbf{R}_{\mathcal{A}}). \quad (10)$$

370 **Size Monotonicity** is natural in portfolio diversification literature (see [Carmichael et al.](#),
 371 [2015](#); [Evans and Archer, 1968](#); [Rudin and Morgan, 2006](#); [Vermorken et al., 2012](#)). It reveals
 372 that increasing portfolio size does not decrease the degree of portfolio diversification. It also
 373 states that increasing portfolio size does not systematically increase the degree of portfolio
 374 diversification.

375 Our next axiom is an adaptation of the risk measure translation invariance axiom. It is
 376 expressed in

377 TRANSLATION INVARIANCE (TI). Let $\mathcal{A}+a = \{A_i+a\}_{i \in \mathcal{I}_N}$ be a universe of assets such that
 378 $R_{A_i+a} = R_{A_i} + a$, for each $i \in \mathcal{I}_N$, $a \in \mathbb{R}$. Then for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}} = \mathbb{W}_{\mathcal{A}+a}$,

$$\Phi(\mathbf{w} | \mathbf{R}_{\mathcal{A}+a}) = \Phi(\mathbf{w} | \mathbf{R}_{\mathcal{A}}). \quad (11)$$

380 The desirability of **Translation Invariance** comes from the translation invariance risk mea-
 381 sure axiom. It implies that adding the same amount of cash to asset returns does not change
 382 the degree of portfolio diversification. Consider the translation invariance risk measure ax-
 383 iom. Assume that $\eta = 0$. In that case, adding the same amount of cash to asset returns does
 384 not affect the degree of portfolio risk. The degree of portfolio diversification is not affected
 385 either. Now, assume that $\eta > 0$. In that case, adding the same amount of cash to asset
 386 returns reduces portfolio risk, but does not affect the degree of portfolio diversification.

387 However, when risk is defined as capital requirement or probability of loss (for example
 388 Expected Shortfall or Conditional Value-at-Risk), **Translation Invariance** can be seen as
 389 counterintuitive. To see this, consider the case where risk $\varrho(\cdot)$ is defined as a capital
 390 requirement verifying the property of translation invariance. Assume that $a = \frac{\varrho(\mathbf{w}^\top \mathbf{R})}{\eta}$ with
 391 $\eta \neq 0$. Then $\varrho(\mathbf{w}^\top \mathbf{R} + a) = 0$, but $\Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}}) \geq 0$. This counterintuitive result
 392 can be viewed as over diversification, and can be interpreted as an extreme precaution
 393 against extreme risk.

394 In the case where $\Phi(\cdot|\mathbf{R})$ is a normalized measure, i.e. when $\Phi(\mathbf{w}|\mathbf{R})$ can be rewritten as
 395 follows

$$\Phi(\mathbf{w}|\mathbf{R}) = \frac{\tilde{\Phi}(\mathbf{w}|\mathbf{R})}{\varrho(\mathbf{w}^\top \mathbf{R})}, \quad (12)$$

397 or equivalently

$$\Phi(\mathbf{w}|\mathbf{R}) = \frac{\tilde{\Phi}(\mathbf{w}|\mathbf{R}) - \varrho(\mathbf{w}^\top \mathbf{R})}{\varrho(\mathbf{w}^\top \mathbf{R})}, \quad (13)$$

399 with $\tilde{\Phi}(\cdot|\mathbf{R})$ the portfolio diversification measure such that $\tilde{\Phi}(\mathbf{w}|\mathbf{R} + a) = \tilde{\Phi}(\mathbf{w}|\mathbf{R})$, the
 400 **Translation Invariance** must be replaced by with the following.

TRANSLATION INVARIANCE-2 (TI2). Let $\mathcal{A} + a = \{A_i + a\}_{i \in \mathcal{I}_N}$ be a universe of assets such
 that $R_{A_i+a} = R_{A_i} + a$, for each $i \in \mathcal{I}_N$ with $a \in \mathbb{R}$. Then for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}+a}$

$$\frac{\partial \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a})}{\partial a} \geq 0, \quad (14)$$

$$\lim_{a \rightarrow -\infty} \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \underline{\Phi}, \quad (15)$$

$$\lim_{a \rightarrow +\infty} \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \underline{\Phi}, \quad (16)$$

$$\lim_{a \rightarrow \frac{\varrho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta}} \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = -\infty, \quad (17)$$

$$\lim_{a \rightarrow \frac{\varrho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta}} \Phi(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \infty. \quad (18)$$

401 The idea behind **Translation Invariance-2** is as follows. **Equation (14)** states that adding

402 cash increases the diversification benefit. This is because adding cash reduces the total risk
 403 $\varrho(\cdot)$ and does not affect diversification $\tilde{\Phi}(\cdot|\mathbf{R})$. **Equations (15)** and **(16)** ensure that when
 404 cash converges to $+\infty$ or $-\infty$, the diversification benefit vanishes because the whole system
 405 becomes homogeneous. **Equations (17)** and **(18)** capture the over diversification behavior
 406 of $\Phi(\cdot|\mathbf{R})$ when risk converges to 0 and diversification becomes unnecessary.

407 Our next axiom is an adaptation of the positive homogeneity of risk measure axiom. It is
 408 expressed in

409 HOMOGENEITY (H). Let $b\mathcal{A} = \{bA_i\}_{i \in \mathcal{I}_N}$ be a universe of assets such that $R_{bA_i} = bR_{A_i}$, for
 410 each $i \in \mathcal{I}_N$ with $b \geq 0$. Then there exists $\kappa \in \mathbb{R}$ such that for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}} = \mathbb{W}_{b\mathcal{A}}$

$$411 \quad \Phi(\mathbf{w}|R_{b\mathcal{A}}) = b^\kappa \Phi(\mathbf{w}|R_{\mathcal{A}}). \quad (19)$$

412 The desirability of **Homogeneity** comes naturally from the homogeneous property of the
 413 risk measure. In the case where $\Phi(\cdot|\mathbf{R})$ is a normalized measure, κ must be equal to zero,
 414 which ensures that $\Phi(\cdot|\mathbf{R})$ must not depend on scalability.

415 Our last axiom presents the behavior of $\Phi(\mathbf{w}|\mathbf{R})$ when R_1, \dots, R_N are exchangeable random
 416 variables. First, let us recall the definition of exchangeable random variables.

417 DEFINITION 1 (EXCHANGEABILITY). The random variables R_1, \dots, R_N are said to be ex-
 418 changeable if and only if their joint distribution $F_{\mathbf{R}}(\mathbf{r})$ is symmetric.

419 A well-known example of an exchangeable sequence of random variables is an independent
 420 and identically distributed sequence of random variables. For more details on exchangeable
 421 random variables, we refer readers to [Aldous \(1985\)](#).

422 Our last axiom is expressed in

423 SYMMETRY (S). If R_1, \dots, R_N are exchangeable, then $\Phi(\mathbf{w}|\mathbf{R})$ is symmetric in \mathbf{w} .

424 **Symmetry** states that a portfolio diversification measure must be symmetric in \mathbf{w} if R_1, \dots, R_N
 425 are exchangeable. The thinking behind Symmetry is that the exchangeable random vari-
 426 ables imply homogeneous risks. Thus, the decision maker must be indifferent in terms of
 427 diversification between \mathbf{w} and $\mathbf{\Pi w}$, where $\mathbf{\Pi}$ is a permutation matrix.

