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Pricing and order quantity of substitutes in two inventory-related markets

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Abstract: We determine optimal pricing and order quantity of two substitute products in two markets, one of them is seasonal, with a decreasing market potential over time, and the other is non-seasonal. The two markets are sealed, that is, the prices in one market are irrelevant to consumers in the other market. Still, the two markets are linked through inventories and the order quantity policy. In the paper, we develop a non-linear model and provide an algorithm to compute the optimal solution. An illustrative example is given along with a sensitivity analysis.

Keywords: Inventory-related markets, pricing, order quantity, product substitution

Résumé : Nous déterminons le prix optimal et la quantité commandée de deux produits substituts offerts dans deux marchés, dont l'un est saisonnier avec un potentiel de marché décroissant dans le temps, et l'autre est non saisonnier. Les deux marchés sont scellés, c'est-à-dire que les prix dans un marché ne sont pas pertinents pour les consommateurs de l'autre marché. Néanmoins, les deux marchés sont couplés par les inventaires et la politique de commande. Dans cet article, nous proposons un modèle non linéaire et fournissons un algorithme de calcul de la solution optimale. Nous illustrons l'approche et l'algorithme par un exemple et procédons à une analyse de sensibilité

Mots clés : Marchés liés aux stocks, tarification, quantité commandée, produits substituts

1 Introduction

In this paper, we consider a firm selling two substitutable products in two markets: a seasonal market where the demand is positive only during part of the planning horizon (e.g., a year), and a nonseasonal (or regular) market with positive sales during the whole period. An illustrative example are markets for heating and cooling systems that are only purchased during a season by households, but are demanded throughout the year by entrepreneurs in the construction sector. To account for consumer's preference for having earlier the product, we let the market potential in the seasonal market to decrease over time. We assume that the firm sells the same set of products in the two (sealed) markets through parallel channels, which means that the prices in one market are irrelevant to the buyers in the other market. However, sales in the two markets are linked through inventories. Consequently, pricing and order quantities must be decided simultaneously.

Our objective is to answer the following research questions:

1. What is the optimal pricing policy in the two markets?
2. What is the ordering quantity policy of the two products?
3. How these policies vary with the model's parameter values?

There is a sizeable literature, not to say it is a field in itself, on each of the keywords involved in this research, namely, pricing, product substitution, inventory management and seasonal market. Clearly, reviewing this literature in detail is not only beyond the scope of the paper, but can mislead the readers by putting them on faraway tracks.[1] Instead, we shall focus on the most relevant contributions to ours. Table 1 provides a list of these papers along some main characteristics. First, in terms of decision variable(s), all retained papers deal with pricing and/or ordering quantity. As in most papers listed in Table 1, here we consider both. Second, in all listed papers, there is at least one seasonal market/product, which can be perishable as in, e.g., Shaikh et al. (2019), Taleizadeh and Rasuli-Baghban (2018) and Lu et al. (2016). In our research, the products are not perishable and are also offered in a regular non-seasonal market. Third, with the exception of Rasouli and Nakhai Kamalabadi (2014), all listed papers do not consider product substitution. Pricing substitutes offered by the same firm renders the pricing policy a bit more complex than in the case of a single-product firm. Offering only one product is nested in our model as it would correspond to a case where the firm sets the price of the substitute at a sufficiently high price to drive its demand to zero. Fourth, the number of price (or order) changes during the planning horizon. Determining this number and the magnitude of each change is a challenging problem in multi-product context. To start with, we let this number be a parameter and determine the best course of action by running a sensitivity analysis. Five, we note that few papers assumed quite realistically a stochastic environment. Here, we assume a deterministic environment for simplicity. Finally, we list some other features that help in highlighting the similarities and the differences between this paper and those appearing in Table 1. To warp up, our main contribution lies in determining optimal pricing and order quantity in two coupled seasonal and non-seasonal markets, which, to the best of our knowledge, has not been considered before despite being a setup that is observed in reality.

The rest of the paper is organized as follows: The mathematical model is described in section 2. The optimal solution and an algorithm are given in Section 3. In section 4, we provide a numerical example along with a sensitivity analysis. Finally, we briefly conclude in Section 5.

[1]For instance, when the seasonal product is perishable and the variable cost is negligible, we are then clearly in the realm of dynamic pricing and revenue management (see, e.g., the surveys in Weatherford and Bodily (1992), Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Talluri and Van Ryzin (2006)). Our products are not perishable per say and the cost is not negligible.

For a survey on inventories for perishable products, see Chaudhary et al. (2018).

Table 1: Summary of most related articles

Paper	Decision variable(s)	Number of markets	Substitution	When to change the price/order quantity	Number of products	Deterministic /stochastic	Other features
Netessine (2006)	Price	One seasonal market	No	Found by enumeration	1	Deterministic	Inventory and capacity constraints No salvage value
You and Chen (2007)	Price and order quantity	One seasonal market	No	Found by enumeration	1	Deterministic	Spot and forward purchases No salvage value
Banerjee and Sharma (2010a)	Price and replenishment cycle	One seasonal market	No	Found by enumeration	1	Deterministic	Partial backlogging
Soysal and Krishnamurthi (2012)	Price	One seasonal market	No	Given	n	Stochastic	Lost sale Strategic customers
Lee (2014)	Price and order quantity	One seasonal market	No	Given	1	Stochastic	Lost sale Salvage value
Shaikh et al. (2019)	Replenishment cycle and time that stock gets zero	One market with perishable goods	No	Variable (Length of replenishment)	1	Deterministic	Discount of purchasing cost Partial backlogging
Rasouli and Nakhai Kamal-abadi (2014)	Price and order quantity	One seasonal market	Yes	Found by enumeration	2(with price and order quantity given for one of them)	Deterministic	No salvage value
Taleizadeh and Rasuli-Baghban (2018)	Price, order quantity and dispatch cycle length	One market with perishable goods	No	No change (Fixed pricing and ordering for whole time horizon)	1	Deterministic	Group dispatching with given number of periods No salvage value
Grewal et al. (2015)	Order quantity and replenishment cycle	One seasonal market	No	Number of periods is given but length of periods for ordering is variable	n	Stochastic	Capacity constraint Simulation-Optimization approach
Banerjee and Sharma (2010b)	Price for both of the markets and order quantity for whole time horizon	Two seasonal markets	No	Found by enumeration	1	Deterministic	Salvage value in the first market and partial backlogging in the second market

Table 1: Summary of most related articles

Paper	Decision variable(s)	Number of markets	Substitution	When to change the price/order quantity	Number of products	Deterministic /stochastic	Other features
Bitran and Mondschein (1997)	Price	One seasonal market	No	Given	1	Stochastic	Initial given inventory Price discount Salvage value
Lu et al. (2016)	Price, order quantity and length of periods	One market with perishable goods	No	Variable	1	Deterministic	No salvage value
You (2005)	Order quantity, replenishment cycle and shortages cycle	One seasonal market	No	Determining the max number of n	1	Deterministic	Backlogging Capacity constrain
You et al. (2010)	Price and order quantity	One seasonal market	No	Given	n	Deterministic	Considering returns from customers Salvage value

2 Model

We consider a firm selling two substitutable products in two markets, one is seasonal, to which we shall refer to by s, and one nonseasonal market, to which we shall refer by ns. The planning horizon is normalized to one (e.g., one year). We denote by S the time during which both markets are active, and by $\bar{S} = 1 - S$ the time during which only the nonseasonal (or regular) market is active. We make the following assumptions:

A1: The market potential in the seasonal market is decreasing over time. The market potential in the nonseasonal (or regular) market is independent of time.

A2: The price in the seasonal market can be changed K times, whereas the price in the regular market does not vary over time.

A3: The same two products are sold in both markets.[2]

A4: The two markets are sealed and cross selling is not possible.[3]

A5: The model's parameter values are deterministic and known.

A6: Shortages and backlogs (at prevailing prices) are not allowed.

The above assumptions are intuitive. Indeed, A1-A2 reflect what is observed in many seasonal markets, namely: (i) consumers prefer to have earlier than later the product to enjoy it for as long as possible; (ii) the duration of the season is well defined; and (iii) prices vary over time (most likely are discounted as the season advances). A3 and A4 allow us to model intra-market substitution and inter-market independence in terms of prices, while being still linked through inventories. A5 and A6 are technical assumptions made to avoid additional complexities that would not add much qualitative insight into the problem we are studying.

2.1 Demand

We assume that the demand function in the nonseasonal market is given by

$$d_i^{ns} = a_i^{ns} - \beta_i^{ns} p_i^{ns} + \lambda^{ns} p_{3-i}^{ns}, \quad i = 1, 2, \tag{1}$$

where $a_i^{ns} > 0$ represents the market potential, $\beta_i^{ns} > 0$ the direct-price sensitivity and $\lambda^{ns} > 0$ is the cross-price parameter. The demand for product i is decreasing in its price and increasing in the price of the substitute, with $\beta_i^{ns} > \lambda^{ns}, i = 1, 2$. The linear form in (1) is often used in the literature and is typically justified on the ground that it can be derived from consumer's maximization of a quadratic utility function.

The market potential in the seasonal market is time-dependent, and the demand varies with both prices. Suppose that the season is divided into K subperiods. At the beginning of each subperiod, the firm orders a quantity of each product and sets its price. We shall index these subperiods by $k = 1, \ldots, K$. We denote by t the time running between k and $k+1$, during which the inventories are decreasing. Following the literature (You and Chen (2007) and Rasouli and Nakhai Kamalabadi (2014)), we adopt the following specification:

$$d_{k,t,i}^s = a_i^s e^{-g_i\left((k-1)\frac{S}{K}+t\right)} - \beta_i^s p_{k,i}^s + \lambda^s p_{k,3-i}^s \qquad t = 0, \ldots, \frac{S}{K}, \qquad k = 1, \ldots, K, \tag{2}$$

where a_i^s is the market potential at the start of the season (i.e., when $t = 0$ and $k = 1$), β_i^s and λ^s are nonnegative parameters having the same meanings as their ns counterparts. The positive parameter g_i measures the speed at which the market potential declines over the season.

