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On Combinatorial Properties of Linear Program Digraphs

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Abstract

An LP-digraph is a directed graph which consists of the vertices and edges of a polytope P directed by a linear function in general position. It is known that an LP-digraph satisfies three necessary conditions: it is acyclic, each face of P has a unique sink (USO), and the Holt-Klee condition holds. These three conditions are not sufficient in general for a directed graph on P to be an LP-digraph. In this paper we survey the relationships between these three condtions and introduce a new necessary condition, the shelling condition. We show that the shelling condition is stronger than a combination of the acyclicity and Holt-Klee conditions for polytopes in dimension at least 4 which are non-simple. In all other cases, we show that it is equivalent to the intersection of these two conditions.

Résumé

Un LP-digraphe est un graphe orienté comprenant l'ensemble des sommets et des arêtes d'un polytope P, dirigé par une fonction linéaire en position générale. Il est connu qu'un LP-digraphe satisfait à trois conditions nécessaires : il est acyclique, il y a un puit unique dans chaque face de P (USO), et la condition Holt-Klee tient. Ces trois conditions ne sont pas suffisantes en général pour qu'un graphe orienté sur P soit un LP-digraphe. Dans cet article nous faisons une étude des liens entre ces condtions et nous introduisons une nouvelle condition nécessaire, la condition de *shelling*. Nous montrons que cette condition est plus forte qu'une combinaison des conditions d'acyclicité et d'être un USO pour les polytopes en dimension d'au moins 4 qui ne sont pas simples. En tout autre cas, elle est équivalente à l'intersection de ces deux conditions.

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1 Introduction

Let P be a d-dimensional convex polytope (d-polytope) in $\Re^d$. We assume that the reader is familiar with polytopes, a standard reference being [9]. The vertices and the edges of P form an (abstract) undirected graph called the skeleton of P. Interest in such graphs stems from the fact that the simplex method with a given pivot rule can be viewed as an algorithm for finding a path in the skeleton of P to a vertex that maximizes a linear function $f(x) = c^T x$ over P. Research continues on pivot rules for the simplex method since they leave open the possibility of finding a strongly polynomial time algorithm for linear programming. Since the simplex method gives an orientation to edges traversed, it is of interest to study directed graphs based on the skeleton. We form a directed graph $G(P)$ by orienting each edge of the skeleton of P in some manner.

We can distinguish four properties that the digraph $G(P)$ may have, each of which has been well studied:

- *Acyclicity*: $G(P)$ has no directed cycles.
- *Unique sink orientation (USO)* [8]: Each subgraph of $G(P)$ induced by a face of P has a unique sink (and hence a unique source).
- *Holt-Klee property* [4]: $G(P)$ has a unique sink orientation, and for every k-dimensional face (k-face) H of P there are k vertex disjoint paths from the unique source to the unique sink of H in the subdigraph $G(P, H)$ of $G(P)$ induced by H.
- *LP digraph*: There is a linear function f such that for each pair of vertices u and v of P that form a directed edge (u, v) in $G(P)$, we have $f(u) < f(v)$. In this case, we denote $G(P)$ by LP(P, f). (This is the same as polytopal digraph in [5].)

The main property of interest is the fourth one, since the digraph LP(P, f) represents the possible pivot operations of the simplex method in maximizing f over P. The other three properties are necessary conditions for $G(P)$ to be an LP-digraph. We note here that Williamson Hoke [3] has defined a property called *complete unimodality* which is equivalent to a combination of acyclicity and unique sink orientation. A definition of complete unimodality and an important result of Williamson Hoke relating to our work are stated in Section 3.2. First, we survey known results on the relationships between the four properties, according to the dimension d of P, collecting them in the following theorem, whose proof along with bibliographic references, will be given in the next section.

Theorem 1 *For a digraph $G(P)$ based on a d-polytope P, the relationships between the properties acyclicity, USO, the Holt-Klee property and LP-digraph are as shown in Figure 1, where the regions A,B,...,J are non-empty. These relationships hold even for simple polytopes. They also hold for cubes, except for $d = 3$, when region E is empty.*

From Figure 1 it can be seen that LP digraphs are completely characterized when $d = 2, 3$ [5]. No such characterization is known yet for higher dimensions. The letters A,B,...,J in the figure refer to a set of ten non-empty regions, as will be demonstrated by specific examples that will be given in the next section.