428 From [Marshall et al. \(2011, C.2. and C.3. Propositions, pp 97-98\)](#), **Symmetry** and **Con-**
 429 **cavity** or **Quasi-Concavity** taken together imply that $\Phi(\mathbf{w}|\mathbf{R})$ is Schur-concave in \mathbf{w} when
 430 R_1, \dots, R_N are exchangeable. As a result, $\Phi(\mathbf{w}|\mathbf{R})$ must be a measure of naive diversification
 431 and the optimal diversified portfolio \mathbf{w}^* must be the naive portfolio $\frac{1}{N}$ when R_1, \dots, R_N are
 432 exchangeable. This result is consistent with the principle that the exchangeability assump-
 433 tion on R_1, \dots, R_N is equivalent to the assumption that the decision maker has no information
 434 about asset risk characteristics \mathbf{R} . Moreover, **Symmetry** and **Concavity** or **Quasi-Concavity**

435 taken together imply the axiom of *market homogeneity* implicitly analyzed in Meucci et al.
 436 (2014, Example 1, pp 4).

437 4 Rationalization

438 This section studies the rationality of our axioms with respect to the two most important
 439 decision theories under risk: the expected utility theory and Yaari's (1987) dual theory.
 440 For each theory, we examine the compatibility of our axioms with investors' preference for
 441 diversification (PFD). We proceed as follows. First, from the notion of PFD, we identify the
 442 measure of portfolio diversification at the core of each model. Next, we test the identified
 443 measure against our axioms. If the identified measure satisfied our axioms, we consider that
 444 our axioms are rationalized by the model.

445 There are several notions of PFD in the theory of choice under risk or uncertainty (see
 446 De Giorgi and Mahmoud, 2016a). We consider the ideas introduced by Dekel (1989) and
 447 extended later by Chateauneuf and Tallon (2002) and Chateauneuf and Lakhnati (2007) to
 448 the space of random variables. Let \geq be the preference relation over \mathcal{R} of a decision maker
 449 (i.e., an investor). The chosen notion of PFD is defined as follows.

450 DEFINITION 2 (CHATEAUNEUF AND TALLON (2002)). The preference relation \geq exhibits
 451 preference for diversification if for any $R_i \in \mathcal{R}$ and $\alpha_i \in [0, 1]$, $i \in \mathcal{I}_N$ such that $\sum_{i=1}^N \alpha_i = 1$,

$$452 \quad R_1 \sim R_2 \sim \dots \sim R_N \Rightarrow \sum_{i=1}^N \alpha_i R_i \geq R_j \quad \text{for each } j \in \mathcal{I}_N. \quad (20)$$

453 Definition 2 states that if assets are equally desirable, then the investor will want to diversify.
 454 This notion of PFD is equivalent to the notion of risk aversion in the expected utility
 455 theory and implies the notion of strong risk aversion (i.e. risk aversion in the sense of mean
 456 preserving spread as defined in De Giorgi and Mahmoud (2016a, Definition 9, pp. 152)) in
 457 Yaari's (1987) dual theory.

458 4.1 Expected Utility Theory

459 Let us first study the rationality of our axioms with respect to the expected utility (EU)
 460 theory. Assume that \geq has an expected utility representation. Then

$$461 \quad R_1 \geq R_2 \iff E_u(R_1) \geq E_u(R_2), \quad (21)$$

462 where $E_u(R) = E(u(R)) = \int u(r) dF_R(r)$ with $u : \mathcal{R} \rightarrow \mathbb{R}$ is an increasing von Neumann-
 463 Morgenstern utility function for wealth. Moreover, $u(\cdot)$ is unique up to positive affine
 464 transformations. The shape of $u(\cdot)$ determines investors' risk attitude and diversification
 465 profile. The investors are risk averse, -neutral and -lover when $u(\cdot)$ is concave, linear and
 466 convex respectively. The investors have a PFD only when $u(\cdot)$ is concave i.e. when they

467 are risk averse. In this paper, we assume that $u(\cdot)$ is concave to be consistent with our
 468 hypothesis of risk averse investors and consequently with the notion of PFD.

469 **Definition 2** is equivalent to the following.

470 **DEFINITION 3.** The preference relation \geq shows a preference for diversification if for any
 471 $R_i \in \mathcal{R}$ and $\alpha_i \in [0, 1]$, $i \in \mathcal{I}_N$ such that $\sum_{i=1}^N \alpha_i = 1$, the following equivalent conditions are
 472 satisfied

$$\begin{aligned}
 473 \quad (i) \quad & E_u(R_1) = \dots = E_u(R_N) \implies E_u\left(\sum_{i=1}^N \alpha_i R_i\right) \geq E_u(R_j) \quad \text{for each } j \in \mathcal{I}_N \\
 474 \quad (ii) \quad & \varrho_{C_u}(R_1) = \dots = \varrho_{C_u}(R_N) \implies \varrho_{C_u}\left(\sum_{i=1}^N \alpha_i R_i\right) \leq \varrho_{C_u}(R_j) \quad \text{for each } j \in \mathcal{I}_N \\
 475 \quad (iii) \quad & \varrho_{\pi_u}(R_1) = \dots = \varrho_{\pi_u}(R_N) \implies \varrho_{\pi_u}\left(\sum_{i=1}^N \alpha_i R_i\right) \leq \varrho_{\pi_u}(R_j) \quad \text{for each } j \in \mathcal{I}_N,
 \end{aligned}$$

476 where $\varrho_{C_u}(R) = -C_u(R)$ is a risk measure induced by the certainty equivalent $C_u(R) =$
 477 $u^{-1}(E_u(R))$ and $\varrho_{\pi_u}(R) = \pi_u(-R)$ is induced by the risk premium $\pi_u(R) = E(R) - C_u(R)$
 478 of $u(\cdot)$.

479 Because diversification is a risk reduction tool, we focus on parts (ii) and (iii) of **Definition 3**.
 480 Multiplying the inequality in (ii) and (iii) by α_j and summing over j , we obtain

$$481 \quad \varrho_l\left(\sum_{i=1}^N \alpha_i R_i\right) \leq \sum_{i=1}^N \alpha_j \varrho_l(R_j) \quad \text{for each } l \in \{C_u, \pi_u\}. \quad (22)$$

482 From (22), following [Embrechts et al. \(1999\)](#) (see item (iii) on [page 20](#)), the gain of diver-
 483 sification in the EU theory can be measured by the difference

$$484 \quad \underline{\varrho}_l(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i \varrho_l(1 + R_i) - \varrho_l\left(1 + \sum_{i=1}^N w_i R_i\right) \quad \text{for each } l \in \{C_u, \pi_u\}. \quad (23)$$

485 The definition of compatibility with the PFD in the EU theory is based on $\underline{\varrho}_l(\mathbf{w}|\mathbf{R})$ for each $l \in$
 486 $\{C_u, \pi_u\}$ and is defined as follows:

487 **DEFINITION 4 (COMPATIBILITY WITH PFD IN THE EU THEORY).** Our axioms are com-
 488 patible with the PFD in the EU theory if and only if they are satisfied by $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ or
 489 $\underline{\varrho}_{\pi_u}(\mathbf{w}|\mathbf{R})$.