[2] There is no conceptual difficulty in extending our approach to more than two products.

[3] One can think of the nonseasonal market as a wholesale market where only enterpreneurs can buy.

To satisfy the demand for the two products in the two markets at each time period, the firm must order the corresponding quantities. At the beginning of period k during season S, the order quantity is given by:

$$q^S_{k,i} = \frac{S}{K} d^{ns}_i + \int_0^{\frac{S}{K}} d^s_{k,t,i} dt, \tag{3}$$

where $\frac{S}{K} d^{ns}_i$ is the fraction of nonseasonal demand occurring in period k, and $\int_0^{\frac{S}{K}} d^s_{k,t,i} dt$ is the cumulative seasonal demand in that period. In the rest of the paper, the cumulative seasonal demand is denoted $d^s_{k,i}$

The ordering policy, i.e., the number of orderings and the quantity to order during $\bar{S}$, will be determined endogenously. Denote by $O^{\bar{S}}_i = \frac{\bar{S} d^{ns}_i}{q^{\bar{S}}_i}$ the number of orderings of product i during the time interval $\bar{S} = 1 - S$. The inventories of product i in the two markets evolve as follows:

$$I^{\bar{S}}_{t,i} = q^{\bar{S}}_i - t d^{ns}_i, \qquad 0 \le t \le \frac{\bar{S}}{O^{\bar{S}}_i}, \tag{4}$$

$$I^S_{k,t,i} = q^S_{k,i} - \int_0^t d^s_{k,t,i} dt - t d^{ns}_i, \qquad 0 \le t \le \frac{S}{K}. \tag{5}$$

Figure 1 shows the evolution over time of the inventory of a product. During the time interval $\bar{S}$, only the nonseasonal market is active and the inventory decreases linearly during this period (see (4)). During period S, both demands are positive and the inventory decreases exponentially over time (see (5)).

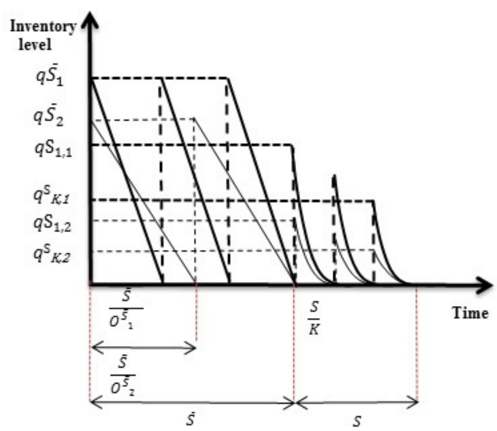

Figure 1: Inventory level of products

Denote by A_i and h_i the per-unit ordering and holding cost of product i, respectively. Let c_i be the purchasing (or production) cost of product i. Assuming profit-maximization behavior, the firm's optimization problem is as follows:

$$\max_{p^s_{k,i},\, p^{ns}_i,\, q^{\bar{S}}_i} \pi = \sum_{i=1}^{2} \sum_{k=1}^{K} \left(\left(p^s_{k,i} - c_i\right) \int_0^{\frac{S}{K}} d^s_{k,t,i} dt \right) + \sum_{i=1}^{2} (p^{ns}_i - c_i) d^{ns}_i - \sum_{i=1}^{2} h_i \sum_{k=1}^{K} \int_0^{\frac{S}{K}} I^S_{k,t,i}\, dt$$
$$- \frac{\bar{S}}{2} \sum_{i=1}^{2} h_i q^{\bar{S}}_i - \sum_{i=1}^{2} A_i \left(\frac{\bar{S} d^{ns}_i}{q^{\bar{S}}_i} + K \right), \tag{6}$$

$$\text{Subject to:} \quad p^{ns}_i \ge 0, \qquad \text{for} \quad i = 1, 2, \tag{7}$$

$$p^s_{k,i} \ge 0, \qquad \text{for} \quad i = 1, 2, \quad k = 1, \ldots, K, \tag{8}$$

$$d_i^{ns} \geq 0, \qquad \text{for} \quad i = 1, 2, \tag{9}$$
$$d^s_{k,\frac{S}{K},i} \geq 0, \qquad \text{for} \quad i = 1, 2, \quad k = 1, \ldots, K. \tag{10}$$

The first and second terms in (6) give the net revenues from selling the two products in the seasonal and nonseasonal market, respectively. The third and fourth terms correspond to the annual inventory holding cost during the time intervals S and $\bar{S}$, respectively. Finally, the last term measures the ordering cost of the two products. The positivity of the variables and demands are stated as constraints in (7)–(10).

Before solving the model, we make the following:

Remark 1 *For simplicity, we abusively refer to $O_i^{\bar{S}}$ as the number of orderings, notwithstanding that $O_i^{\bar{S}}$ is not an integer number. We note that $O_i^{\bar{S}}$ is not directly optimized but results from computing the ratio $\frac{\bar{S}d_i^{ns}}{q_i^{\bar{S}}}$.*

Remark 2 *As $d^s_{k,t,i}$ is decreasing over time, the demand reaches its minimum $d^s_{k,\frac{S}{K},i}$ in each sub period. Therefore, it is sufficient to consider the constraint $d^s_{k,\frac{S}{K},i} \geq 0$.*

Table 2 summarizes the notation used in the paper.

Table 2: Notation

Indices and labels	
i	Product ($i = 1, 2$)
k	Subperiod ($k = 1, \ldots, K$)
s	Seasonal market
ns	Nonseasonal market
t	Time
S	Percentage of time during which both markets are active
$\bar{S}$	Percentage of time during which only the nonseasonal market is active
Demand Parameters	
a_i^s (a_i^{ns})	Market potential of product i in market s (ns)
β_i^s (β_i^{ns})	Direct-price effect, product i in market s (ns)
λ^s (λ^{ns})	Cross-price effect in market s (ns)
g_i	Speed of decrease of seasonal market potential of product i
Cost parameters	
c_i	Unit purchasing cost of product i
A_i	Unit ordering cost of product i
h_i	Unit holding cost of product i
Decision variables	
$p^s_{k,i}$	Price of product i in subperiod k in seasonal market
p_i^{ns}	Price of product i in nonseasonal market
$q_i^{\bar{S}}$	Order quantity of product i for time interval $\bar{S}$
Other variables	
$q^S_{k,i}$	Order quantity of product i in subperiod k for time interval S
$d^s_{k,t,i}$	Demand at time t during subperiod k of product i in seasonal market
d_i^{ns}	Demand of product i in nonseasonal market
$I^{\bar{S}}_{t,i}$	Available inventory of product i at time t for time interval $\bar{S}$
$I^S_{k,t,i}$	Available inventory of product i at time t in period k for time interval S
$O_i^{\bar{S}}$	Number of orderings of product for time interval $\bar{S}$
π	Total profit

3 Optimal solution

In this section, we characterize analytically the properties of the optimal solution and next propose a simple algorithm to compute it. To start with, we have the following:

Lemma 1 *If $A_i > \frac{\bar{S}^4 \beta_i^{ns} h_i^2}{8(O_i^{\bar{S}})^3}, i = 1, 2,$ then the objective function in (6) is concave.*

Proof. See Appendix. □

The condition in the statement of the above lemma, which is not restrictive in view of $\bar{S} < 1$ and $O_i^{\bar{S}} \geq 1$, will be enforced in the numerical examples.

To obtain the optimal solution, we proceed in two steps. First, we assume that the solution is interior and next complete the analysis by accounting for the constraints that may be binding. Differentiating the profit function with respect to the decision variables and equating to zero, we get for $i = 1, 2$ and $k = 1, \ldots, K$ the following expressions:

$$\frac{\partial \pi}{\partial p_i^{ns}} = a_i^{ns} - 2\beta_i^{ns} p_i^{ns} + 2\lambda^{ns} p_{3-i}^{ns} + A_i \bar{S} \frac{\beta_i^{ns}}{q_i^{\bar{S}}} - A_{3-i} \bar{S} \frac{\lambda^{ns}}{q_{3-i}^{\bar{S}}} + \frac{S^2}{2K}(h_i \beta_i^{ns} - h_{3-i}\lambda^{ns}) + c_i \beta_i^{ns} - c_{3-i}\lambda^{ns} = 0, \quad \text{for } i = 1, 2, \tag{11}$$

$$\frac{\partial \pi}{\partial p_{k,i}^{s}} = \frac{e^{\frac{-kSg_i}{K}} \left(-1 + e^{\frac{Sg_i}{K}}\right) a_i^s}{g_i} + \frac{S(-2\beta_i^s p_{k,i}^s + 2\lambda^s p_{k,3-i}^s)}{K} + \frac{S}{K}(c_i \beta_i^s - c_{3-i}\lambda^s) + \frac{S^2}{2K^2}(h_i \beta_i^s - h_{3-i}\lambda^s) = 0, \quad \text{for } i = 1, 2, \quad k = 1, \ldots, K, \tag{12}$$

$$\frac{\partial \pi}{\partial q_i^{\bar{S}}} = \frac{-1}{2}\bar{S} h_i + A_i \frac{\bar{S} d_i^{ns}}{(q_i^{\bar{S}})^2} = 0, \quad \text{for } i = 1, 2. \tag{13}$$

