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EXTENSIONS OF DAGUM'S GINI DECOMPOSITION

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Abstract: The purpose of this paper is to extend Dagum's Gini decomposition ("A New Approach to the Decomposition of the Gini Income Inequality Ratio", *Empirical Economics* 22(4), 515-531, 1997a) following three types of theoretical modelisation. The first one deals with a "poor/non-poor" decomposition within a sub-group multilevel framework. The second one exhibits the multi-decomposition technique, that is, the combination of the sub-group and the income source decomposition. Finally, we provide a parametric multi-decomposition in order to capture different dimensions of income inequality within groups and between groups.

Keywords and phrases: Gini, Income Source Decomposition, Multi-decomposition, Poverty, Sub-group Decomposition.

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I. Introduction

For a long time the Gini decomposition has been considered as a non-appealing method. In the context of income inequality decomposition, the comparison of the Gini coefficient with the measures derived from entropy, as the Theil index (1967), highlights questions about its particular structure.

Nevertheless, many authors have continued to develop new Gini decompositions, which [following Shorrocks (1999)] exercised their mind. The first one can be attributed to Soltow (1960). Then, Bhattacharya and Mahalanobis (1967) introduced a Gini decomposition on the basis of Gini's mean difference index (1912) in order to study regional disparities. Afterwards, Rao (1969) first proposed the two existing procedures to decompose an income inequality measure, that is: the sub-group (sub-population) decomposition; and, the income source decomposition (factor component). His sub-group decomposition is an atypical one. He offers a two-term decomposition with a quadratic form, which points out the fundamental difference that prevails between the Gini coefficient and the entropy measures. The Gini ratio provides an unusual between-group component because it measures the income inequalities between each and every pair of sub-populations, whereas entropy and most of between-group inequality measures yield the income inequalities in mean between the sub-populations. Rao's decomposition by factor component makes possible the computation of each income source contribution (such as wages, fringe benefits, child support benefits, transfers, etc.) in the amount of the overall inequality.

Mookherjee and Shorrocks (1982) demonstrated that Gini index can be rewritten in order to bring out a between-group Gini index in mean plus a residual term, say "interaction term". Their conclusion has incited many researchers to reject the Gini coefficient as a good decomposable measure. Contrary to this, the entropic measures were increasingly used, generalized [see Bourguignon (1979), Shorrocks (1980), Cowell (1980a, 1980b)], and defined as sub-group consistent.

The sub-group consistency is a new decomposition property introduced by Shorrocks (1984, 1988) in the field of income inequality measurement and particularly *via* the entropic indices. Let us remember. Imagine a change in incomes of group *A*, *ceteris paribus*, such that mean incomes and population shares remain unchanged. If the inequality increases (decreases) in group *A*, then the overall inequality rises (decreases). This property comes from the notion of sub-group monotonicity, introduced by Foster, Greer and Thobercke (1984).

A problem arises with the sub-group consistency property since it leads inevitably to a between-group component, which represents the inequalities in mean between the groups. In this sense, Dagum (1997a) advanced some critical insights. Indeed, this kind of measures needs an equally distributed income vector within each group that is equal to the mean income of the corresponding group. Then, inequalities in mean between the groups are not valid and cannot characterize a good statistical measure since it is very similar to the one-way variance analysis (ANOVA), for which: (*i*) the sub-groups have equal variances; (*ii*) the observations are statistically independent; and (*iii*) the distributions are equally distributed.

Following these statistical limitations concerning the sub-group consistency property, we investigate new kinds of decomposition using the Gini coefficient and extending Dagum's Gini decomposition (1997a, 1997b). After reviewing Dagum's approach (*Section 2*), we deal with our initial model, the multilevel Gini decomposition, that is, a sequence of overlapping sub-group decompositions with the case of "poor/non-poor" decomposition (*Section 3*). The second modelisation exposes a mixture of decomposition, the so-called Gini multi-decomposition, which merges the sub-group and the income source Gini decompositions (*Section 4*). This technique is apprehended within a multidimensional framework and then combined with the multilevel Gini decomposition to yield our third model (*Section 5*). Finally, *Section 6* is devoted to the concluding remarks.