490 Using **Definition 4**, we establish the necessary and sufficient conditions for the compatibility
 491 of our axioms with the PFD in the EU theory. Two cases are considered.

492 **4.1.1 Case 1: Risk is Small**

493 In this first case, we assume that risk is small in the sense of [Pratt \(1964\)](#) i.e. measured
 494 by σ_i^2 , for each $i \in \mathcal{I}_N$. We refer to this compatibility as local compatibility. The following
 495 proposition establishes the necessary and sufficient conditions.

496 PROPOSITION 1 (LOCAL COMPATIBILITY WITH PFD IN THE EU THEORY). *If risk is small,*
 497 *then our axioms are compatible with the PFD in the EU theory if and only if the absolute*
 498 *risk aversion of $u(\cdot)$ is constant; formally, $k(x) = -\frac{u''(x)}{u'(x)} = c$, $c \in \mathbb{R}$, where $u'(\cdot)$ and $u''(\cdot)$*
 499 *are the first and second derivatives of $u(\cdot)$.*

500 Proposition 1 shows that our axioms can be rationalized by the EU theory if risk is small
 501 and the absolute risk aversion of $u(\cdot)$ is constant. The negative exponential utility function,
 502 $u(x) = -\exp(-\lambda x)$ with $\lambda > 0$, the investor's risk aversion coefficient is the only example
 503 of a concave utility function that implies constant absolute risk aversion. Thus, if risk is
 504 small, our axioms can be rationalized by the EU theory if and only if $u(\cdot)$ is the negative
 505 exponential utility.

506 4.1.2 Case 2: Location-Scale Family of Distributions

507 In the second case, we assume that each distribution of asset returns belongs to the location-
 508 scale family. This family of distributions includes, among others, the normal, student's t
 509 and all other elliptical distributions. For more details see Meyer et al. (1987). The following
 510 proposition establishes the necessary and sufficient conditions.

511 PROPOSITION 2 (COMPATIBILITY WITH PFD IN THE EU THEORY: LOCATION-SCALE FAMILY).
 512 *If each asset returns distribution belongs to the location-scale family, then our axioms are*
 513 *compatible with the PFD in the EU theory if and only if the certainty equivalent has the*
 514 *following additive separable form*

$$515 \quad C_u(R) = \mu - g(\sigma) \quad (24)$$

516 *for a strictly increasing continuous and homogeneous function $g(\cdot)$ on \mathbb{R}_+ .*

517 Below, we present an example of location-scale distribution and utility function for which
 518 Proposition 2 is valid.

519 EXAMPLE 3 (NORMAL DISTRIBUTION AND NEGATIVE EXPONENTIAL UTILITY). Assume that
 520 the asset returns are normally distributed and the utility function is negative exponential.
 521 It is proven in the literature that

$$522 \quad C_u(R(\mathbf{w})) = \mathbf{w}^\top \boldsymbol{\mu} - \lambda \sigma^2(\mathbf{w}). \quad (25)$$

523 Proposition 2 also implies that our axioms can also be rationalized by the additive sepa-
 524 rable mean-variance utility functions (including Markowitz (1952)'s mean-variance utility)
 525 axiomatized by Nakamura (2015, Theorem 4., pp. 544).

526 Propositions 1 and 2 jointly represent the necessary and sufficient conditions of our axioms to
 527 be compatible with the PFD in the EU theory. In Propositions 1 and 2 we have compatibility

528 only when risk is measured by variance, so the conditions might be thought to be restrictive,
 529 thereby considerably weakening the desirability of our axioms. However, this is not the case,
 530 because the majority of our axioms remain compatible with the PFD in the EU theory when
 531 we consider other standard utility functions. For example, exploiting the results in Müller
 532 (2007), if we consider the negative exponential utility with a non-Location-Scale family of
 533 distributions, one can verify that $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies all the axioms except **Homogeneity**.
 534 In the case of the power or logarithmic utility function, one can verify that $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R})$
 535 satisfies all the axioms, except **Concavity**, **Quasi-Concavity**, **Translation Invariance** and
 536 **Homogeneity**. Second, as we show in the next subsection, our axioms are also relevant
 537 when risk is not completely captured by the variance.

538 In sum, **Propositions 1** and **2** provide a decision-theoretic foundation of our axioms and
 539 consequently strengthen their desirability, reasonableness and relevance.

540 4.2 Yaari's (1987) Dual Theory

541 Despite the importance of the EU theory in the theory of rational choice under risk, it has
 542 been shown that it often fails to describe and predict peoples' choices properly (see Allais,
 543 1953; Kahneman and Tversky, 1979). As a consequence, alternative theories of choice were
 544 proposed; see Schoemaker (1982), Machina (1987) and Starmer (2000) for comprehensive
 545 reviews. In this section, we study the rationality of our axioms with respect to one of the
 546 most successful of them: Yaari's (1987) dual (DU) theory of choice, which is a special case
 547 of the rank dependent utility theory of Quiggin (1982); see also Quiggin (2012).

548 Yaari's (1987) DU theory was constructed from the EU theory by replacing the independence
 549 axiom by the dual independence axiom, which states that for any $X, Y, Z \in \mathcal{R}$, if X is
 550 preferred to Y , then $(\alpha F_X^{-1} + (1 - \alpha)F_Z^{-1})^{-1}$ is preferred to $(\alpha F_Y^{-1} + (1 - \alpha)F_Z^{-1})^{-1}$. Doing
 551 so, Yaari (1987) obtained a preference functional that is linear concerning payoffs and
 552 nonlinear concerning probability, which is the opposite of the EU theory in which the
 553 preference functional is nonlinear concerning payoffs and linear concerning probability.

554 More formally, assume that \geq has a DU theory representation. Then, from Yaari (1987)
 555 (see also Tsanakas and Desli, 2003),

$$556 \quad R_i \geq R_j \Leftrightarrow E_{\bar{h}}(R_i) \geq E_{\bar{h}}(R_j), \quad (26)$$

557 where

$$558 \quad E_{\bar{h}}(R) = \int_{-\infty}^0 (\bar{h}(\bar{F}_R(r)) - 1) dr + \int_0^{\infty} \bar{h}(\bar{F}_R(r)) dr = \int_{-\infty}^{\infty} r dh(F_R(r)) \quad (27)$$

559 with $\bar{h}, h : [0, 1] \mapsto [0, 1]$ being increasing functions satisfying $\bar{h}(0) = h(0) = 0$ and $\bar{h}(1) =$
 560 $h(1) = 1$ such that $\bar{h}(u) = 1 - h(1 - u)$. $\bar{h}(\cdot)$ is called the probability distortion function and

561 $h(\cdot)$ represents its dual.