Solving these equations yields

$$p_i^{ns*} = \frac{\bar{S} A_i}{2 q_i^{\bar{S}*}} + \frac{1}{4}\left(\frac{2a_i^{ns}\beta_{3-i}^{ns} + 2a_{3-i}^{ns}\lambda^{ns}}{\beta_i^{ns}\beta_{3-i}^{ns} - (\lambda^{ns})^2} + 2c_i + \frac{S^2 h_i}{K}\right) \quad \text{for } i = 1, 2, \tag{14}$$

$$p_{k,i}^{s*} = \frac{e^{\frac{-kSg_{3-i}}{K}} \left(-1 + e^{\frac{Sg_{3-i}}{K}}\right) a_{3-i}^s K g_i \lambda^s + e^{\frac{-kSg_i}{K}} \left(-1 + e^{\frac{Sg_i}{K}}\right) a_i^s K g_{3-i} \beta_{3-i}^s}{2S g_i g_{3-i} (\beta_i^s \beta_{3-i}^s - (\lambda^s)^2)} + \frac{2Kc_i + Sh_i}{4K}, \quad \text{for } i = 1, 2, \quad k = 1, \ldots, K, \tag{15}$$

$$q_i^{\bar{S}*} = \sqrt{\frac{2A_i d_i^{ns*}}{h_i}}, \quad \text{for } i = 1, 2. \tag{16}$$

These results call for three comments. First, we make the implicit assumption that $q_i^{\bar{S}*} \neq 0$; otherwise, the above equations are not valid. The order quantity $q_i^{\bar{S}*}$ is defined, only if the demand in market ns is non negative. Second, for all parameter values, the prices p_i^{ns*} and $p_{k,i}^{s*}$ are strictly positive. Third, the optimal prices in each market are interdependent, which affects the ordering quantity of each product. This result is of course due to the fact that, in each market, the two products are substitutes. Fourth, the optimal prices in the nonseasonal market, and hence the demands for the two products in this market, also depend on K, which is the number of sub periods in the seasonal market. Therefore, the two markets are coupled.

Substituting for the prices in the demand functions yields

$$d_i^{ns*} = \frac{a_i^{ns} - \left(c_i + \frac{S^2}{2K}h_i + \bar{S}\frac{A_i}{q_i^{\bar{S}*}}\right)\beta_i^{ns} + \left(c_{3-i} + \frac{S^2}{2K}h_{3-i} + \bar{S}\frac{A_{3-i}}{q_{3-i}^{\bar{S}*}}\right)\lambda^{ns}}{2}, \qquad \text{for } i = 1, 2, \tag{17}$$

$$d_{k,\frac{S}{K},i}^{s*} = \frac{e^{\frac{-kSg_i}{K}} a_i^s \left(2Sg_i - \left(-1 + e^{\frac{Sg_i}{K}}\right) K\right)}{2Sg_i} - \frac{(2Kc_i + Sh_i)\beta_i^s - (2Kc_{3-i} + Sh_{3-i})\lambda^s}{4K}, \quad \text{for } i = 1, 2, \quad k = 1, \ldots, K. \tag{18}$$

The expressions derived so far were obtained under the assumption of an interior solution, which is not yet fully established. The seasonal demand only depends on the parameter values. We shall not impose further conditions on these values to insure its positivity, but take care of it in the optimization problem defined below. The demand in the nonseasonal market given in (17) cannot be written in closed form. Indeed, it involves $q_i^{\bar{S}^*}$ that depends on d_i^{ns*}, which is function of p_i^{ns*} that, in turn, depends on $q_i^{\bar{S}^*}$. Instead of adding a non-negativity constraint, which would lead to many cases to analyze due to the possible multiplicity of solutions, we shall show that this demand is positive under a mild condition involving the average cost.

Denote by AC_i^{ns} the average cost of a product sold in the nonseasonal market. For $i = 1, 2$, the average cost is given by

$$AC_i^{ns} = (\text{purchasing cost})_i + (\text{average holding cost})_i^{ns} + (\text{average ordering cost})_i^{ns}. \tag{19}$$

Let us consider the three items on the right-hand side. The purchasing cost is c_i. The average holding cost is defined as follows:

$$\begin{aligned}&(\text{average holding cost})_i^{ns}\\ &\quad= \frac{1}{(\text{total demand})_i^{ns}} \Big(((\text{total demand in period } S)_i^{ns} \times (\text{average holding cost in period } S)_i^{ns}\\ &\qquad\qquad + (\text{total demand in period } \bar{S})_i^{ns} \times (\text{average holding cost in period } \bar{S})_i^{ns}\Big).\end{aligned} \tag{20}$$

Further, the average holding costs for periods S and $\bar{S}$ are given by

$$(\text{average holding cost in period } S)_i^{ns} = \frac{(\text{total holding cost in period } S)_i^{ns}}{(\text{total demand in period } S)_i^{ns}} = \frac{\frac{S^2 h_i d_i^{ns}}{2K}}{S d_i^{ns}}. \tag{21}$$

$$(\text{average holding cost in period } \bar{S})_i^{ns} = \frac{(\text{total holding cost in period } \bar{S})_i^{ns}}{(\text{total demand in period } \bar{S})_i^{ns}} = \frac{\frac{\bar{S} h_i q_i^{\bar{S}}}{2}}{\bar{S} d_i^{ns}}. \tag{22}$$

Substituting for (21) and (22) in (20), we get

$$(\text{average holding cost})_i^{ns} = S^2 \frac{h_i}{2K} + \bar{S} \frac{h_i q_i^{\bar{S}}}{2 d_i^{ns}} = S^2 \frac{h_i}{2K} + \bar{S}^2 \frac{h_i}{2 O_i^{\bar{S}}}. \tag{23}$$

The last equality follows from $O_i^{\bar{S}} = \frac{\bar{S} d_i^{ns}}{q_i^{\bar{S}}}$. Now, we determine the average ordering cost in the nonseasonal market, which is given by

$$\begin{aligned}&(\text{average ordering cost})_i^{ns}\\ &\quad= \frac{1}{(\text{total demand})_i^{ns}} ((\text{total demand in period } S)_i^{ns} \times (\text{average ordering cost in period } S)_i^{ns}\\ &\qquad\qquad + (\text{total demand in period } \bar{S})_i^{ns} \times (\text{average ordering cost in period } \bar{S})_i^{ns}).\end{aligned} \tag{24}$$

For the average ordering costs in periods S and $\bar{S}$, we have

$$\begin{aligned}(\text{average ordering cost in period S})_i^{ns} &= \frac{(\text{total ordering cost of period } S)_i}{(\text{total demand of period } S)_i}\\ &= \frac{A_i K}{\bar{S} d_i^{ns} + \sum_{k=1}^{K} \int_0^{\frac{S}{K}} d_{k,t,i}^s dt}.\end{aligned} \tag{25}$$

$$(\text{average ordering cost in period } \bar{S})_i^{ns} = \frac{(\text{total ordering cost in period } \bar{S})_i^{ns}}{(\text{total demand in period } \bar{S})_i^{ns}} = \frac{A_i O_i^{\bar{S}}}{\bar{S} d_i^{ns}}. \tag{26}$$

Substituting for (25) and (26) in (24), we obtain

$$(\text{average of ordering cost})_i^{ns} = \frac{S d_i^{ns} \left(\frac{A_i K}{\bar{S} d_i^{ns} + \sum_{k=1}^{K} \int_0^{\frac{S}{K}} d_{k,t,i}^s dt} \right) + \bar{S} d_i^{ns} \left(\frac{A_i O_i^{\bar{S}}}{\bar{S} d_i^{ns}} \right)}{d_i^{ns}}$$

$$= S\left(\frac{A_iK}{\bar{S}d_i^{ns} + \sum_{k=1}^{K}\int_0^{\frac{S}{K}} d_{k,t,i}^s dt}\right) + \bar{S}\left(\frac{A_iO_i^{\bar{S}}}{\bar{S}d_i^{ns}}\right)$$

$$= S\left(\frac{A_iK}{\bar{S}d_i^{ns} + \sum_{k=1}^{K}\int_0^{\frac{S}{K}} d_{k,t,i}^s dt}\right) + \bar{S}\left(\frac{A_i}{q_i^{\bar{S}}}\right), \tag{27}$$

where again the last equality follows from $O_i^{\bar{S}} = \frac{\bar{S}d_i^{ns}}{q_i^{\bar{S}}}$. Finally, substituting for (23) and (27) in (19), we get

$$AC_i^{ns} = c_i + S^2\left(\frac{h_i}{2K}\right) + \bar{S}^2\left(\frac{h_i}{2O_i^{\bar{S}}}\right) + S\left(\frac{A_iK}{\bar{S}d_i^{ns} + \sum_{k=1}^{K}\int_0^{\frac{S}{K}} d_{k,t,i}^s dt}\right) + \bar{S}\left(\frac{A_i}{q_i^{\bar{S}}}\right).$$

Proposition 1 *For $i = 1, 2$, if $d_i^{ns}\left(AC_i^{ns}, AC_{3-i}^{ns}\right) \geq 0$, then the optimal nonseasonal demand d_i^{ns*} is positive.*

Proof. See Appendix. □

The above proposition shows that if the demand evaluated at the average cost is positive, then the optimal demand given in (17) is also positive. The assumption is mild because if the price does not cover the average cost, then there is no reason to continue doing business.

The only pending non-negativity constraints are the ones in (10), that is, $d^{s*}_{k,\frac{S}{K},i} \geq 0$, for $i = 1, 2$ and $k = 1, \ldots, K$. To solve the corresponding constrained optimization problem, we introduce the Lagrangian

$$L = \pi(p_{k,i}^s,\, p_i^{ns},\, q_i^{\bar{S}}) + \sum_{i=1}^{2}\sum_{k=1}^{K} u_{k,i} d^s_{k,\frac{S}{K},i}$$

where $u_{k,i}$ is the Lagrange multiplier appended to the constraint $d^s_{k,\frac{S}{K},i} \geq 0$, for $i = 1, 2$ and $k = 1, \ldots, K$.