II. DAGUM'S GINI DECOMPOSITION BY SUB-POPULATION (1997A, 1997B)

Let P be a population with n income units: x_i (i, r = 1,…, n). P is partitioned into k sub-populations P_j (j=1,…, k) of size n_j, with cumulative distribution function $F_j(x)$, and mean income μ_j. Let $F(x)$ and μ be respectively the cumulative distribution function and the mean income of P. The income level of the i-th individual that belongs to the j-th group is x_{ij} (i, r =1,…, n_j). Then, the overall income vector of P can be written as:

$$\left(x_1,\ldots,x_i,\ldots,x_n\right)=\left(\left(x_{11},\ldots,x_{n_1 1}\right),\ldots,\left(x_{1j},\ldots,x_{n_j j}\right),\ldots,\left(x_{1k},\ldots,x_{n_k k}\right)\right) \tag{1}$$

The Gini index, based on Gini's mean difference (1912), is given by:

$$G=\frac{\sum_{i=1}^{n}\sum_{r=1}^{n}\left|x_i-x_r\right|}{2n^2\mu} \;. \tag{2}$$

Definition 1. *The Gini index associated with sub-group P_j yields the income inequalities within P_j:*

$$G_{jj}=\frac{\sum_{i=1}^{n_j}\sum_{r=1}^{n_j}\left|x_{ij}-x_{rj}\right|}{2n_j^2\mu_j} \;. \tag{3}$$

Definition 2. *The Gini index associated with two sub-groups P_j and P_h* [*Dagum* (1987)] *quantifies the income inequalities between P_j and P_h:*

$$G_{jh}=\frac{\sum_{i=1}^{n_j}\sum_{r=1}^{n_h}\left|x_{ij}-x_{rh}\right|}{(\mu_j+\mu_h)\;n_j\,n_h}\;,\;\forall\, j,h=1,\ldots,k \;. \tag{4}$$

The G_{jh} index involves the $n_j \times n_h$ binary income differences between P_j and P_h, whereas G_{jj} involves the $n_j \times n_j$ binary income differences within P_j. Gathering these income differences within groups and between groups, we bring out a two-term Gini decomposition by sub-population:

$$G = \frac{\sum_{j=1}^{k}\left(\sum_{i=1}^{n_j}\sum_{r=1}^{n_j}\left|x_{ij}-x_{rj}\right|\right)}{2\mu n^2} + \frac{2\sum_{j=2}^{k}\sum_{h=1}^{j-1}\left(\sum_{i=1}^{n_j}\sum_{r=1}^{n_h}\left|x_{ij}-x_{rh}\right|\right)}{2\mu n^2} \tag{5}$$

$$= G^w + G^{gb}. \tag{6}$$

G^w characterizes the within-group Gini index, and G^{gb} represents the gross between-group Gini index that gauges the inequalities between each and every pair of sub-groups [in the same way than Rao (1969)]. The expression "gross between-group" singles out G^{gb} from the measures between the groups in mean and signifies that G^{gb} yields a net income inequality index between groups.

The Gini index can be formulated as a weighted average of the Gini indices associated with one group [G_{jj}], and with two groups [G_{jh}]. The weights are population shares and income shares, respectively:

$$p_j = \frac{n_j}{n}, \quad s_j = \frac{n_j \mu_j}{n\,\mu}. \tag{7}$$

Then, the Gini index is:

$$\begin{aligned} G &= \sum_{j=1}^{k} G_{jj}\, p_j s_j + \sum_{j=2}^{k}\sum_{h=1}^{j-1} G_{jh}\left(p_j s_h + p_h s_j\right) \\ &= G^w + G^{gb}. \end{aligned} \tag{8}$$

Let us introduce the gross economic affluence and the first-order moment of transvariation in order to implement a decomposition in three components [see also Silber (1989), Lerman and Yitzhaki (1991), etc.].

Lemma 1. *The gross economic affluence d_{jh} is a weighted average of the income differences $x_{ij}-x_{rh}$ for each x_{ij} of P_j which is higher than x_{rh} of P_h, given that P_j is in mean more "affluent" than P_h ($\mu_j > \mu_h$):*

$$d_{jh} = \int_0^\infty dF_j(x) \int_0^x (x-y)\, dF_h(y). \tag{9}$$

Lemma 2. *The first-order moment of transvariation p_{jh} between the j-th and the h-th sub-populations (such as $\mu_j > \mu_h$) is:*

$$p_{jh} = \int_0^\infty dF_h(x) \int_0^x (x-y)\, dF_j(y). \tag{10}$$

The transvariation [Gini (1916), Dagum (1959, 1960, 1961)] stands to the fact that income differences are of opposite sign compared with the difference of their corresponding mean incomes.