562 Like in the EU theory, in the DU theory, the investor's risk profile can be characterized by
 563 some conditions on $\bar{h}(\cdot)$. However, the notions of risk aversion are not equivalent in the DU
 564 theory. The investor is risk averse in the sense of $E_{\bar{h}}(R) \leq E_{\bar{h}}(E(R)) = E(R)$ if and only
 565 if $h(u) \leq u$, $\forall u \in [0, 1]$. The investor is risk averse in the sense of mean preserving spread
 566 as defined in [De Giorgi and Mahmoud \(2016a, Definition 9, pp. 152\)](#) if and only if $\bar{h}(\cdot)$ is
 567 convex and $\bar{h}(u) \neq u$ or $h(\cdot)$ is concave and $h(u) \neq u$. In this paper, to be consistent both
 568 with our hypothesis of risk averse investors and our notion of PFD, we assume that $\bar{h}(\cdot)$ is
 569 convex and $\bar{h}(u) \neq u$ or equivalently $h(\cdot)$ is concave and $h(u) \neq u$.

570 Like in the EU theory, the gain of diversification in the DU theory can be measured by the
 571 difference

$$572 \quad \underline{\varrho}_l(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i \varrho_l(1 + R_i) - \varrho_l\left(1 + \sum_{i=1}^N w_i R_i\right) \quad \text{for each } l \in \{C_{\bar{h}}, \pi_{\bar{h}}\}, \quad (28)$$

573 where $C_{\bar{h}}(\cdot)$ and $\pi_{\bar{h}}(\cdot)$ are respectively the certainty equivalent and the risk premium asso-
 574 ciated to the DU utility function $E_{\bar{h}}(\cdot)$. The certainty equivalent $C_{\bar{h}}(\cdot)$ is $E_{\bar{h}}(\cdot)$ itself (see
 575 [Yaari, 1987](#), pp. 101) and the risk premium is $\pi_{\bar{h}}(R) = E(R) - E_{\bar{h}}(R)$ (see [Denuit et al.,](#)
 576 [1999](#)). The risk premium can also be derived from $E_{\bar{h}}(\cdot)$ using the indifference arguments as
 577 in [Denuit et al. \(2006\)](#) and [Tsanakas and Desli \(2003\)](#). Formally, the risk premium $\varrho_{\pi_h}(R)$
 578 is determined such that

$$579 \quad E_{\bar{h}}(r) = E_{\bar{h}}(r - R + \varrho_{\pi_h}(R)). \quad (29)$$

580 From [\(29\)](#), one obtains

$$581 \quad \pi_h(R) = -E_{\bar{h}}(-R) = E_h(R) = \int_{-\infty}^0 (h(\bar{F}_R(r)) - 1) dr + \int_0^{\infty} h(\bar{F}_R(x)) dx. \quad (30)$$

582 $\pi_h(\cdot)$ is also known as the distortion risk measure and is equivalent to the spectral risk
 583 measure (see [Gzyl and Mayoral, 2008](#); [Sereda et al., 2010](#)). $\varrho_{C_{\bar{h}}} = -C_{\bar{h}}(R)$ is the risk measure
 584 induced by $C_{\bar{h}}(\cdot)$, $\varrho_{C_{\bar{h}}} = \pi_{\bar{h}}(-R)$ is the risk measure induced by $\pi_{\bar{h}}(\cdot)$ and $\varrho_{\pi_h} = \pi_h(-R)$ is
 585 the risk measure induced by $\pi_h(R)$.

586 The definition of compatibility with the PFD in the DU theory is based on $\underline{\varrho}_l(\mathbf{w}|\mathbf{R})$, for
 587 each $l \in \{C_{\bar{h}}, \pi_{\bar{h}}\}$ and is defined as follows.

588 **DEFINITION 5 (COMPATIBILITY WITH PFD IN THE DU THEORY).** Our axioms are com-
 589 patible with the PFD in the DU theory if they are satisfied by $\underline{\varrho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ or $\underline{\varrho}_{\pi_{\bar{h}}}(\mathbf{w}|\mathbf{R})$.

590 **Proposition 3** examines this compatibility.

591 **PROPOSITION 3 (COMPATIBILITY WITH PFD IN THE DU THEORY).** *Our axioms are com-*
 592 *patible with the PFD in the DU theory if and only if $\bar{h}(\cdot)$ is convex or $h(\cdot)$ is concave.*

593 **Proposition 3** shows that our axioms can be rationalized by the DU theory of choice if
 594 and only if $\bar{h}(\cdot)$ is convex or equivalently $h(\cdot)$ is concave. This result provides another
 595 decision-theoretic foundation of our axioms and consequently strengthens their desirability,
 596 reasonableness and relevance. It also implies that any concave distortion risk measure
 597 induces a coherent diversification measure.

598 5 Existing Diversification Measures

599 In this section, we explore whether some useful methods of measuring *correlation* diversifica-
 600 tion satisfy our axioms. We consider the four *correlation* diversification measures used most
 601 frequently on the marketplace and by academic researchers in both finance and insurance:

(i) Portfolio variance

$$\sigma^2(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \Sigma \mathbf{w}.$$

(ii) *Diversification ratio* (DR)

$$\text{DR}(\mathbf{w}|\mathbf{R}) = \frac{\mathbf{w}^\top \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}}.$$

(iii) Embrechts et al.'s (2009) class of measures

$$D_\varrho(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N \varrho(w_i R_i) - \varrho(\mathbf{w}^\top \mathbf{R}).$$

602

(iv) Tasche's (2007) class of measures

$$\text{DR}_\varrho(\mathbf{w}|\mathbf{R}) = \frac{\varrho(\mathbf{w}^\top \mathbf{R})}{\sum_{i=1}^N \varrho(w_i R_i)}.$$

603 Portfolio variance is the risk measure in the mean-variance model. It is usually used to
 604 quantify the benefit of diversification (Markowitz, 1952, 1959; Sharpe, 1964), and is formally
 605 analyzed as a portfolio diversification measure in Frahm and Wiechers (2013).

606 The diversification ratio (DR) is a diversification measure introduced by Choueifaty and
 607 Coignard (2008); see also Choueifaty et al. (2013). An intuitive interpretation of the DR
 608 is the Sharpe ratio when each asset's volatility is proportional to its expected premium i.e.
 609 $E(R_i) - R_N = \delta \sigma_i$, for each $i \in \mathcal{I}_{N-1}$ where $\delta > 0$ and R_N is the rate of the risk-free asset.
 610 DR is used in the finance industry by the french firm TOBAM to manage billions worth of
 611 assets via its Anti-Benchmark[®] strategies in Equities and Fixed Income.

Embrechts et al.'s (2009) class (D_ϱ) is the class of diversification measures induced by a risk
 measure $\varrho(\cdot)$. An intuitive interpretation can be provided to D_ϱ when $\varrho(\cdot)$ is homogeneous

of degree one. In that case,

$$D_\varrho(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i (\varrho(R_i) - \varrho(\mathbf{w}^\top \mathbf{R})).$$

612 The term in parentheses, $\varrho(R_i) - \varrho(\mathbf{w}^\top \mathbf{R})$, measures the benefit of diversification, in terms
 613 of risk reduction, of holding portfolio \mathbf{w} instead of concentrating on single-asset i . It follows
 614 that D_ϱ quantifies the average benefit of diversification. [Tasche's \(2007\)](#) class of diversi-
 615 fication measures (DR_ϱ) is a normalized version of D_ϱ . Some authors (see [Degen et al.](#),
 616 [2010](#); [Mao et al., 2012](#)) refer to DR_ϱ as a measure of concentration risk and $1 - DR_\varrho$ as
 617 a measure of diversification benefit. D_ϱ and DR_ϱ are the most commonly used in the fi-
 618 nance and insurance literature (see [Bignozzi et al., 2016](#); [Dhaene et al., 2009](#); [Embrechts](#)
 619 [et al., 2013, 2015](#); [Tong et al., 2012](#); [Wang et al., 2015](#)) and recommended implicitly in some
 620 international regulatory frameworks (see [Basel Committee on Banking Supervision, 2010](#);
 621 [Committee on Risk Management and Capital Requirements, 2016](#)).