The Karush-Kuhn-Tucker (KKT) necessary and sufficient (the objective being concave) conditions are as follows:

$$\frac{\partial L}{\partial p_{k,1}^s} = \frac{\partial \pi}{\partial p_{k,1}^s} - \beta_1^s u_{k,1} + \lambda^s u_{k,2} = 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{28}$$

$$\frac{\partial L}{\partial p_{k,2}^s} = \frac{\partial \pi}{\partial p_{k,2}^s} - \beta_2^s u_{k,2} + \lambda^s u_{k,1} = 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{29}$$

$$u_{k,1} d^s_{k,\frac{S}{K},1} = 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{30}$$

$$u_{k,2} d^s_{k,\frac{S}{K},2} = 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{31}$$

$$d^s_{k,\frac{S}{K},1},\, d^s_{k,\frac{S}{K},2} \geq 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{32}$$

$$u_{k,1},\, u_{k,2} \geq 0, \qquad \text{for} \quad k = 1, \ldots, K \tag{33}$$

To characterize the optimal solution, it is useful to introduce the following auxiliary function:

$$\psi_{k,i} = \frac{e^{\frac{-kSg_i}{K}} a_i^s\left(2Sg_i - \left(-1 + e^{\frac{Sg_i}{K}}\right)K\right)}{2Sg_i} - \frac{(2Kc_i + Sh_i)\beta_i^s - (2Kc_{3-i} + Sh_{3-i})\lambda^s}{4K},$$
$$\text{for} \quad i = 1, 2, \quad k = 1, \ldots, K \tag{34}$$

Proposition 2 *The optimal prices $p^{s*}_{k,i}$ in the seasonal market are as follows:*

1. *If* $\psi(k,i) \geq 0$ *for* $i = 1,2$ *and* $k = 1,\ldots,K$, *then the optimal retail prices are given by (15).*
2. *If* $\psi(k,i) < 0$ *for* $i = 1,2$ *and* $k = 1,\ldots,K$, *then*

$$p^s_{k,i} = \frac{e^{\frac{-kSg_i}{K}} a^s_i \beta^s_{3-i} + e^{\frac{-kSg_{3-i}}{K}} a^s_{3-i}\lambda^s}{\beta^s_i \beta^s_{3-i} - (\lambda^s)^2}. \tag{35}$$

3. *If* $\psi_{k,1} \geq 0$ *and* $\psi_{k,2} < 0$ *for each* k, *then*

   (a) *If* $\psi_{k,1} \geq -\psi_{k,2}\frac{\lambda^s}{\beta^s_2}$, *then* $p^s_{k,1}$ *is given by* (15) *and* $p^s_{k,2}$ *is by*

$$p^s_{k,2} = \frac{e^{\frac{-kSg_2}{K}} a^s_2}{\beta^s_2} + \frac{\lambda^s K(e^{\frac{-kSg_2}{K}}(-1+e^{\frac{Sg_2}{K}})g_1 a^s_2 \lambda^s + e^{\frac{-kSg_1}{K}}(-1+e^{\frac{Sg_1}{K}})g_2 a^s_1 \beta^s_2)}{2Sg_1g_2\beta^s_2(\beta^s_1\beta^s_2 - (\lambda^s)^2)} + \frac{\lambda^s(2Kc_1 + Sh_1)}{4K\beta^s_2}. \tag{36}$$

   (b) *If* $\psi_{k,1} < -\psi_{k,2}\frac{\lambda^s}{\beta^s_2}$, *then* $p^s_{k,1}$ *and* $p^s_{k,2}$ *are given by* (35).

4. *If* $\psi_{k,1} < 0$ *and* $\psi_{k,2} \geq 0$ *for each* k, *then:*

   (a) *If* $\psi_{k,2} \geq -\psi_{k,1}\frac{\lambda^s}{\beta^s_1}$, *then* $p^s_{k,2}$ *is given by* (15)*and* $p^s_{k,1}$ *by*

$$p^s_{k,1} = \frac{e^{\frac{-kSg_1}{K}} a^s_1}{\beta^s_1} + \frac{\lambda^s K(e^{\frac{-kSg_1}{K}}(-1+e^{\frac{Sg_1}{K}})g_2 a^s_1 \lambda^s + e^{\frac{-kSg_2}{K}}(-1+e^{\frac{Sg_2}{K}})g_1 a^s_2 \beta^s_1)}{2Sg_1g_2\beta^s_1(\beta^s_1\beta^s_2 - (\lambda^s)^2)} + \frac{\lambda^s(2Kc_2 + Sh_2)}{4K\beta^s_1}. \tag{37}$$

   (b) *If* $\psi_{k,2} < -\psi_{k,1}\frac{\lambda^s}{\beta^s_1}$, *then* $p^s_{k,1}$ *and* $p^s_{k,2}$ *are given by* (35).

Proof. See Appendix. □

Based on the above proposition and the concavity of the objective function, the following algorithm is proposed to find the optimal solution:

4 Numerical example

In this section, we provide a numerical example to illustrate the model and the algorithm. Also, we investigate the impact of varying the parameter values on the optimal solution. The base case parameter values are provided in Table 3, with $S = \frac{1}{3}$(4 months) and $K = 2$.

Table 3: Parameter values

	$c_i \times 10^3$	$h_i \times 10^2$	$A_i \times 10^3$	$a^{ns}_i \times 10^3$	$a^s_i \times 10^3$	β^{ns}_i	β^s_i	g_i	λ^{ns}	λ^s
$i=1$	36	2.6	30	38	29	1.40	0.70	0.07	0.8	0.40
$i=2$	39	3.0	30	33	26	1.20	0.66	0.05	0.8	0.40

Using our algorithm, we obtain the optimal solution in Table 4.

We make the following two straightforward observations. First, although the price is slightly declining over time in the seasonal market, the demand is also lower in the second period. This result can be explained by the decline in the market size potential. Second, the prices are higher in the seasonal market than in the regular market, which is essentially a by-product of lower consumer sensitivity to price.

Algorithm 1

Step 1: *For* $i = 1, 2$, *set* $q_i^{\bar{S}*}$ *as* (16) *and* p_i^{ns*} *as* (14).

Step 2: Set $k = 1$.

Step 3: For $i = 1, 2$, compute $\psi_{k,i}$ using (34).

- **(a):** If $\psi_{k,1}$ and $\psi_{k,2} \geq 0$, then $p_{k,i}^s$, for $i = 1, 2$, is given by (15).
- **(b):** If $\psi_{k,1}$ and $\psi_{k,2} < 0$, then for $i = 1, 2$, $p_{k,i}^s$is given by (35).
- **(c):** If $\psi_{k,1} \geq 0$ and $\psi_{k,2} < 0$, then
  - **(c1):** If $\psi_{k,1} \geq -\psi_{k,2} \frac{\lambda^s}{\beta_2^s}$, then $p_{k,1}^s$ is given by (15) and $p_{k,2}^s$ by (36).
  - **(c2):** If $\psi_{k,1} < -\psi_{k,2} \frac{\lambda^s}{\beta_2^s}$, then for $i = 1, 2$, $p_{k,i}^s$ is given by (35).
- **(d):** If $\psi_{k,1} < 0, \psi_{k,2} \geq 0$, then
  - **(d1):** If $\psi_{k,2} \geq -\psi_{k,1} \frac{\lambda^s}{\beta_1^s}$, then $p_{k,1}^s$ is given by (37) and $p_{k,2}^s$ by (15).
  - **(d2):** If $\psi_{k,2} < -\psi_{k,1} \frac{\lambda^s}{\beta_1^s}$, then for $i = 1, 2$, $p_{k,i}^s$ is given by (35).

Step 4: For $i = 1, 2$, set $q_{k,i}^S$ by (3).

Step 5: Set $k = k + 1$. If $k \leq K$, go to Step 3; otherwise go to Step 6.

Step 6: End.

Table 4: Optimal solution

	$p_i^{ns} \times 10^3$	$p_{k,i}^s \times 10^3$		$q_i^{\bar{S}}$	$q_{k,i}^S$		$\pi \times 10^7$
		$k=1$	$k=2$		$k=1$	$k=2$	
$i=1$	52.63	66.66	66.15	1472	3167	3139	**45.2**
$i=2$	56.35	68.61	68.14	1223	2460	2442	

4.1 Sensitivity analysis

In this section, we have done two sensitivity analyses in order to investigate the effect of varying the parameters on the optimal solution. In the first analysis, the main parameters of the model are changed by -30% to +30%. The detailed results are given in Table 5. In the second analysis, we look at the impact of varying K in Table 6, which is the number of price changes in the seasonal market. The results are summarized as follows:

Impact on p_i^{ns} : The optimal price in the nonseasonal market is only affected by the demand parameters in this market, that is, β_i^{ns}, λ^{ns}, and by the purchasing cost c_i. A higher direct-price sensitivity β_i^{ns} leads to a lower price, while a a higher cross-price sensitivity λ^{ns}, and similarly a higher unit purchasing cost c_i, results in a higher price. As expected, the most important effect is due to β_i^{ns} (see Figure 2).

Impact on $p_{k,i}^s$: The optimal retail price in the seasonal market is affected positively by λ^s and c_i and negatively by β_i^s and g_i, and is independent of the changes in the other parameter values. The only difference with respect to the result concerning p_i^{ns} is the impact of g_i, which measures the decrease in the market potential in the seasonal market. When the market shrinks, the optimal response is to reduce the price. Figure 3 illustrates the results.