In this paper we introduce another necessary property for $G(P)$ to be an LP digraph, based on *shelling*, which is one of the fundamental tools of polytope theory. A formal definition of shelling is given in Section 3. Suppose a polytope P has m vertices labelled $v_1, v_2, ..., v_m$, and let $G(P)$ be a digraph based on its skeleton. A permutation r of the vertices is a *topological*

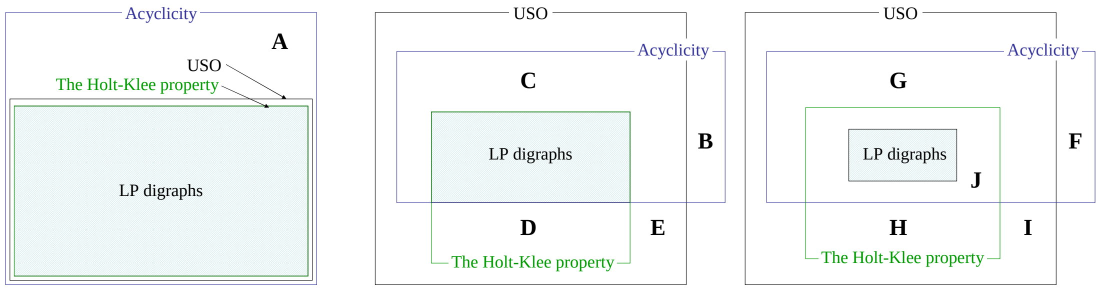

Figure 1: The relationships for d-polytopes: $d = 2$[left], $d = 3$[middle] and $d \geq 4$[right]

sort of $G(P)$ if, whenever (v_i, v_j) is a directed edge of $G(P)$, v_i precedes v_j in the permutation r. Let $L(P)$ be the face lattice of P. A polytope P^* whose face lattice is $L(P)$ "turned upside-down" is called a *combinatorially polar* polytope of P. Combinatorial polarity interchanges vertices of P and with facets of P^*. We denote by r^* the facet ordering of P^* corresponding to vertex ordering of P given by r.

- *Shelling property*: There exists a topological order r of $G(P)$ such that the facets of P^* ordered by r^* are a shelling of P^*.

In Section 3 we analyze the relationship between the shelling property and the four previously defined properties. The results are summarized in the following theorem.

Theorem 2 *For digraphs $G(P)$ based on a d-polytopes P, the relationships between the properties acyclicity, USO, the Holt-Klee property, LP-digraph and the shelling property are as shown in Figure 2, where the regions A,B,...,J,X,Y are non-empty.*

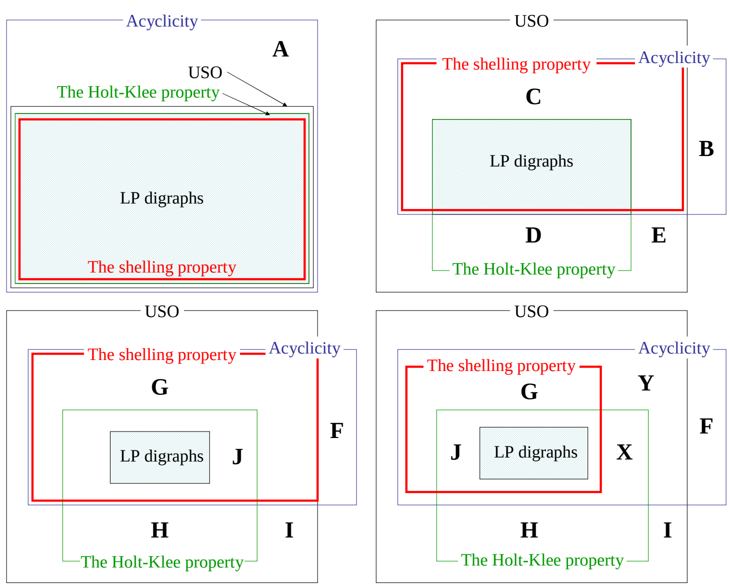

Figure 2: The relationships when $d = 2$[upperleft] and $d = 3$[upperright] and the relationships when for $d \geq 4$, P is simple[lowerleft] and general[lowerright]

2 Proof of Theorem 1

2.1 The case when P is a cube

Let P be the d-dimensional cube C_d. Specialized to C_d, Theorem 1 reduces to Figure 3.