The gross economic affluence and the first-order moment of transvariation yield the relative economic affluence (directional economic distance ratio, "economic distance" from now on) between the *j*-th and the *h*-th sub-populations:

$$D_{jh} = \frac{d_{jh} - p_{jh}}{d_{jh} + p_{jh}} \ . \tag{11}$$

D_{jh} is included in the close interval [0,1]. D_{jh} = 0 when $\mu_j = \mu_h$, and D_{jh} = 1 when the probability functions of P_j and P_h do not overlap. It is a normalized measure of dimension zero since d_{jh} and p_{jh} have the same dimension. The relative economic affluence permits the gross between-group inequality to be separated in two contributions:

- the net contribution of the extended Gini between groups to the overall Gini index

$$G^{nb} = \sum_{j=2}^{k} \sum_{h=1}^{j\text{-}1} G_{jh} \ D_{jh} \left(p_j s_h + p_h s_j \right) \ ; \tag{12}$$

- and the contribution of the intensity of transvariation between groups to the global Gini index

$$G^{t} = \sum_{j=2}^{k} \sum_{h=1}^{j\text{-}1} G_{jh} \ \left(1 - D_{jh} \right) \left(p_j s_h + p_h s_j \right). \tag{13}$$

Indeed, the gross between-group Gini index can be expressed as follows:

$$\begin{aligned} G^{gb} &= \sum_{j=2}^{k} \sum_{h=1}^{j\text{-}1} G_{jh} \ \left(p_j s_h + p_h s_j \right) \\ &= \sum_{j=2}^{k} \sum_{h=1}^{j\text{-}1} G_{jh} \ D_{jh} \left(p_j s_h + p_h s_j \right) + \sum_{j=2}^{k} \sum_{h=1}^{j\text{-}1} G_{jh} \ \left(1 \text{-} D_{jh} \right) \left(p_j s_h + p_h s_j \right) \\ &= G^{nb} + G^{t} \ . \end{aligned} \tag{14}$$

The transvariation intensity measures the disparities of overlap between the distributions, that is, the inequalities inherent to the high incomes of the poorest sub-populations (in mean). The net between-group Gini measures the inequalities of non-overlap between the income distributions, that is, those generated by the high incomes of the richest sub-groups (in mean) as well as the inequalities in mean (since G_{nb} depends on D_{jh}, where $D_{jh} = 0$ if $\mu_j = \mu_h$).

Theorem [Dagum (1997a)]. *The Gini coefficient computed on P of size n, partitioned into k sub-groups of size n_j (j = 1,…, k) is given by:*

$$G = G^w + G^{nb} + G^t . \tag{15}$$

III. The multilevel Gini decomposition

III.1. The general framework

Cowell (1985), Wodon (1999) and Salas (2002) introduced the multilevel decomposition technique. The overall population is divided into a first partition K, with k groups P_j (j =1,…, k). Let S be a second partition. Suppose that each group P_j is separated into q_j sub-groups (w, z =1,…, q_j ; hereafter the groups are associated with the first partition K and the sub-groups with S).

Salas's result about the multilevel entropy decomposition is defined as follows:

$$I = I^{b,K} + I^{b,SK} + I^{w,S} , \tag{16}$$

where $I^{b,K}$ is the between-group inequalities (in mean) among the groups of the first partition; $I^{b,SK}$ is the between-group measure (in mean) among the sub-groups of the second partition S; and, $I^{w,S}$ is the inequalities within the sub-groups of the second partition.

Let us extend Dagum's Gini decomposition and introduce the multilevel framework. We name n_z and μ_z (and n_w, μ_w) the number of individuals and the mean income of the z-th sub-group (w-th) of the second partition, respectively. Therefore, we can calculate the inequalities within the z-th sub-group of the j-th group [$G_{j,zz}$]:

$$G_{j,zz} = \frac{\sum_{i=1}^{n_z}\sum_{r=1}^{n_z}\left|x_{iz} - x_{rz}\right|}{2 n_z^2 \mu_z} . \tag{17}$$

Also, it is possible to estimate the inequalities between the z-th and the w-th sub-groups of group P_j of the first partition [$G_{z,wz}$]:

$$G_{j,wz} = \frac{\sum_{i=1}^{n_w}\sum_{r=1}^{n_z}\left|x_{iw} - x_{rz}\right|}{\left(\mu_z + \mu_w\right) n_z\, n_w} . \tag{18}$$

Proposition 1. *When the population P is separated in two group partitions, we have a multilevel Gini decomposition such as:*

$$G = \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j} G_{j,zz}\, p'_z\, s'_z\right) p_j s_j \qquad (G^{w,S}) \tag{19}$$

$$+ \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w\, s'_z + p'_z\, s'_w\right)\right) p_j s_j \qquad (G^{gb,SK})$$

$$+ \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right). \qquad (G^{gb,K})$$

(i) $G^{,w,S}$ *is the contribution of the inequalities along the second partition S within K;*
(ii) $G^{gb,SK}$ *is the gross contribution of the inequalities between the sub-populations of the second partition S within the first partition K* (*the gross second-order between-group inequality along S within K*)*;*
(iii) $G^{gb,K}$ *is the gross contribution of the between-group inequality along the first partition K* (*the gross first-order between-group inequality along K*).