Impact on $q_i^{\bar{S}}$: The optimal order quantity in time interval $\bar{S}$ exhibits the same variations as p_i^{ns} with respect to $\beta_i^{ns}, \lambda^{ns}$ and c_i. In terms of the inventory parameters, this quantity increases with the acquisition cost and decreases with the holding cost. It is independent of demand parameters of the seasonal market. Figure 4 exhibits the results.

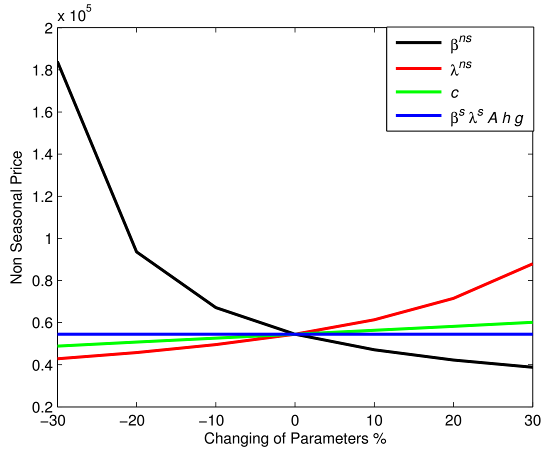

Figure 2: The impact of parameters on the optimal wholesale price

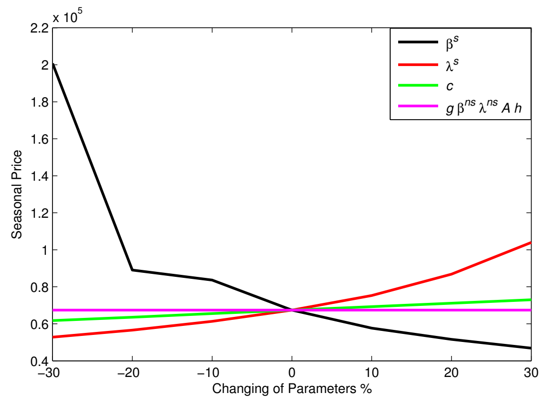

Figure 3: The impact of parameters on the retail price

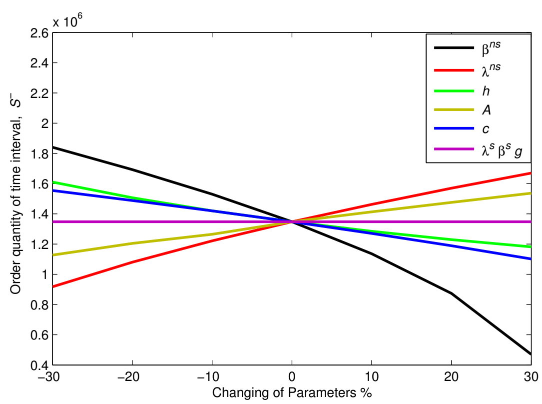

Figure 4: The impact of parameters on the order quantity of time interval $\bar{S}$

Impact on $q^{S}_{k,i}$: The demand parameters act in a similar way as for $q^{\bar{S}}_{i}$. The interesting results to report here are (i) the dependence with respect to the demand parameters of both markets, and (ii) the independence with respect to the ordering and holding costs. The reason is that the order quantity in each period is the sum of the demands in that period, and therefore the costs of ordering and inventory do not affect directly $q^{S}_{k,i}$. See Figure 5 for the details.

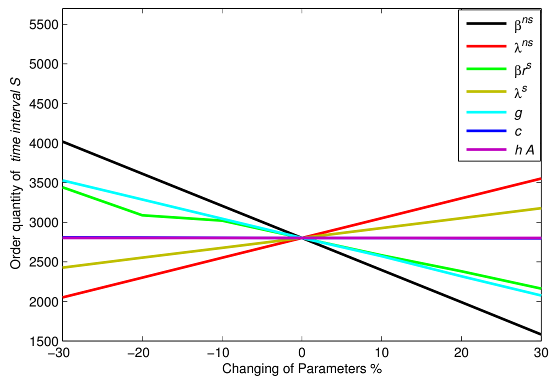

Figure 5: The impact of parameters on the order quantity of time interval S

In a nutshell, the sensitivity analysis confirms economic and inventory management intuitions: in any market, direct-price sensitivity and production (or purchasing) cost lead to higher prices and lower demand (and hence order quantity), whereas a higher cross-price sensitivity leads to the opposite effect. A higher holding cost induces the firm to order less, while a higher acquisition cost is an incentive to increase the order to minimize the number of orderings during the planning horizon. Finally, we note that the impact of varying the parameters on the profit is very much in line with the observations made above (see Figure 6).

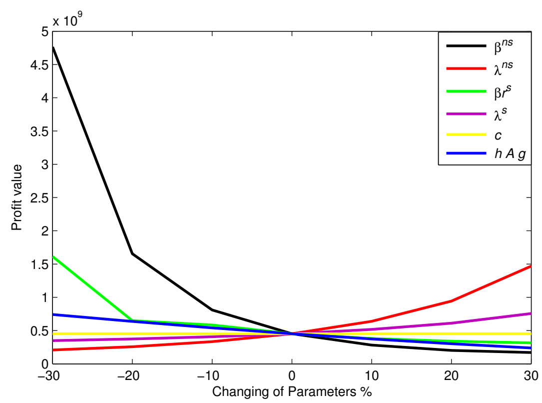

Figure 6: The impact of parameters on the profit value

Impact of varying K **:** The impact of varying the number of price settings on the optimal decision and profit values is reported in Table 6 and Figure 6. Our key findings are as follows: (i) The price and order quantity in the nonseasonal market are robust (insensitive) to a change in K. This result is due to the fact that the two markets are separated in terms of their demands. (ii) Increasing K, leads to higher prices in the seasonal market and a consequent lower order quantity. One explanation is that a larger value of K gives more opportunities to the firm to adapt its pricing to market conditions, and in particular to the speed of change in the market potential. Figure 7 gives an insight on the relationship between K and the profit. As we see, it is a concave function reaching its maximum at $K = 3$. This result, which is in line with You and Chen (2007) and Rasouli and Nakhai Kamalabadi (2014), means that it is not in the best interest to the firm to change too many times its prices during the selling season. Of course this claim is only valid for our parameter constellation.

Conceptually, K should be a decision variable. However, solving an optimal K would greatly complicate the determination of the optimal solution to the whole problem. To keep the model manageable and the approach easy to implement, we proceeded in a different way. We assumed that K is a parameter, and by enumeration we obtained its "optimal" value.

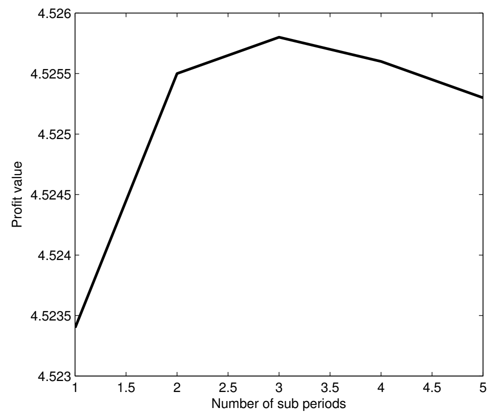

Figure 7: The impact of price setting number on profit value

Table 5: Sensitivity analysis of the parameters

	%change	$p_i^{ns} \times 10^3$		$p_{k,i}^s \times 10^3$				$q_i^{\bar{S}}$		$q_{k,i}^S$				$\pi \times 10^7$
				$k=1$		$k=2$				$k=1$		$k=2$		
	$i=1,2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	
β_i^{ns}	+30	37.51	40.11	66.66	68.61	66.15	68.14	649	291	1906	1282	1878	1264	**17.01**
	+20	40.82	43.65	66.66	68.61	66.15	68.14	1001	747	2326	1678	2298	1660	**20.06**
	+10	45.50	48.68	66.66	68.61	66.15	68.14	1258	1014	2746	2069	2718	2051	**28.16**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	64.80	69.45	66.66	68.61	66.15	68.14	1658	1402	3587	2850	3560	2832	**80.8**
	-20	90.26	96.91	66.66	68.61	66.15	68.14	1825	1560	4008	3241	3980	3223	**165.46**
	-30	177.19	190.76	66.66	68.61	66.15	68.14	1978	1703	4428	3631	4400	3613	**476.35**
λ^{ns}	+30	84.79	91.11	66.66	68.61	66.15	68.14	1802	1537	3948	3180	3920	3162	**146.91**
	+20	68.97	74.03	66.66	68.61	66.15	68.14	1699	1440	3687	2940	3660	2922	**94.22**
	+10	59.23	63.49	66.66	68.61	66.15	68.14	1589	1336	3427	2700	3399	2682	**63.95**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	47.87	51.19	66.66	68.61	66.15	68.14	1344	1099	2906	2219	2879	2201	**33.31**
	-20	44.28	47.28	66.66	68.61	66.15	68.14	1202	952	2646	1979	2618	1961	**25.65**
	-30	41.47	44.22	66.66	68.61	66.15	68.14	1041	794	2386	1738	2358	1720	**20.88**
β_i^s	+30	52.63	56.35	46.24	47.69	45.94	47.43	1472	1223	2536	1809	2509	1791	**31.41**
	+20	52.63	56.35	50.90	52.55	50.56	52.24	1472	1223	2747	2037	2719	2019	**34.06**
	+10	52.63	56.35	57.06	58.64	56.65	58.27	1472	1223	2957	2232	2929	2214	**38.07**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	52.63	56.35	82.64	85.27	81.97	84.64	1472	1223	3377	2687	3349	2669	**58.33**
	-20	52.63	56.35	111.17	114.66	110.75	113.73	1472	1223	3587	2882	3559	2864	**83.19**
	-30	52.63	56.35	198.59	204.68	196.75	202.85	1472	1223	3797	3110	3769	3092	**161.34**
λ^s	+30	52.63	56.35	102.80	105.93	101.94	105.09	1472	1223	3557	2820	3529	2802	**75.58**
	+20	52.63	56.35	85.92	88.51	85.22	87.74	1472	1223	3427	2700	3399	2682	**61.04**
	+10	52.63	56.35	74.68	76.90	74.09	75.35	1472	1223	3297	2579	3269	2562	**51.67**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	52.63	56.35	60.65	62.38	60.20	61.98	1472	1223	3037	2339	3009	2321	**40.67**
	-20	52.63	56.35	55.98	57.54	55.58	57.18	1472	1223	2907	2219	2879	2208	**37.32**
	-30	52.63	56.35	52.26	53.65	51.89	53.34	1472	1223	2777	2099	2749	2081	**34.82**
h_i	+30	52.63	56.35	66.66	68.61	66.15	68.14	1290	1073	3166	2459	3138	2441	**45.24**
	+20	52.63	56.35	66.66	68.61	66.15	68.14	1343	1116	3166	2459	3139	2441	**45.24**
	+10	52.63	56.35	66.66	68.61	66.15	68.14	1403	1166	3166	2459	3139	2441	**45.25**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	52.63	56.35	66.66	68.61	66.15	68.14	1551	1289	3167	2460	3139	2442	**45.26**
	-20	52.63	56.35	66.66	68.61	66.15	68.14	1645	1368	3167	2460	3139	2442	**45.26**