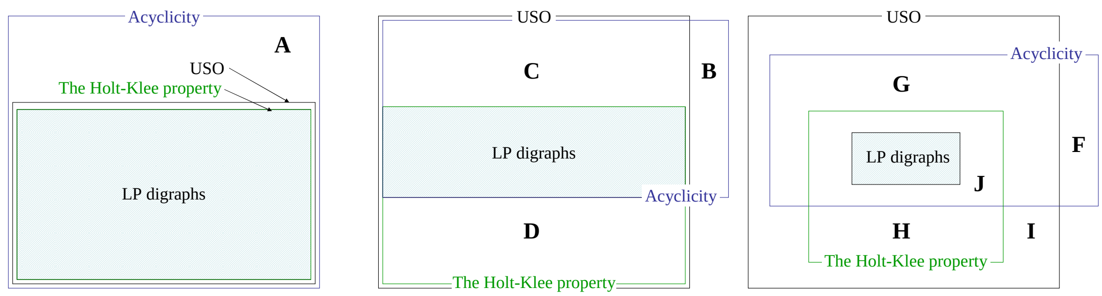

Figure 3: Possible orientations of the cube C_d: $d = 2$[left], $d = 3$[middle] and $d \geq 4$[right]

For each of the marked regions we will give a corresponding example based on a cube. When $d = 2$, it is immediate that the digraph $G(C_2)$ is a USO if and only if it satisfies the Holt-Klee property and is acyclic. On the other hand, there exists a $G(C_2)$ which is not a USO, does not satsify Holt-Klee, but is acyclic. This is example A of Figure 4. Based on this example, we also generate examples for the regions B and F, as shown in the same figure. These two examples are acyclic, but are not USOs.

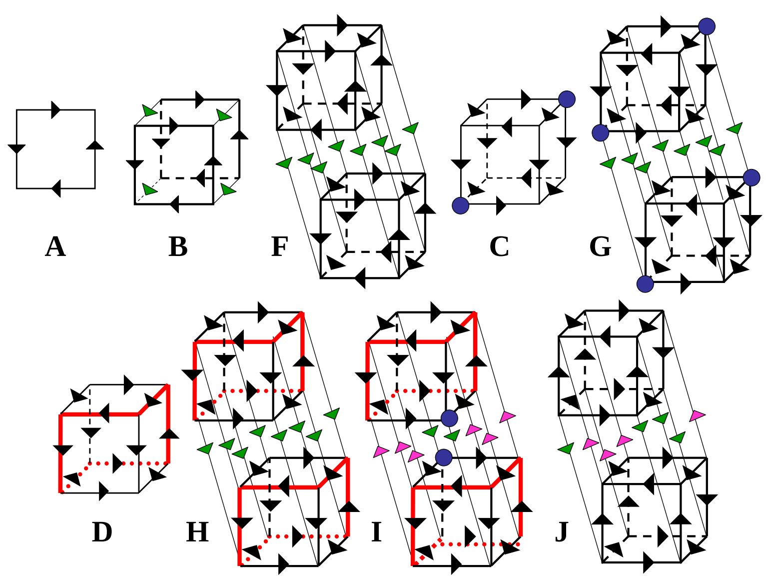

Figure 4: Examples corresponding to non-empty regions of Figure 1

When $d = 3$, there is no cyclic orientation of $G(C_3)$ which is a USO but does not satisfy the Holt-Klee property [7], so region E of Figure 1 is empty for cubes. On the other hand, there are two distinct orientations of C_3 which are acyclic USOs, but do not satisfy the Holt-Klee property [7]. Example C of Figure 4 is one such example, and can be used to generate example G, showing the corresponding region of Figure 3 is non-empty. In Figure 4 a cut set of size two separating source and sink of C is highlighted. There are two facets in G each isomorphic to C, the Holt-Klee condition fails in each of them and so fails for G. also.

Furthermore, C_3 admits only one cyclic orientation which is a USO and satisfies the Holt-Klee property [7]. This is example D in Figure 4, which can be used to generate examples to show that regions H and I are non-empty, shown in the same figure. The cycles of examples D, H and I are highlighted. In I a cut set of size two between source and sink is highlighted, showing the Holt-Klee property fails.

On the other hand, there exists an orientation of C_4 which satisfies the three properties, but is not an LP digraph [6]. This is example J of Figure 4.

The examples F,G,H,I and J readily extend to cubes in higher dimensions: duplicate the figure, add an edge between each pair of corresponding vertices and then direct them all from the first copy to the second.

2.2 The case when P is a simple polytope

Since C_d is simple, all of the preceeding examples apply to this case. It remains to show that the region E may be non-empty for simple 3-polytopes. Indeed, Figure 5 shows a 3-polytope with a cyclic USO which does not satisfy the Holt-Klee property. There is a cut set of size 2: top vertex and bottom left vertex, so the Holt-Klee condition is violated. There is a cycle of length 6 along the boundary of the region left over when the top left vertex is deleted. These are highlighted in Figure 5.