622 The following proposition analyzes these four measures in the light of our axioms.

623 PROPOSITION 4. *The following statements hold.*

- 624 (i) *Portfolio variance satisfies our axioms if and only if assets have the same standard*
 625 *deviation i.e $\sigma_i = \sigma_j$, $i, j = 1, \dots, N$.*
- 626 (ii) *The diversification ratio satisfies our axioms.*
- 627 (iii) *[Embrechts et al.'s \(2009\)](#) class of measures satisfies our axioms if and only if $\varrho(\cdot)$ is*
 628 *convex (or quasi-convex), positively homogeneous, translation invariant and reverse*
 629 *lower comonotonic additive.*
- 630 (iv) *[Tasche's \(2007\)](#) class of measures satisfies our axioms if and only if $\varrho(\cdot)$ is convex*
 631 *(or quasi-convex), positively homogeneous, translation invariant and reverse lower*
 632 *comonotonic additive.*

633 Part (i) of [Propositions 4](#) shows that the portfolio variance satisfies our axioms, but under
 634 very restrictive (if not impossible) conditions that assets have identical variances. More
 635 specifically, portfolio variance satisfies our axioms, except [Size Degeneracy](#), [Risk Degeneracy](#)
 636 and [Reverse Risk Degeneracy](#). This result, rather than weakening our axioms, reveals the
 637 limits of portfolio variance as an adequate measure of diversification in the mean-variance
 638 model.

639 Part (ii) of [Propositions 4](#) shows that the DR is coherent. Parts (iii) and (iv) show that [Em-](#)
 640 [brechts et al.'s \(1999\)](#) and [Tasche's \(2006\)](#) classes of measures also are coherent, but under
 641 the same conditions that the risk measure $\varrho(\cdot)$ is convex (or quasi-convex), homogeneous,

642 translation invariant and reverse lower comonotonic additive.⁴ These findings strengthen
 643 both the axioms and the measures. The axioms are applicable to a number of measures
 644 that we have considerable experience with. The measures have several properties whose de-
 645 sirability can be rationalized by the expected utility theory and Yaari's (1987) dual theory,
 646 and their popular use in empirical works and on the marketplace can be defended by our
 647 axioms.

648 Because Embrechts et al.'s (1999) and Tasche's (2006) classes of measures are diversification
 649 measures induced by risk measure, parts (iii) and (iv) also establish the conditions under
 650 which a coherent risk measure induces a coherent diversification measure.

651 COROLLARY 1 (COHERENT RISK MEASURE (ARTZNER ET AL., 1999)). *A coherent risk mea-*
 652 *sure induces a coherent diversification measure if and only if the coherent risk measure is*
 653 *reverse lower comonotonic additive.*

654 Corollary 1 follows from the fact that any coherent risk measure is convex, positively ho-
 655 mogeneous (with $\kappa = 1$), translation invariant and all coherent risk measures are not reverse
 656 lower comonotonic additive. An example of a coherent risk measure that is not reverse lower
 657 comonotonic additive is the expectation risk measure i.e. $\varrho(X) = E(X)$. An example of a
 658 coherent risk measure that is reverse lower comonotonic additive is any concave distortion
 659 risk measure as implied by Proposition 3. It follows that the expected shortfall, which is
 660 chosen over Value-at-Risk in Basel III (see Basel Committee on Banking Supervision, 2013),
 661 induces a coherent diversification measure in the case of continuous distribution.

662 Parts (iii) and (iv) of Propositions 4 also imply that the family of deviation risk measures of
 663 Rockafellar et al. (2006) induces a family of coherent diversification measures, but not the
 664 family of convex risk measures (see Follmer and Schied, 2002; Frittelli and Gianin, 2002,
 665 2005).

666 Part (iv) of Propositions 4 also supports the findings of Flores et al. (2017) that the *diversifi-*
 667 *cation delta* of Vermorken et al. (2012) is an inadequate measure of portfolio diversification.

668 6 Representation

669 To close this paper, we examine whether or not our axioms imply a family of representations.
 670 As mentioned in Section 3, from Marshall et al. (2011, C.2. and C.3. Propositions, pp
 671 97-98), Symmetry and Concavity or Quasi-Concavity taken together imply that $\Phi(\mathbf{w}|\mathbf{R})$ is
 672 Schur-concave in \mathbf{w} when the sequence R_1, \dots, R_N is exchangeable. Therefore, from Marshall
 673 et al. (2011, B.1. Proposition, pp 393), $\Phi(\mathbf{w}|\mathbf{R})$ can have the following representation form

⁴Embrechts et al.'s (1999) and Tasche's (2006) classes of measures satisfy our axioms under the same conditions because the Tasche's (2006) class of measures is a normalized version of the Embrechts et al.'s (1999) class of measures.

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675

$$\Phi(\mathbf{w}|\mathbf{R}) = \mathbb{E}(\phi(\mathbf{w}, \mathbf{R})), \quad (31)$$

676

where $\phi(\mathbf{w}, \mathbf{R})$ satisfies [Size Degeneracy](#), [Risk Degeneracy](#), [Reverse Risk Degeneracy](#), [Duplication Invariance](#), [Size Monotonicity](#), [Translation Invariance](#), [Homogeneity](#), and the following additional properties

678

679

(i) $\phi(\mathbf{w}, \mathbf{R})$ is concave in \mathbf{w} for each fixed $\mathbf{R} \in \mathcal{R}^N$;

680

(ii) $\phi(\mathbf{\Pi w}, \mathbf{\Pi R}) = \phi(\mathbf{w}, \mathbf{R})$ for all permutations $\mathbf{\Pi}$;

681

(iii) $\phi(\mathbf{w}, \mathbf{R})$ is Borel-measurable in \mathbf{R} for each fixed \mathbf{w} .

682

The first two properties ensure that $\phi(\mathbf{w}, \mathbf{R})$ is concave in \mathbf{w} and symmetric in \mathbf{w} when \mathbf{R} is exchangeable, therefore satisfying [Concavity](#) and [Symmetry](#). Below, we present two examples of $\phi(\mathbf{w}, \mathbf{R})$.