Table 5: Sensitivity analysis of the parameters

	%change	$p_i^{ns} \times 10^3$		$p_{k,i}^s \times 10^3$				$q_i^{\bar{S}}$		$q_{k,i}^S$				$\pi \times 10^7$
				$k=1$		$k=2$				$k=1$		$k=2$		
	$i=1,2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	
	-30	52.63	56.35	66.66	68.61	66.15	68.14	1759	1462	3167	2460	3139	2442	**45.27**
A_i	+30	52.63	56.35	66.66	68.61	66.15	68.14	1678	1395	3167	2460	3139	2442	**45.24**
	+20	52.63	56.35	66.66	68.61	66.15	68.14	1612	1340	3167	2460	3139	2442	**45.24**
	+10	52.63	56.35	66.66	68.61	66.15	68.14	1543	1283	3167	2460	3139	2442	**45.25**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	52.63	56.35	66.66	68.61	66.15	68.14	1369	1160	3167	2460	3139	2442	**45.26**
	-20	52.63	56.35	66.66	68.61	66.15	68.14	1315	1094	3167	2460	3139	2442	**45.26**
	-30	52.63	56.35	66.66	68.61	66.15	68.14	1231	1023	3167	2460	3139	2442	**45.27**
c_i	+30	58.03	62.20	72.06	74.46	71.55	73.99	1225	978	2447	1726	2419	1708	**23.82**
	+20	56.23	60.25	70.26	72.51	69.75	72.04	13122	1066	2687	1971	2659	1953	**30.13**
	+10	54.43	58.30	68.46	70.56	67.95	70.09	1394	1147	2972	2215	2899	2197	**37.28**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	50.83	54.40	64.86	66.66	64.35	66.19	1545	1295	3407	2704	3379	2686	**54.05**
	-20	49.03	52.45	63.06	64.71	62.25	64.24	1615	1362	3647	2949	3619	2931	**63.68**
	-30	47.23	50.50	61.26	62.76	60.75	62.29	1682	1427	3887	3139	3859	3175	**74.14**
g_i	+30	52.63	56.35	66.58	68.54	65.93	67.93	1472	1223	3163	2457	3127	2434	**45.09**
	+20	52.63	56.35	66.61	68.56	66.00	68.00	1472	1223	3164	2458	3131	2436	**45.14**
	+10	52.63	56.35	66.63	68.59	66.07	68.07	1472	1223	3165	2459	3135	2439	**45.20**
	0	52.63	56.35	66.66	68.61	66.15	68.14	1472	1223	3167	2460	3139	2442	**45.25**
	-10	52.63	56.35	66.68	68.63	66.23	68.21	1472	1223	3168	2461	3143	2444	**45.31**
	-20	52.63	56.35	66.71	68.66	66.30	68.28	1472	1223	3170	2462	3147	2447	**45.36**
	-30	52.63	56.35	66.73	68.68	66.38	68.35	1472	1223	3171	2463	3151	2450	**45.42**

Table 6: Impact of varying K

	$p_i^{ns} \times 10^3$	$p_{k,i}^s \times 10^3$					$q_i^{\bar{S}}$	$q_{k,i}^S$					$\pi \times 10^7$
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	
$i=1$	52.62	66.41					1472	6306					**45.234**
$i=2$	56.34	68.39					1224	4901					
							$K=1$						
$i=1$	52.63	66.66	66.15				1472	3167	3139				**45.255**
$i=2$	56.35	68.61	68.14				1223	2460	2442				
							$K=2$						
$i=1$	52.64	66.74	66.40	66.06			1471	2114	2102	2089			**45.258**
$i=2$	56.36	68.68	68.37	68.06			1223	1641	1633	1625			
							$K=3$						
$i=1$	52.65	66.78	66.52	66.27	66.01		1471	1586	1579	1572	1565		**45.256**
$i=2$	56.36	68.72	68.48	68.25	68.00		1223	1231	1227	1222	1218		
							$K=4$						
$i=1$	52.65	66.80	66.60	66.40	66.19	65.99	1471	1269	1265	1260	1256	1251	**45.253**
$i=2$	56.37	68.74	68.55	68.37	68.18	67.99	1222	985	982	979	976	974	
							$K=5$						

5 Conclusion

In this paper, we proposed a model of dynamic pricing and of order quantity for a firm that sells substitutable products in two markets, with one of them being seasonal. We assumed that the substitution is price-driven and not due to shortages. Our results show that considering a dynamic-pricing policy for the seasonal market, i.e., $K > 1$, leads to higher profit than a fixed-pricing policy ($K = 1$). Another interesting result is the dependency of nonseasonal market prices and order quantity to the number of price changes in seasonal market.

As in any prescriptive research, we made some simplifying assumptions to be able to focus on those issues that we considered to be the main ones. Taking stock on what we obtained, it is of interest to relax some of our assumptions in future work and to extend the model. Here are some suggestions: (i) The assumption that the parameter values are known with certainty can be dropped. (ii) In our framework, the number of times the price is changed is given at first and then computed by enumeration. It is clearly of intellectual interest to have it endogenous. (iii) It is worth it to investigate the role of strategic consumers in the seasonal market. These consumers can anticipate the price decrease and their buying decision is forward looking, that is, they do not necessarily buy at the first occasion in which they have a surplus, but at the time where this surplus is maximum. This behavior is expected to affect not only the price but also the ordering policy. Finally, (iv) the model could be extended to a multi-level supply chain, which would allow to account for vertical strategic interactions that are present in most markets.

Appendix

A Proof of Lemma 1

The Hessian matrix is of dimension $4 + 2K \times 4 + 2K$ and is given by

$$
H = \begin{bmatrix}
\frac{\partial^2\pi}{\partial(p_1^{ns})^2} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial p_1^{ns}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial p_1^{ns}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial p_2^{ns}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial(p_2^{ns})^2} & \frac{\partial^2\pi}{\partial p_2^{ns}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_2^{ns}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_2^{ns}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial p_2^{ns}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial p_2^{ns}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial p_2^{ns}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial(q_1^{\bar{S}})^2} & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial q_1^{\bar{S}}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial(q_2^{\bar{S}})^2} & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial q_2^{\bar{S}}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial(p_{1,1}^{s})^2} & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial p_{1,1}^{s}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial(p_{1,2}^{s})^2} & \dots & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial p_{1,2}^{s}\partial p_{K,2}^{s}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial(p_{K,1}^{s})^2} & \frac{\partial^2\pi}{\partial p_{K,1}^{s}\partial p_{K,2}^{s}} \\
\frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial p_1^{ns}} & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial p_2^{ns}} & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial q_1^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial q_2^{\bar{S}}} & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial p_{1,1}^{s}} & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial p_{1,2}^{s}} & \dots & \frac{\partial^2\pi}{\partial p_{K,2}^{s}\partial p_{K,1}^{s}} & \frac{\partial^2\pi}{\partial(p_{K,2}^{s})^2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-2\beta_1^{ns} & 2\lambda^{ns} & \frac{-A_1\beta_1^{ns}\bar{S}}{(q_1^{\bar{S}})^2} & \frac{A_2\lambda^{ns}\bar{S}}{(q_2^{\bar{S}})^2} & 0 & 0 & \dots & 0 & 0 \\
2\lambda^{ns} & -2\beta_2^{ns} & \frac{A_1\lambda^{ns}\bar{S}}{(q_1^{\bar{S}})^2} & \frac{-A_2\beta_2^{ns}\bar{S}}{(q_2^{\bar{S}})^2} & 0 & 0 & \dots & 0 & 0 \\
\frac{-A_1\beta_1^{ns}\bar{S}}{(q_1^{\bar{S}})^2} & \frac{A_1\lambda^{ns}\bar{S}}{(q_1^{\bar{S}})^2} & \frac{-2A_1\, d_1^{ns}\bar{S}}{(q_1^{\bar{S}})^3} & 0 & 0 & 0 & \dots & 0 & 0 \\
\frac{A_2\lambda^{ns}\bar{S}}{(q_2^{\bar{S}})^2} & \frac{-A_2\beta_2^{ns}\bar{S}}{(q_2^{\bar{S}})^2} & 0 & \frac{-2A_2\, d_2^{ns}\bar{S}}{(q_2^{\bar{S}})^3} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-2S\beta_1^s}{K} & \frac{4S\lambda^s}{K} & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4S\lambda^s}{K} & \frac{-2S\beta_2^s}{K} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{-2S\beta_1^s}{K} & \frac{4S\lambda^s}{K} \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{4S\lambda^s}{K} & \frac{-2S\beta_2^s}{K}
\end{bmatrix}
$$