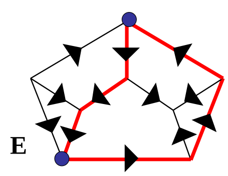

Figure 5: An example for region E of Figure 1

3 The shelling property and proof of Theorem 2

3.1 Shellings and Line shellings

We first give a formal defintion of shelling, in order to make the definition of shelling property in Section 1 precise. Let P be a d-polytope in $\Re^d$. Let s be a sequence of all m facets of P and denote the i-th facet of s by F_i for $i = 1, \cdots, m$. A sequence s is called a *shelling* of P if the intersection $\overline{F_k} \cap \bigcup_{i=1}^{k-1} \overline{F_i}$ is homeomorphic to a $(d-2)$-dimensional ball, where $\overline{H}$ is the set of the faces of P included in H. Any polytope has at least one shelling because of existence of *line shellings* [1], described below.

Not every sequence s can be a shelling of P. Consider a 3-dimensional cube C_3 with the facets F_i for $i = 1, \cdots, 6$ as in Figure 6. Let s be a sequence of all F_i's whose first three facets are F_6, F_2 and F_5. If the fourth facet of s is F_3, the intersection $\left(\overline{F_6} \cup \overline{F_2} \cup \overline{F_5}\right) \cap \overline{F_3}$ is not homeomorphic to a 2-dimensional ball.

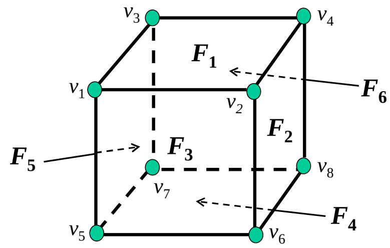

Figure 6: A 3-dimensional cube C_3 with the six facets

Let P be a d-polytope with m facets in $\Re^d$. A directed straight line L that intersects the interior of P and the affine hull of each facet of P at distinct points is called *generic* with respect to P. We choose a generic line L and label a point interior to P on L as x. Starting at x, we number consecutively the intersection points along L with facets as $x_1, x_2, ..., x_m$, wrapping around at infinity, as in Figure 7, The ordering of the corresponding facets of P is the *line shelling* of P generated by L. Every line shelling is a shelling of P (see, e.g., [9]).

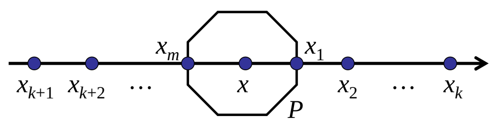

Figure 7: The intersection points along a directed straight line L

Using these definitions we can prove the following proposition.

Proposition 3 *If $G(P)$ is an LP-digraph, it satisfies the shelling property.*

Proof. Choose c such that $G(P) = LP(P, c^T x)$, and suppose P has m vertices. Consider the m distinct parallel hyperplanes with normal c that each contain a vertex of P. Choose a line L parallel to c that intersects the interior of P. Then the intersection of the m hyperplanes with L gives an ordering r of the vertices which is a topological sort of $G(P)$. The corresponding ordering r^* of the facets of the polar P^* is a line shelling of P^*. Therefore $G(P)$ satsifies the shelling property. □

3.2 Acyclicity and Unique sink orientations

In this subsection we complete the proof of Theorem 2. We begin with the following proposition.

Proposition 4 *If $G(P)$ satisfies the shelling property, it is an acyclic USO.*

Proof. Since $G(P)$ satisfies the shelling condition, there exists an order r such that r^* is a shelling of P^*. It follows that $G(P)$ is acyclic.

Next, we prove that $G(P)$ is a USO. Let F_i be the i-th facet of r^*. The acyclicity of $G(P)$ implies that for any face H of P, the induced subdigraph $G(P, H)$ of $G(P)$ has no directed cycles. Therefore, there exists at least one source in $G(P, H)$. When $\dim(H) = 1$ trivially $G(P, H)$ has only one source. We prove that when $\dim(H) > 1$, $G(P, H)$ also has only one source. Let U be the set of the indices of the vertices of P included in H. Assume that there exist more than one source in $G(P, H)$. We denote the set of the indices of the sources in $G(P, H)$ by S, and let k be the smallest index of S. No two sources are adjacent in $G(P, H)$. In the polar, if $j \in S$ and $j \neq k$, no ridges are included in the intersection $\overline{F_j} \cap \bigcup \overline{F_i}$ for $i \in U \backslash S$ and $i < j$. However, the intersection is not empty but includes at least $\overline{H}$, hence the intersection cannot be homeomorphic to a $(d-2)$-dimensional ball. This contradicts the fact that r^* is a shelling of P^*. Therefore every face subdigraph for $G(P)$ has a unique source. □

In the next two propositions we show that if P is simple, or if $d = 2, 3$, then the converse of this proposition is true. A *completely unimodal numbering* of the m vertices of a simple d-polytope P is a numbering $1, \cdots, m$ of the vertices such that there is exactly one local minimum on every k-face F of P, for $k = 2, 3, ..., d$. That is, on F there is only one vertex with no lower-numbered neighbours on F.