684

685

EXAMPLE 4 (RAO'S QUADRATIC ENTROPY). Consider $\phi(\mathbf{w}, \mathbf{R})$ such that

686

$$\phi(\mathbf{w}, \mathbf{R}) = \sum_{i,j=1}^N |(R_i - \mu_i) - (R_j - \mu_j)|^q w_i w_j, \quad 0 < q \leq 2 \quad (32)$$

687

Obviously, $\phi(\mathbf{w}, \mathbf{R})$ satisfies the three above properties and [Size Degeneracy](#), [Risk Degeneracy](#), [Reverse Risk Degeneracy](#), [Duplication Invariance](#), [Size Monotonicity](#), [Translation Invariance](#), and [Homogeneity](#). As a consequence $\Phi(\mathbf{w}|\mathbf{R}) = \mathbb{E}(\phi(\mathbf{w}, \mathbf{R}))$ is a coherent class of portfolio diversification measures analyzed in [Carmichael et al. \(2015\)](#) and [Carmichael et al. \(2018\)](#) under the name of *Rao's Quadratic Entropy*. In the case where $q = 2$, $\Phi(\mathbf{w}|\mathbf{R})$ coincides with the *diversification returns*, we see obtain the popular diversification measure analyzed in [Willenbrock \(2011\)](#), [Chambers and Zdanowicz \(2014\)](#), [Bouchev et al. \(2012\)](#), [Qian \(2012\)](#) and in [Fernholz \(2010\)](#) under the name *excess growth rate*.

692

695

EXAMPLE 5 ([EMBRECHTS ET AL.'S \(2009\)](#) CLASS OF DIVERSIFICATION MEASURES).

696

Consider $\phi(\mathbf{w}, \mathbf{R})$ such that

697

$$\phi(\mathbf{w}, \mathbf{R}) = \sum_{i,j=1}^N \max(w_i(-R_i - \text{VaR}_\theta(-R_i)), 0) - \max(-\mathbf{w}^\top \mathbf{R} - \text{VaR}_\theta(-\mathbf{w}^\top \mathbf{R}), 0) \quad p \in (0, 1), \quad (33)$$

698

where $\text{VaR}_p(X)$ is the Value-at-risk of X at level p . As we can see, $\phi(\mathbf{w}, \mathbf{R})$ satisfies the three above properties and [Size Degeneracy](#), [Risk Degeneracy](#), [Reverse Risk Degeneracy](#), [Duplication Invariance](#), [Size Monotonicity](#), [Translation Invariance](#), and [Homogeneity](#). As a consequence $\Phi(\mathbf{w}|\mathbf{R}) = \mathbb{E}(\phi(\mathbf{w}, \mathbf{R}))$ is a coherent class of portfolio diversification measure, and a special case of [Embretchs et al.'s \(2009\)](#) class of measures induced by the expected shortfall when F_X is a continuous distribution.

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703

704

Is the representation form in [\(31\)](#) unique? The following example provides evidence that it is not. It presents a diversification measure that satisfies our axioms, but does not have

705

706 the representation form (31).

707 EXAMPLE 6 (EMBRECHTS ET AL.'S (2009) CLASS OF DIVERSIFICATION MEASURES).

708 Consider $\Phi(\mathbf{w}|\mathbf{R})$ such that

$$709 \quad \Phi(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \boldsymbol{\sigma} - \sigma(\mathbf{w}). \quad (34)$$

710 It is straightforward to verify that $\Phi(\mathbf{w}|\mathbf{R})$ in (34) satisfies our axioms, but does not have
711 the representation form (31).

712 In sum, we have the following representation theorem.

713 PROPOSITION 5 (REPRESENTATION THEOREM). *If $\Phi(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then $\Phi(\mathbf{w}|\mathbf{R})$
714 can take the following representation form*

$$715 \quad \Phi(\mathbf{w}|\mathbf{R}) = \mathbb{E}(\phi(\mathbf{w}, \mathbf{R})), \quad (35)$$

716 where $\phi(\mathbf{w}, \mathbf{R})$ satisfies *Size Degeneracy, Risk Degeneracy, Reverse Risk Degeneracy, Du-*
717 *plication Invariance, Size Monotonicity, Translation Invariance, Homogeneity and the fol-*
718 *lowing additional properties*

- 719 (i) $\phi(\mathbf{w}, \mathbf{R})$ is concave in \mathbf{w} for each fixed $\mathbf{R} \in \mathcal{R}^N$;
- 720 (ii) $\phi(\mathbf{\Pi w}, \mathbf{\Pi R}) = \phi(\mathbf{w}, \mathbf{R})$ for all permutations $\mathbf{\Pi}$;
- 721 (iii) $\phi(\mathbf{w}, \mathbf{R})$ is Borel-measurable in \mathbf{R} for each fixed \mathbf{w} .

722 7 Concluding Remarks and Future Research

723 This paper provides an axiomatic foundation of the measurement of *correlation* diversifica-
724 tion in a one-period portfolio theory under the assumption that the investor has complete
725 information about the joint distribution of asset returns. We have specified a set of mini-
726 mum desirable axioms for measures of *correlation* diversification, and named the measures
727 satisfying these axioms coherent diversification measures.

728 We have shown that these axioms can be rationalized by (a) the expected utility theory if
729 and only if one of the following conditions is satisfied: (i) risk is small in the sense of Pratt
730 (1964) and absolute risk aversion is constant, or (ii) each asset returns distribution belongs
731 to a location-scale family and the certainty equivalent has a particular additive separable
732 form; (b) Yaari's (1987) dual theory if and only if its probability distortion function is
733 convex. These results provide the decision-theoretic foundations of our axiomatic system,
734 and consequently strengthen their desirability, reasonableness and relevance.

735 We have explored whether portfolio diversification measures such as portfolio variance,
736 diversification ratio, Embrechts et al.'s class of diversification measures and Tasche's class
737 of diversification measures, which are used on the marketplace to manage millions of US
738 dollars and are also in use in the academic world, satisfy those axioms. We have shown that

739 (i) portfolio variance satisfies our axioms, but under the very restrictive (if not impossible)
740 condition that the assets have identical variance; (ii) the diversification ratio satisfies our
741 axioms; (iii) Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification measures
742 satisfy our axioms, but under the conditions that the underlying risk measure is convex (or
743 quasi-convex), homogeneous, translation invariant and reverse lower comonotonic additive.
744 These results strengthen both the axioms and such measures as the diversification ratio and
745 Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification measures. However,
746 they reveal the limits of portfolio variance as an adequate measure of diversification in the
747 mean-variance model.

748 Finally, we have investigated whether or not our axioms have functional representations.
749 We have shown that our axioms imply a family of representations, but this family is not
750 unique.

751 Our objective is to offer the first step towards a rigorous theory of *correlation* diversification
752 measures. We believe that with our axiomatic system this is the case. A feasible and
753 desirable direction for future research is to investigate what further axioms could be added
754 or relaxed in order to provide a unique family of representations because our axiomatic
755 system does not.

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759 **A Appendix: Proofs**

 760 **A.1 Proposition 1**

761 Assume that risk is small. According to Pratt (1964), the approximation of the local risk
 762 premium of $u(\cdot)$ is $\pi_u(X) \simeq \frac{1}{2}\text{Var}(X)k(1 + E(X))$, where $k(x) = -\frac{u''(x)}{u'(x)}$ is the measure of
 763 local risk aversion of $u(\cdot)$ in a small risk scenario. It follows that $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) = -\underline{\rho}_{\pi_u}(\mathbf{w}|\mathbf{R}) =$
 764 $\frac{1}{2}(\sum_{i=1}^N w_i \sigma_i^2 k(1 + \mu_i) - \sigma^2(\mathbf{w})k(1 + \mu(\mathbf{w})))$. Now let us show that $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies our
 765 axioms if and only if $k(x)$ is a constant function.