The minor determinants of above matrix are as follows:

$$
\begin{aligned}
|H|_1 =& -2\beta_1^{ns} < 0,\\
|H|_2 =& 4(\beta_1^{ns}\beta_2^{ns} - (\lambda^{ns})^2) > 0,\\
|H|_3 =& \frac{1}{(q_1^{\bar{S}})^4} 2\bar{S}A_1(\beta_1^{ns}\beta_2^{ns} - (\lambda^{ns})^2)\left(\bar{S}A_1\beta_1^{ns} - 4q_1^{\bar{S}}d_1^{ns}\right),\\
|H|_4 =& \frac{1}{(q_1^{\bar{S}}q_2^{\bar{S}})^4}\bar{S}^2A_1A_2\left(\beta_1^{ns}\beta_2^{ns} - (\lambda^{ns})^2\right)\times\\
& \left(\bar{S}^2A_1A_2(\beta_1^{ns}\beta_1^{ns} - (\lambda^{ns})^2) + 4q_2^{\bar{S}}d_2^{ns}(2q_1^{\bar{S}}d_1^{ns} - \bar{S}A_1\beta_1^{ns}) + 4q_1^{\bar{S}}d_1^{ns}(2q_1^{\bar{S}}d_2^{ns} - \bar{S}A_2\beta_2^{ns})\right),\\
|H|_5 =& |H|_4 \times -\frac{1}{K}2S\beta_1^s,\\
|H|_6 =& |H|_4 \times \frac{1}{(K)^2}(2S)^2(\beta_1^s\beta_2^s - (\lambda^{ns})^2),\\
& \dots\\
|H|_{4+2K} =& |H|_4 \times \frac{1}{(K)^{2K}}(2S)^{2S}(\beta_1^s\beta_2^s - (\lambda^{ns})^2)^K.
\end{aligned}
$$

Positivity of $|H|_2$ follows from our assumption $\beta_i^{ns} > \lambda^{ns}, i = 1, 2$. Let us now show that if $A_i > \frac{\bar{S}^4\beta_i^{ns}h_i^2}{8(O_i^{\bar{S}})^3}$, then $|H|_4$ is positive. Note that $d_i^{ns} = \frac{q_i^{\bar{S}}O_i^{\bar{S}}}{\bar{S}}$ and $q_i^{\bar{S}} = \sqrt{\frac{2A_id_i^{ns}}{h_i}}$. Consequently,

$$\bar{S}d_i^{n*} = q_i^{\bar{S}}O_i^{\bar{S}} = O_i^{\bar{S}}\sqrt{\frac{2A_id_i^{ns}}{h_i}} \Leftrightarrow d_i^{ns} = \frac{2A_i\left(O_i^{\bar{S}}\right)^2}{\bar{S}^2h_i}.$$

Now, consider $2q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns}$. We have

$$2q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns} = 2\left(q_i^{\bar{S}}\right)^2\frac{O_i^{\bar{S}}}{\bar{S}} - \bar{S}A_i\beta_i^{ns} = A_i\left(\frac{4d_i^{ns}}{h_i}\frac{O_i^{\bar{S}}}{\bar{S}} - \bar{S}\beta_i^{ns}\right).$$

Therefore,

$$sign\ \left(2q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns}\right) = sign\ A_i\left(\frac{4d_i^{ns}}{h_i}\frac{O_i^{\bar{S}}}{\bar{S}} - \bar{S}\beta_i^{ns}\right).$$

Substituting for d_i^{ns} by its value, we get

$$sign\ \left(2q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns}\right) = sign\ A_i\left(\frac{8A_i(O_i^{\bar{S}})^3}{\bar{S}^3h_i^2} - \bar{S}\beta_i^{ns}\right) > 0.$$

Positivity follows from our assumption $A_i > \frac{\bar{S}^4\beta_i^{ns}h_i^2}{8(O_i^{\bar{S}})^3}$. Now, $2q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns} > 0$ implies that $4q_i^{\bar{S}}d_i^{ns} - \bar{S}A_i\beta_i^{ns} > 0$ and consequently $|H|_3 < 0$.

Now, given that to $\beta_i^s > \lambda^s > 0$, we can show that $|H|_5 < 0, |H|_6 > 0$ and $|H|_{4+2K} > 0$. Since xth $(1 \leq x \leq 4 + 2K)$ minor determinant of the above Hessian matrix has $(-1)^x$ sign, the proof is complete.

B Proof of Proposition 1

By assumption, we have

$$d_i^{ns}\left(AC_i^{ns}, AC_{3-i}^{ns}\right) = a_i^{ns} - \beta_i^{ns} AC_i^{ns} + \lambda_{3-i}^{ns} AC_{3-i}^{ns} > 0, \quad i = 1, 2.$$

$$d_i^{ns*} = \frac{a_i^{ns} - \left(c_i + \frac{S^2}{2K} h_i + \bar{S}\frac{A_i}{q_i^{\bar{S}*}}\right)\beta_i^{ns} + \left(c_{3-i} + \frac{S^2}{2K} h_{3-i} + \bar{S}\frac{A_{3-i}}{q_{3-i}^{\bar{S}*}}\right)\lambda^{ns}}{2}$$

Rewrite the demand in (17) as

$$d_i^{ns*} = \frac{1}{2} d_i^{ns}(x_i, x_{3-i}), \qquad \text{for } i = 1, 2,$$

where

$$x_i = c_i + \frac{S^2}{2K} h_i + \bar{S}\frac{A_i}{q_i^{\bar{S}*}}, \qquad x_{3-i} = c_{3-i} + \frac{S^2}{2K} h_{3-i} + \bar{S}\frac{A_i}{q_i^{\bar{S}*}}.$$

Clearly, we have

$$x_i < AC_i^{ns}.$$

The demand is higher at a lower price. Therefore,

$$d_i^{ns}(x_i, x_{3-i}) > d_i^{ns}\left(AC_i^{ns}, AC_{3-i}^{ns}\right) > 0,$$

and finally

$$d_i^{ns*} = \frac{1}{2} d_i^{ns}(x_i, x_{3-i}) > 0.$$

C Proof of Proposition 2

The proof proceeds by considering the different combinations of values of the Lagrange multipliers.

1. If $u_{k,1} = u_{k,2} = 0$, then we conclude from (28) and (29) that $\frac{\partial L}{\partial p_{k,i}^s} = \frac{\partial \pi}{\partial p_{k,i}^s} = 0, i = 1, 2$ and therefore $p_{k,i}^s = p_{k,i}^{s*}$ for $i = 1, 2$ as given by (15). Substituting in the demand, we get

$$\begin{aligned} d^s_{k,\frac{S}{K},i} &= \frac{e^{\frac{-kSg_i}{K}} a_i^s (2Sg_i - (-1 + e^{\frac{Sg_i}{K}})K)}{2Sg_i} - \frac{(2Kc_i + Sh_i)\beta_i^s - (2Kc_{3-i} + Sh_{3-i})\lambda^s}{4K} \\ &= \psi_{k,i} \geq 0 \qquad\qquad \text{for} \quad i = 1, 2, \quad k = 1, \ldots, K. \end{aligned} \tag{38}$$

2. Let $u_{k,1}, u_{k,2} \neq 0$. From (30) and (31), we have $d^s_{k,\frac{S}{K},1} = d^s_{k,\frac{S}{K},2} = 0$. Solving, we obtain the prices

$$p_{k,i}^s = \frac{e^{\frac{-kSg_i}{K}} a_i^s}{\beta_i^s} + \frac{\lambda^s}{\beta_i^s} p_{k,3-i}^s \qquad \text{for} \quad i = 1, 2, \quad k = 1, \ldots, K. \tag{39}$$

   By solving system of equations (two equations and two variables), we have:

$$p_{k,i}^s = \frac{e^{\frac{-kSg_i}{K}} a_i^s \beta_{3-i}^s + e^{\frac{-kSg_{3-i}}{K}} a_{3-i}^s \lambda^s}{\beta_i^s \beta_{3-i}^s - (\lambda^s)^2} \geq 0 \qquad \text{for} \quad i = 1, 2. \tag{40}$$

   Solving for the multipliers, we get

$$\begin{aligned} u_{k,i} =& \frac{e^{\frac{-kSg3-i}{K}}(-1 + e^{\frac{Sg_{3-i}}{K}})Ka_{3-i}^s g_i \lambda^s + e^{\frac{-kSg_i}{K}}(-2Sg_i + (-1 + e^{\frac{Sg_i}{K}})K)a_i^s g_{3-i}\beta_{3-i}^s}{Kg_i g_{3-i}(\beta_i^s \beta_{3-i}^s - \lambda^{s2})} \\ &- \frac{2e^{\frac{-kSg3-i}{K}} Sa_{3-i}^s \lambda^s}{K(\beta_i^s\beta_{3-i}^s - \lambda^{s2})} + \frac{S(2Kc_i + Sh_i)}{2K^2} = \frac{-2S(\psi_{k,i}\beta_{3-i}^s + \psi_{k,3-i}\lambda^s)}{K(\beta_i^s\beta_{3-i}^s - \lambda^{s2})} > 0, \\ & \qquad\qquad \text{for} \quad i = 1, 2, \quad k = 1, \ldots, K. \end{aligned} \tag{41}$$