Proposition 5 *Let P be a simple d-polytope. If $G(P)$ is an acyclic USO then it satisfies the shelling property.*

Proof. Williamson Hoke [3] has shown that the completely unimodal numberings of P correspond to line shellings of P^*. Since G(P) is acyclic we may choose a topological sort of $G(P)$. Since $G(P)$ is a USO, it follows that this ordering of the vertices is completely unimodal. By Williamson Hoke's result it corresponds to a line shelling of P^* and so $G(P)$ satisfies the shelling property. □

Proposition 6 *Let P be a d-polytope in $\Re^d$, with d=2,3. If $G(P)$ is an acyclic USO, then it satisfies the shelling property.*

Proof. Let P satisfy the conditions of the theorem. Acyclicity implies that there exists a topological sort of $G(P)$. When $d = 2$, it is easy to check that every topological sort r of $G(P)$ induces a shelling r^* of P^*.

Now suppose $d = 3$. Let F_i be the i-th facet of r^*. Since $G(P)$ is a USO, the intersection $\overline{F_j} \cap \bigcup_{i=1}^{j-1} \overline{F_i}$ has at least one 1-face for every $j \geq 2$. Next we prove that for $j \geq 2$, the intersection $I(j) := \overline{F_j} \cap \bigcup_{i=1}^{j-1} \overline{F_i}$ is homeomorphic to a 1-dimensional ball, i.e. the intersection consists of only one connected component. By way of contradiction, let us assume that for some k, $I(k)$ consists of $n \geq 2$ components. When $2 \leq j \leq k-1$, every intersection $I(j)$ consists of only one connected component. Here we consider the *Euler characteristics* of P [9]. It is defined as $\chi(P) := v - e + f$, where v, e and f are the number of vertices, edges and facets of P, respectively. We compute $\chi(P)$ by updating its value when each facet F_j is added according to r^*. The initial contribution of the facet F_1 to $\chi(P)$ is one. Let $x(j)$ be the number of the vertices newly added by F_j. For $2 \leq j \leq k-1$, the contribution of F_j to $\chi(P)$ is equal to $x(j) - (x(j) + 1) + 1 = 0$. Hence the contribution of the F_j's to $\chi(P)$ for $1 \leq j \leq k-1$ is equal to one. On the other hand, the contribution of F_k to $\chi(P)$ is equal to $x(k) - (x(k) + n) + 1 = 1 - n$, hence the contribution of F_j's for $1 \leq j \leq k$ becomes $1 + 1 - n = 2 - n$. We note that $\chi(P) = 0$ when $d = 3$ [9]. It follows that for $h > k$, the number of facets F_h's such that $I(h)$ is the whole boundary of F_h should be at least n, i.e. $G(P)$ has at least n sinks, and so is not a USO. This completes the proof. □

Proposition 6 cannot be extended to the case where $d \geq 4$ as we now show. Let Δ be the 4-dimensional polytope in Figure 8 with 40 vertices and 54 facets, as shown in Table 2. We checked using the software PORTA [2] that Δ is convex when the coordinates of the 40 vertices of Δ are given as Table 1. For the coordinates of the 40 vertices in Table 1, the