 766 **A.1.1 Sufficiency**

767 Suppose that $k(x) = c$, $c > 0$. Then $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) = c(\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w}))$. Therefore, we consider
 768 $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})$ for the proof.

769 (C)- Since $\sigma^2(\mathbf{w})$ is convex on \mathbb{W} , $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R})$ is concave on \mathbb{W} .

770 (SD)- It is straightforward to verify that $\underline{\rho}_{C_u}(\boldsymbol{\delta}_i|\mathbf{R}) = \sigma_i^2 - \sigma_i^2 = 0 = \underline{\Phi}$, for each $i \in \mathcal{I}_N$.

771 (RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then, $\sigma_i = \sigma_j = \underline{\sigma}$ and $\rho_{ij} = 1$ for each $i, j \in \mathcal{I}_N$
 772 with $\underline{\sigma} > 0$. It follows that $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \underline{\sigma}^2(\sum_{i=1}^N w_i)^2 = 0 = \underline{\Phi}$.

773 (RRD)- Since $w_i \geq 0$, for each $i \in \mathcal{I}_N$ and $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i \|R_i - R(\mathbf{w})\|_2^2$, $\underline{\rho}_{C_u}(\mathbf{w}|\mathbf{R}) =$
 774 $0 \Leftrightarrow R_i = R(\mathbf{w})$, for each $i \in \mathcal{I}_N$. The result follows.

(DI)- Since $A_{N+1}^+ = A_k$, $A_i^+ = A_i$ for each $i \in \mathcal{I}_N$,

$$\begin{aligned} \underline{\rho}_{C_u}(\mathbf{w}_{A^+}|\mathbf{R}_{A^+}) &= \sum_{i=1}^{N+1} w_{A_i^+} \sigma_{A_i^+}^2 - \sum_{i,j=1}^{N+1} w_{A_i^+} w_{A_j^+} \sigma_{A_i^+} \sigma_{A_j^+} \\ &= \sum_{i \neq k=1}^{N-1} w_{A_i^+} \sigma_{A_i^+}^2 + (w_{A_{N+1}^+} + w_{A_k^+}) \sigma_{A_k^+}^2 \\ &\quad - \sum_{i,j \neq k=1}^{N-2} w_{A_i^+} w_{A_j^+} \sigma_{A_i^+} \sigma_{A_j^+} - \sum_{i=1}^{N-2} w_{A_i^+} (w_{A_k^+} + w_{A_{N+1}^+}) \sigma_{A_i^+} \sigma_{A_k^+}. \end{aligned}$$

Let $\mathbf{w}_{A^+}^{**} = (w_{A_1^+}^*, \dots, w_{A_{k-1}^+}^*, w_{A_k^+}^* + w_{A_{N+1}^+}^*, w_{A_{k+1}^+}^*, \dots, w_{A_N^+}^*)$ and $\mathbf{w}_{A^+}^* = (w_{A_1}^*, \dots, w_{A_{k-1}}^*, \frac{w_{A_k}^*}{2}, w_{A_{k+1}}^*, \dots, \frac{w_{A_k}^*}{2})$.
 It follows that

$$\begin{aligned} \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^*|\mathbf{R}_{A^+}) &= \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^{**}|\mathbf{R}_A) \leq \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^*|\mathbf{R}_A), \\ \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^*|\mathbf{R}_A) &= \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^{**}|\mathbf{R}_{A^+}) \leq \underline{\rho}_{C_u}(\mathbf{w}_{A^+}^*|\mathbf{R}_{A^+}). \end{aligned}$$

Then

$$\begin{aligned} \underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}^+}^* | \mathbf{R}_{\mathcal{A}^+}) &= \underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}}^* | \mathbf{R}_{\mathcal{A}}) \\ w_{A_i}^* &= w_{A_i^+}^*, \text{ for each } i \neq k, i \in \mathcal{I}_N \\ w_{A_k}^* &= w_{A_k^+}^* + w_{A_{N+1}^+}^*. \end{aligned}$$

775 (M)- Consider a portfolio $\mathbf{w}_{\mathcal{A}^{++}} = (\mathbf{w}_{\mathcal{A}}^*, 0)$. Portfolio $\mathbf{w}_{\mathcal{A}^{++}}$ is an element of $\mathbb{W}_{\mathcal{A}^{++}}^{N+1}$, so
 776 $\underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}^{++}}^* | \mathbf{R}_{\mathcal{A}^{++}}) \geq \underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}^{++}} | \mathbf{R}_{\mathcal{A}^{++}})$. Since $\underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}^{++}} | \mathbf{R}_{\mathcal{A}^{++}}) = \underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}}^* | \mathbf{R}_{\mathcal{A}})$, $\underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}^{++}}^* |$
 777 $\mathbf{R}_{\mathcal{A}^{++}}) \geq \underline{\rho}_{C_u}(\mathbf{w}_{\mathcal{A}}^* | \mathbf{R}_{\mathcal{A}})$.

778 (TI,H)- Because covariance is translation invariant and homogeneous of degree two.

779 (S)- Since $\sigma_i = \sigma_j = \underline{\sigma}$ and $\rho_{ij} = \underline{\rho}$ for each $i, j \in \mathcal{I}_N$ when R_1, \dots, R_N is exchangeable,
 780 $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R}) = \underline{\sigma}^2 - \underline{\sigma}^2 (\sum_{i=1}^N w_i^2 + \underline{\rho} \sum_{i,j=1}^N w_i w_j)$. It is straightforward to verify that $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R})$
 781 is symmetric.

782 A.1.2 Necessity

783 For the converse, suppose that $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R})$ satisfies our axioms and show that $k(x)$ is a
 784 constant function. To do so, we proceed by contradiction. Suppose that $k(x)$ is not a
 785 constant function. It is straightforward to verify that $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R})$ satisfies translation in-
 786 variance and homogeneity if and only if $k(1+x)$ is translation invariant and homogeneous,
 787 which is the case only if $k(x)$ is a constant function. This contradicts our hypothesis that
 788 $k(x)$ is constant. As a consequence, $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R})$ satisfies our axioms, which implies that
 789 $k(x) = c, c > 0$.

790 A.2 Proposition 2

791 A.2.1 Sufficiency

792 Follow from the proof of the sufficiency part of [Proposition 1](#).

793 A.2.2 Necessity

794 Since asset i returns distributions belong to the location-scale family, form [Meyer et al.](#)
 795 (1987), $C_u(R) = u^{-1}(U(\mu, \sigma))$, where $E_u(R) = U(\mu, \sigma) = \int u(\mu + \sigma x) dx$ and $u^{-1}(\cdot)$ is
 796 the inverse of $u(\cdot)$. It is obvious that if $\underline{\rho}_{C_u}(\mathbf{w} | \mathbf{R})$ satisfies our axioms, then $C_u(R) =$
 797 $u^{-1}(U(\mu, \sigma)) = \mu - g(\sigma)$ with $g(\cdot)$ is a strictly increasing, continuous and homogeneous
 798 function on \mathbb{R}_+ .