3. If $\psi_{k,1} \geq 0$ and $\psi_{k,2} < 0$ for each k, we have two cases:

   (a) $\psi_{k,1} \geq -\psi_{k,2}\frac{\lambda^s}{\beta_2^s}$. Let $u_{k,1} = 0$ and $u_{k,2} \neq 0$. Solving for $d^s_{k,\frac{S}{K},2} = 0$, we conclude that $p^s_{k,2}$ is given by (39) which is depended on $p^s_{k,1}$. Substituting in (29) and solving, we obtain

$$\begin{aligned}u_{k,2} =& \frac{e^{\frac{-kSg_2}{K}}((-1+e^{\frac{Sg_2}{K}})K - 2Sg_2)a_2^s}{Kg_2\beta_2^s} + \frac{S((2Kc_2+Sh_2)\beta_2^s - (2Kc_1+Sh_1)\lambda^s)}{2(K)^2\beta_2^s}\\ =& \frac{-2S}{K\beta_2^s}\psi_{k,2} > 0. \end{aligned} \tag{42}$$

   Inserting in (28) and solving $\frac{\partial \pi}{\partial p^s_{k,1}} = -\lambda^s u_{k,2}$, yields the value in (15) for $p^s_{k,1}$. Substituting $p^s_{k,1}$in $p^s_{k,2}$ gives:

$$\begin{aligned}p^s_{k,2} =& \frac{e^{\frac{-kSg_2}{K}}a_2^s}{\beta_2^s} + \frac{\lambda^s K(e^{\frac{-kSg_2}{K}}(-1+e^{\frac{Sg_2}{K}})g_1a_2^s\lambda^s + e^{\frac{-kSg_1}{K}}(-1+e^{\frac{Sg_1}{K}})g_2a_1^s\beta_2^s)}{2Sg_1g_2\beta_2^s(\beta_1^s\beta_2^s - (\lambda^s)^2)}\\ &+ \frac{\lambda^s(2Kc_1+Sh_1)}{4K\beta_2^s} \geq 0. \end{aligned} \tag{43}$$

   Substituting the prices in $d^s_{k,\frac{S}{K},1}$ gives:

$$\begin{aligned}d^s_{k,\frac{S}{K},1} =& \frac{e^{\frac{-kSg_1}{K}}(2Sg_1 - (-1+e^{\frac{Sg_1}{K}})K)a_1^sg_2\beta_2^s - e^{\frac{-kSg_2}{K}}(-1+e^{\frac{Sg_2}{K}})Ka_2^sg_1\lambda^s}{2Sg_1g_2\beta_2^s}\\ &+ \frac{e^{\frac{-kSg_2}{K}}\lambda^s a_2^s}{\beta_2^s} - \frac{(2Kc_1+Sh_1)(\beta_1^s\beta_2^s - \lambda^{s2})}{4K\beta_2^s} = \psi_{k,1} + \psi_{k,2}\frac{\lambda^s}{\beta_2^r} \geq 0 \end{aligned} \tag{44}$$

   (b) If $\psi_{k,1} < -\psi_{k,2}\frac{\lambda^s}{\beta_2^s}$, the solution is the same as in 2.

4. If $\psi_{k,1} < 0$ and $\psi_{k,2} \geq 0$ for each k, then we have two cases:

   (a) $\psi_{k,2} \geq -\psi_{k,1}\frac{\lambda^s}{\beta_1^s}$. Let $u_{k,1} \neq 0$ and $u_{k,2} = 0$. Solving $d^s_{k,\frac{S}{K}1} = 0$, yields the value in (39) for $p^s_{k,1}$. By solving (28), we obtain

$$\begin{aligned}u_{k,1} =& \frac{e^{\frac{-kSg_1}{K}}((-1+e^{\frac{Sg_1}{K}})K - 2Sg_1)a_1^s}{Kg_1\beta_1^s} + \frac{S((2Kc_1+Sh_1)\beta_1^s - (2Kc_2+Sh_2)\lambda^s)}{2(K)^2\beta_1^s}\\ =& \frac{-2S}{K\beta_1^s}\psi_{k,1} > 0. \end{aligned} \tag{45}$$

   Solving (29), yields the value in (15) for $p^s_{k,2}$. Substituting $p^s_{k,2}$ in $p^s_{k,1}$ (39) gives:

$$\begin{aligned}p^s_{k,1} =& \frac{e^{\frac{-kSg_1}{K}}a_1^s}{\beta_1^s} + \frac{\lambda^s K(e^{\frac{-kSg_1}{K}}(-1+e^{\frac{Sg_1}{K}})g_2a_1^s\lambda^s + e^{\frac{-kSg_2}{K}}(-1+e^{\frac{Sg_2}{K}})g_1a_2^s\beta_1^s)}{2Sg_1g_2\beta_1^s(\beta_1^s\beta_2^s - (\lambda^s)^2)}\\ &+ \frac{\lambda^s(2Kc_2+Sh_2)}{4K\beta_1^s} \geq 0 \end{aligned} \tag{46}$$

   Substituting for the prices in $d^s_{k,\frac{S}{K}2}$ gives

$$\begin{aligned}d^s_{k,\frac{S}{K},2} =& \frac{e^{\frac{-kSg_2}{K}}(2Sg_2 - (-1+e^{\frac{Sg_2}{K}})K)a_2^sg_1\beta_1^s - e^{\frac{-kSg_1}{K}}(-1+e^{\frac{Sg_1}{K}})Ka_1^sg_2\lambda^s}{2Sg_1g_2\beta_1^s}\\ &+ \frac{e^{\frac{-kSg_1}{K}}\lambda^s a_1^s}{\beta_1^s} - \frac{(2Kc_2+Sh_2)(\beta_1^s\beta_2^s - \lambda^{s2})}{4K\beta_1^s} = \psi_{k,2} + \psi_{k,1}\frac{\lambda^s}{\beta_1^r} \geq 0 \end{aligned} \tag{47}$$

   (b) If $\psi_{k,2} < -\psi_{k,1}\frac{\lambda^s}{\beta_1^s}$, the solution is the same as in 2.

References

Snigdha Banerjee and Ashish Sharma. Optimal procurement and pricing policies for inventory models with price and time dependent seasonal demand. Mathematical and Computer Modelling, 51(5):700–714, 2010a.

Snigdha Banerjee and Ashish Sharma. Inventory model for seasonal demand with option to change the market. Computers & Industrial Engineering, 59(4):807–818, 2010b.

Gabriel R. Bitran and Rene Caldentey. An overview of pricing models for revenue management. Manufacturing & Service Operations Management, 5(3):203–209, 2003.

Gabriel R. Bitran and Susana V. Mondschein. Periodic pricing of seasonal products in retailing. Management Science, 43(1):64–79, 1997.

Vaibhav Chaudhary, Rakhee Kulshrestha, and Srikanta Routroy. State-of-the-art literature review on inventory models for perishable products. Journal of Advances in Management Research, 15(3):306–346, 2018.

Wedad Elmaghraby and Pınar Keskinocak. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. Management Science, 49(10):1287–1309, 2003.

Chandandeep S. Grewal, S.T. Enns, and Paul Rogers. Dynamic reorder point replenishment strategies for a capacitated supply chain with seasonal demand. Computers & Industrial Engineering, 80:97–110, 2015.

Jinpyo Lee. Dynamic pricing inventory control under fixed cost and lost sales. Applied Mathematical Modelling, 38(2):712–721, 2014.

Lihao Lu, Jianxiong Zhang, and Wansheng Tang. Optimal dynamic pricing and replenishment policy for perishable items with inventory-level-dependent demand. International Journal of Systems Science, 47(6): 1480–1494, 2016.

Serguei Netessine. Dynamic pricing of inventory/capacity with infrequent price changes. European Journal of Operational Research, 174(1):553–580, 2006.

Nadia Rasouli and Isa Nakhai Kamalabadi. Joint pricing and inventory control for seasonal and substitutable goods mentioning the symmetrical and asymmetrical substitution. International Journal of Engineering, Transactions B: Applications, 27(9):1385–1394, 05 2014.

Ali Akbar Shaikh, Md. Al-Amin Khan, Gobinda Chandra Panda, and Ioannis Konstantaras. Price discount facility in an eoq model for deteriorating items with stock-dependent demand and partial backlogging. International Transactions in Operational Research, 26(4):1365–1395, 2019.

Gonca P. Soysal and Lakshman Krishnamurthi. Demand dynamics in the seasonal goods industry: An empirical analysis. Marketing Science, 31(2):293–316, 2012.

Ata Allah Taleizadeh and Arezu Rasuli-Baghban. Pricing and lot sizing of a decaying item under group dispatching with time-dependent demand and decay rates. Scientia Iranica, 25(3):1656–1670, 2018.

Kalyan T Talluri and Garrett J Van Ryzin. The theory and practice of revenue management, volume 68. 2006.

Lawrence R. Weatherford and Samuel E. Bodily. A taxonomy and research overview of perishable-asset revenue management: Yield management, overbooking, and pricing. Operations Research, 40:831–844, 1992.

Peng Sheng You. Optimal replenishment policy for product with season pattern demand. Operations Research Letters, 33(1):90–96, 2005.

Peng-Sheng You and Ta Cheng Chen. Dynamic pricing of seasonal goods with spot and forward purchase demands. Computers & Mathematics with Applications, 54(4):490–498, 2007.

Peng-Sheng You, Seizo Ikuta, and Yi-Chih Hsieh. Optimal ordering and pricing policy for an inventory system with trial periods. Applied Mathematical Modelling, 34(10):3179–3188, 2010.