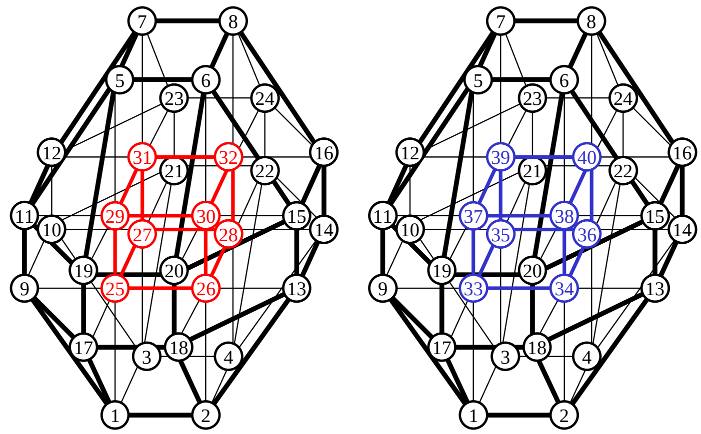

Figure 8: A 4-dimensional polytope Δ with 40 vertices and 54 facets

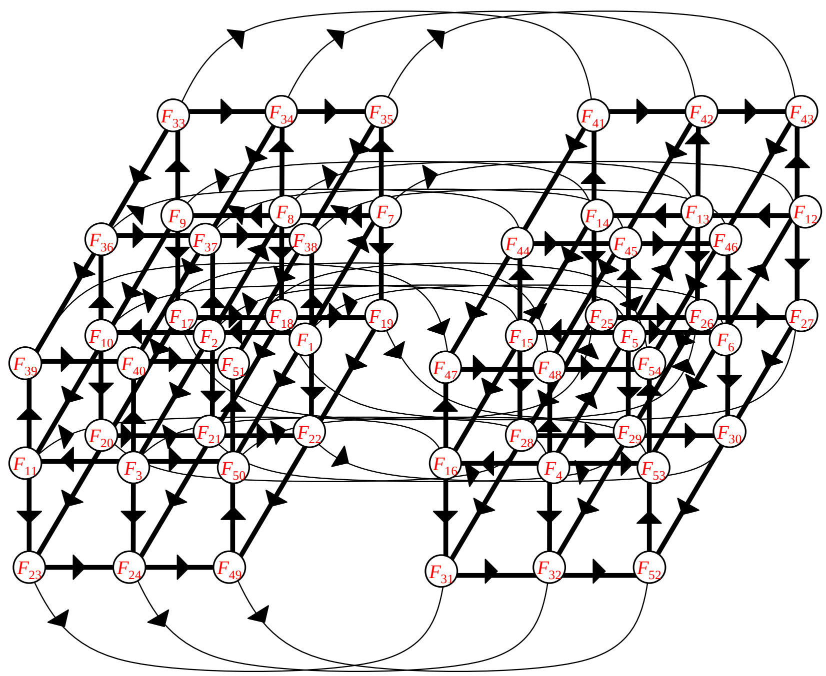

Figure 9: The graph $G(\Delta^*)$

supporting hyperplanes of the 54 facets of Δ are given as $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \leq b$ with the coefficients in Table 2. Let Δ^* be a combinatorial polar of Δ. An orientation $G(\Delta^*)$ of the skeleton of Δ^* is given in Figure 9.

Proposition 7 *$G(\Delta^*)$ is an acyclic USO that satisfies the Holt-Klee property, but not the shelling property.*

Proof. We prove that the graph $G(\Delta^*)$ does not satisfy the shelling property. In every topological sort of $G(\Delta^*)$ the first six facets are always F_1, F_2, F_3, F_4, F_5 and F_6. The

Table 1: The coordinates of the 40 vertices of the 4-dimensional polytope Δ

vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)	vertex	(x_1, x_2, x_3, x_4)
1	$(-1, -1, -3, 0)$	2	$(1, -1, -3, 0)$	3	$(-1, 1, -3, 0)$	4	$(1, 1, -3, 0)$
5	$(-1, -1, 3, 0)$	6	$(1, -1, 3, 0)$	7	$(-1, 1, 3, 0)$	8	$(1, 1, 3, 0)$
9	$(-3, -1, -1, 0)$	10	$(-3, 1, -1, 0)$	11	$(-3, -1, 1, 0)$	12	$(-3, 1, 1, 0)$
13	$(3, -1, -1, 0)$	14	$(3, 1, -1, 0)$	15	$(3, -1, 1, 0)$	16	$(3, 1, 1, 0)$
17	$(-1, -3, -1, 0)$	18	$(1, -3, -1, 0)$	19	$(-1, -3, 1, 0)$	20	$(1, -3, 1, 0)$
21	$(-1, 3, -1, 0)$	22	$(1, 3, -1, 0)$	23	$(-1, 3, 1, 0)$	24	$(1, 3, 1, 0)$
25	$(-1, -1, -1, 1)$	26	$(1, -1, -1, 1)$	27	$(-1, 1, -1, 1)$	28	$(1, 1, -1, 1)$
29	$(-1, -1, 1, 1)$	30	$(1, -1, 1, 1)$	31	$(-1, 1, 1, 1)$	32	$(1, 1, 1, 1)$
33	$(-1, -1, -1, -1)$	34	$(1, -1, -1, -1)$	35	$(-1, 1, -1, -1)$	36	$(1, 1, -1, -1)$
37	$(-1, -1, 1, -1)$	38	$(1, -1, 1, -1)$	39	$(-1, 1, 1, -1)$	40	$(1, 1, 1, -1)$