799 **A.3 Proposition 3**

800 We focus only on $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$.

801 **A.3.1 Sufficiency**

802 Suppose that $\bar{h}(\cdot)$ is convex and let us show that $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

803 (C)- Since $\bar{h}(\cdot)$ is convex, $C_{\bar{h}}(\cdot)$ is convex on \mathcal{R} (Tsanakas and Desli, 2003). It follows that
804 $C_{\bar{h}}(\mathbf{w}|\mathbf{R})$ is convex on \mathbb{W} and consequently, $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ is concave.

805 (SD)- Let $\delta_i \in \mathbb{W}$ be a single-asset i portfolio. It is straightforward to show that $\underline{\rho}_{C_{\bar{h}}}(\delta_i|\mathbf{R}) =$
806 $C_{\bar{h}}(R_i) - C_{\bar{h}}(R_i) = 0 = \underline{\Phi}$.

807 (RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then, $\underline{\rho}_{C_{\bar{h}}}(R_i) = \underline{\rho}_{C_{\bar{h}}}(R_j)$ for each $i, j \in \mathcal{I}_N$. It
808 follows that $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R}) = C_{\bar{h}}(R) - C_{\bar{h}}(R) = 0 = \underline{\Phi}$.

809 (RRD)- Since $C_{\bar{h}}(R)$ is coherent, comonotonic additive and non-independent additive,
810 $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ satisfies **Reverse Risk Degeneracy**.

811 (DI)- Follows the proof of **Proposition 1**.

812 (M)- Follows the proof of **Proposition 1**.

813 (TI,H)- Since $\bar{h}(\cdot)$ is convex, $C_{\bar{h}}(R)$ is translation invariant and homogeneous of degree
814 one. Therefore $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ is translation invariant and homogeneous of degree one.

815 (S)- Suppose that R_1, \dots, R_N is exchangeable. It is straightforward to verify that $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$
816 is symmetric.

817 **A.3.2 Necessity**

818 For the converse, suppose that $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ satisfies our axioms and let us show that $\bar{h}(\cdot)$
819 is convex. To do so, we proceed by contradiction. Suppose that $h(\cdot)$ is not convex. It is
820 straightforward to verify that $\underline{\rho}_{C_{\bar{h}}}(\mathbf{w}|\mathbf{R})$ is not concave (Wang et al., 1997).

821 **A.4 Proposition 4**822 **A.4.1 Portfolio variance**

823 **A.4.1.1 Sufficiency** Suppose that assets have identical variances and show that the port-
824 folio variance satisfies our axioms. It is straightforward to verify that if assets have identical
825 variances i.e $\sigma_i^2 = \underline{\sigma}^2$, then

$$826 \quad \mathbf{w}^\top \underline{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}). \quad (36)$$

827 From (36) and Proposition 1, it follows that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

828 **A.4.1.2 Necessity** For the converse, suppose that $\sigma^2(\mathbf{w}|\mathbf{R})$ our axioms and show that
 829 assets have identical variances. To do so, we proceed by contradiction. Suppose that asset
 830 variances are not identical and without the loss of generality that $N = 2$ such that $\sigma_1^2 < \sigma_2^2$.
 831 Then $\sigma^2(\delta_1|\mathbf{R}) < \sigma^2(\delta_2|\mathbf{R})$. Thus $\sigma^2(\mathbf{w}|\mathbf{R})$ fails **Size Degeneracy**. From the failure of
 832 **Size Degeneracy**, it is straightforward to prove that $\sigma^2(\mathbf{w}|\mathbf{R})$ also fails **Risk Degeneracy**
 833 and **Reverse Risk Degeneracy**. This contradicts our hypothesis that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our
 834 axioms. As a consequence, if $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then assets have identical
 835 variances.

836 A.4.2 Diversification ratio

837 Because the standard-deviation is convex, positive homogeneous (with $\kappa = 1$), translation
 838 invariant (with $\eta = 0$) and is reverse lower comonotonic additive, from part (iv) of Proposi-
 839 tion 4, DR_σ satisfies our axioms. It follows that $\text{DR}(\mathbf{w}|\mathbf{R}) = \frac{1}{\text{DR}_\sigma}(\mathbf{w}|\mathbf{R})$ also satisfies our
 840 axioms.

841 A.4.3 Embrechts et al.'s (2009) class measures

842 See the proof of Proposition 3.

843 A.4.4 Tasche's (2007) class measures

844 (QC)- Since $\varrho(\cdot)$ is convex and $\sum_{i=1}^N w_i \varrho(R_i)$ is linear on \mathbb{W} , from Avriel et al. (2010),
 845 $\text{DR}_\varrho(\mathbf{w}|\mathbf{R})$ is quasi-concave.

846 (SD)- $\text{DR}_\varrho(\delta_i|\mathbf{R}) = \frac{\varrho(R_i)}{\varrho(R_i)} = 1$, for each $i \in \mathcal{I}_N$.

847 (RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then $\text{DR}_\varrho(\delta_i|\mathbf{R}) = \frac{\varrho(R)}{\varrho(R)} = 1 = \underline{\Phi}$ for each $i \in \mathcal{I}_N$.

848 (RRD)- By assumption that $\varrho(\cdot)$ is reverse upper comonotonic additive.

849 (DI)- Follows the proof of Proposition 1.

850 (M)- Follows the proof of Proposition 1.

(TI)- Since $\varrho(\cdot)$ is translation invariant i.e. $\varrho(R + a) = \varrho(R) - \eta a$ with R is a random variable,

$$\text{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \frac{\varrho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}) - \eta a}{\mathbf{w}^\top \varrho(\mathbf{R}_{\mathcal{A}}) - \eta a}.$$

851 If $\eta = 0$,

$$852 \text{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \text{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}}).$$

If $\eta \neq 0$,

$$\begin{aligned} \frac{\partial \text{DR}_\rho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a})}{\partial a} &= \frac{\eta(\rho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}) - \mathbf{w}^\top \rho(\mathbf{R}_{\mathcal{A}}))}{(\mathbf{w}^\top \rho(\mathbf{R}_{\mathcal{A}}) - \eta a)^2} \leq 0, \\ \lim_{a \rightarrow -\infty} \text{DR}_\rho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) &= 1, \\ \lim_{a \rightarrow +\infty} \text{DR}_\rho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) &= 1, \\ \lim_{\substack{a \rightarrow \frac{\rho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta} \\ a > \frac{\rho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta}}} \text{DR}_\rho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) &= -\infty, \\ \lim_{\substack{a \rightarrow \frac{\rho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta} \\ a < \frac{\rho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}})}{\eta}}} \text{DR}_\rho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) &= \infty. \end{aligned}$$

(H)- If $\rho(\cdot)$ is homogeneous i.e. $\rho(bR) = b^\kappa \rho(R)$ with R is a random variable,

$$\begin{aligned} \text{DR}_\rho(\mathbf{w}|b\mathbf{R}) &= \frac{\rho(b\mathbf{w}^\top \mathbf{R})}{\mathbf{w}^\top \rho(b\mathbf{R})}, \\ &= \frac{b^\kappa \rho(\mathbf{w}^\top \mathbf{R})}{b^\kappa \mathbf{w}^\top \rho(\mathbf{R})}, \\ \text{DR}_\rho(\mathbf{w}|b\mathbf{R}) &= \text{DR}_\rho(\mathbf{w}|\mathbf{R}). \end{aligned}$$

853 (S)- Follows the proof of [Proposition 3](#).

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