Table 2: The 54 facets of the 4-dimensional polytope Δ in Figure 8 and the coefficients of their supporting hyperplanes for the coordinates of the 40 vertices in Table 1

facet	vertices	(a_1, a_2, a_3, a_4, b)	facet	vertices	(a_1, a_2, a_3, a_4, b)
F_1	13, 14, 15, 16, 26, 28, 30, 32	$(1, 0, 0, 2, 3)$	F_2	25, 26, 27, 28, 29, 30, 31, 32	$(0, 0, 0, 1, 1)$
F_3	17, 18, 19, 20, 25, 26, 29, 30	$(0, -1, 0, 2, 3)$	F_4	17, 18, 19, 20, 33, 34, 37, 38	$(0, -1, 0, -2, 3)$
F_5	33, 34, 35, 36, 37, 38, 39, 40	$(0, 0, 0, -1, 1)$	F_6	13, 14, 15, 16, 34, 36, 38, 40	$(1, 0, 0, -2, 3)$
F_7	14, 16, 22, 24, 28, 32	$(1, 1, 0, 2, 4)$	F_8	21, 22, 23, 24, 27, 28, 31, 32	$(0, 1, 0, 2, 3)$
F_9	10, 12, 21, 23, 27, 31	$(-1, 1, 0, 2, 4)$	F_{10}	9, 10, 11, 12, 25, 27, 29, 31	$(-1, 0, 0, 2, 3)$
F_{11}	9, 11, 17, 19, 25, 29	$(-1, -1, 0, 2, 4)$	F_{12}	14, 16, 22, 24, 36, 40	$(1, 1, 0, -2, 4)$
F_{13}	21, 22, 23, 24, 35, 36, 39, 40	$(0, 1, 0, -2, 3)$	F_{14}	10, 12, 21, 23, 35, 39	$(-1, 1, 0, -2, 4)$
F_{15}	9, 10, 11, 12, 33, 35, 37, 39	$(-1, 0, 0, -2, 3)$	F_{16}	9, 11, 17, 19, 33, 37	$(-1, -1, 0, -2, 4)$
F_{17}	3, 10, 21, 27	$(-1, 1, -1, 2, 5)$	F_{18}	3, 4, 21, 22, 27, 28	$(0, 1, -1, 2, 4)$
F_{19}	4, 14, 22, 28	$(1, 1, -1, 2, 5)$	F_{20}	1, 3, 9, 10, 25, 27	$(-1, 0, -1, 2, 4)$
F_{21}	1, 2, 3, 4, 25, 26, 27, 28	$(0, 0, -1, 2, 3)$	F_{22}	2, 4, 13, 14, 26, 28	$(1, 0, -1, 2, 4)$
F_{23}	1, 9, 17, 25	$(-1, -1, -1, 2, 5)$	F_{24}	1, 2, 17, 18, 25, 26	$(0, -1, -1, 2, 4)$
F_{25}	3, 10, 21, 35	$(-1, 1, -1, -2, 5)$	F_{26}	3, 4, 21, 22, 35, 36	$(0, 1, -1, -2, 4)$
F_{27}	4, 14, 22, 36	$(1, 1, -1, -2, 5)$	F_{28}	1, 3, 9, 10, 33, 35	$(-1, 0, -1, -2, 4)$
F_{29}	1, 2, 3, 4, 33, 34, 35, 36	$(0, 0, -1, -2, 3)$	F_{30}	2, 4, 13, 14, 34, 36	$(1, 0, -1, -2, 4)$
F_{31}	1, 9, 17, 33	$(-1, -1, -1, -2, 5)$	F_{32}	1, 2, 17, 18, 33, 34	$(0, -1, -1, -2, 4)$
F_{33}	7, 12, 23, 31	$(-1, 1, 1, 2, 5)$	F_{34}	7, 8, 23, 24, 31, 32	$(0, 1, 1, 2, 4)$
F_{35}	8, 16, 24, 32	$(1, 1, 1, 2, 5)$	F_{36}	5, 7, 11, 12, 29, 31	$(-1, 0, 1, 2, 4)$
F_{37}	5, 6, 7, 8, 29, 30, 31, 32	$(0, 0, 1, 2, 3)$	F_{38}	6, 8, 15, 16, 30, 32	$(1, 0, 1, 2, 4)$
F_{39}	5, 11, 19, 29	$(-1, -1, 1, 2, 5)$	F_{40}	5, 6, 19, 20, 29, 30	$(0, -1, 1, 2, 4)$
F_{41}	7, 12, 23, 39	$(-1, 1, 1, -2, 5)$	F_{42}	7, 8, 23, 24, 39, 40	$(0, 1, 1, -2, 4)$
F_{43}	8, 16, 24, 40	$(1, 1, 1, -2, 5)$	F_{44}	5, 7, 11, 12, 37, 39	$(-1, 0, 1, -2, 4)$
F_{45}	5, 6, 7, 8, 37, 38, 39, 40	$(0, 0, 1, -2, 3)$	F_{46}	6, 8, 15, 16, 38, 40	$(1, 0, 1, -2, 4)$
F_{47}	5, 11, 19, 37	$(-1, -1, 1, -2, 5)$	F_{48}	5, 6, 19, 20, 37, 38	$(0, -1, 1, -2, 4)$
F_{49}	2, 13, 18, 26	$(1, -1, -1, 2, 5)$	F_{50}	13, 15, 18, 20, 26, 30	$(1, -1, 0, 2, 4)$
F_{51}	6, 15, 20, 30	$(1, -1, 1, 2, 5)$	F_{52}	2, 13, 18, 34	$(1, -1, -1, -2, 5)$
F_{53}	13, 15, 18, 20, 34, 38	$(1, -1, 0, -2, 4)$	F_{54}	6, 15, 20, 38	$(1, -1, 1, -2, 5)$

intersection $\overline{F_6} \cap \bigcup_{i=1}^{5} \overline{F_i}$ is not homeomorphic to a 2-dimensional ball. Hence there exists no topological sort of $G(\Delta^*)$ which is a shelling of Δ. This complete the proof. $\square$

Next we form a new orientation $G'(\Delta^*)$ of the skeleton of Δ^* by exchanging the orientations of the two edges (F_7, F_8) and (F_{21}, F_{22}) of $G(\Delta^*)$. We have the following result.

Proposition 8 *$G'(\Delta^*)$ is an acyclic USO that does not satisfy either the Holt-Klee property or the shelling property.*

Proof. Firstly we observe that the 3-dimensional cube shown in Figure 10 does not satisfy the Holt-Klee property, as the vertices F_2 and F_{22} are a two point cut-set. This cube is a

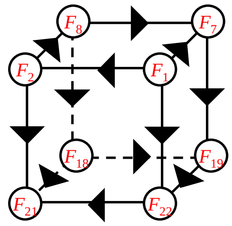

Figure 10: The directed graph of a 3-dimensional cube not satisfying the Holt-Klee property

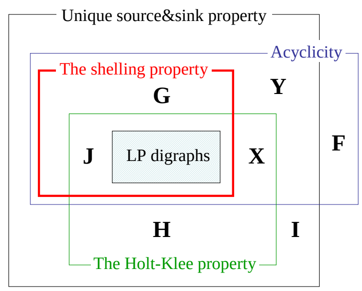

Figure 11: The relationships when P is general polytope and $d \geq 4$

3-face of $G'(\Delta^*)$. Moreover, $G'(\Delta^*)$ does not satisfy the shelling property by the same proof as that of Proposition 7. $\square$

Collecting the above propostions, we have verified the relationships between the five properties shown in Figure 11 for non-simple polytopes of dimension at least four. The examples G and J are orientationss of a cube, hence they satisfy the shelling property [3]. The examples $G(\Delta^*)$ and $G'(\Delta^*)$ above are the examples X and Y, respectively. Combined with the results for $d = 2, 3$, we have proved Theorem 2.

4 Conclusion

In this paper, we introduced a new necessary property for a directed graph of a d-polytope P to be an LP digraph, called the shelling property. We analyzed the relationships between the shelling property and the three existing propeties, acyclicity, unique sink orientation, and the Holt-Klee property. We proved that if $G(P)$ satisfies the shelling property, it is an acyclic USO. The converse is true for simple polytopes but not in general if the dimension is at least four. In showing this we gave two examples, one satisfying the Holt-Klee property and one not. This establishes the independence of the shelling condition from the other conditions.

Moreover, when $d = 2, 3$, we also proved that if $G(P)$ is an acyclic USO then it also satisfies the shelling property. This is an extension of Williamson-Hoke's result [3], which states that when P is simple, $G(P)$ is an acyclic USO if and only if it satisfies the shelling property.

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