Proof. The Gini decomposition in two elements proposed by Dagum (1997a, 1997b) is:

$$G = \sum_{j=1}^{k} G_{jj}\, p_j s_j + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right). \tag{20}$$

This holds for the first group partition K. It is also feasible for the second group partition S within P_j:

$$G_{jj} = \sum_{z=1}^{q_j} G_{j,zz}\; p'_z\; s'_z + \sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right), \tag{21}$$

where the population share and the income share of the z-th sub-group (in P_j) are:

$$p'_z = \frac{n_z}{n_j}\ ;\ s'_z = \frac{n_z\ \mu_z}{n_j\ \mu_j}\ . \tag{22}$$

Insert (21) in (20), the proof is completed. ■

When the Gini decomposition (8) is directly implemented on the second partition, it is possible to find a standard decomposition in two components, that are: (i) the contribution of the inequalities along the second partition S within the overall population P [$G^{,w,S}$]; (ii) and the gross between-group inequalities along the partition S within P [$G^{,gb,S}$]. This means that the overall inequality is defined with Gini coefficients related to the last partition.

Let us now generalise proposition 1.

Corollary 1. *The multilevel Gini decomposition based on two group partitions, K and S, can be generalized to a finite α number of partitions.*

Proof. Let α be the number of sub-partitions. Let $G^{\alpha}_{j,z}$ be the Gini ratio associated with the z-th sub-group of the α-th partition (issued from the j-th group of the first partition) and $G^{\alpha}_{j,wz}$ be the Gini ratio associated with the w-th and the z-th sub-groups of the α-th partition. Equation (19) can be recursively rewritten as:

$$
\begin{aligned}
G = {} & \sum_{j=1}^{k}\left(\overbrace{\sum_{z=1}^{q_j} \cdots \Big(\sum_{z=1}^{q_j^{\alpha}}}^{\alpha-1\text{ times}} G^{\alpha}_{j,z}\, p^{\alpha}_{j,z} s^{\alpha}_{j,z}\Big) \cdots\, p_z s_z\right) p_j s_j & (G^{w,\alpha}) \qquad (23)\\
& + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right) & (G^{gb,K})\\
& + \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j^2}\sum_{w=1}^{z-1} G^{2}_{j,wz}\,\varpi^{2}_{j,wz}\, p^{2}_{z} s^{2}_{z}\right) p_j s_j & (G^{gb,2K})\\
& \vdots & \vdots\\
& + \sum_{j=1}^{k}\left(\overbrace{\sum_{z=1}^{q_j} \cdots \Big(\sum_{z=1}^{q_j^{\alpha-1}}}^{\alpha-2\text{ times}} \Big(\sum_{z=1}^{q_j^{\alpha}}\sum_{w=1}^{z-1} G^{\alpha}_{j,wz}\,\varpi^{\alpha}_{j,wz}\Big) p^{\alpha-1}_{j,z} s^{\alpha-1}_{j,z}\Big) \cdots\, p^{2}_{z} s^{2}_{z}\right) p_j s_j\ , & (G^{gb,\alpha K})
\end{aligned}
$$

where $p^{\alpha}_{j,z}$ and $s^{\alpha}_{j,z}$ are the population share and the income share of the z-th sub-group of the α-th partition within the j-th first partition, and where the weights associated with the Gini index between the w-th and the z-th groups of the α-th partition are:

$$
\varpi^{\alpha}_{j,wz} = p^{\alpha}_{j,w} s^{\alpha}_{j,z} + p^{\alpha}_{j,z} s^{\alpha}_{j,w}\,. \tag{24}
$$

Expression $G^{w,\alpha}$ stands for the within-group inequalities along the α-th partition within K and $G^{gb,2K}$ for the gross second-order between-group inequality along S within K, which is equivalent to $G^{gb,SK}$ in (19). $G^{gb,\alpha K}$ is the gross α-order between-group inequality along the α-th partition within K. Consequently, we obtain the generalization of the three-term Gini decomposition (15) in a multilevel context. ■

The within-group Ginis [G^{w} and $G^{w,\alpha}$] converge toward 0 when the number of groups is equal to the number of individuals. The between-group Ginis [$G^{gb,K}$, $G^{gb,\alpha K}$] converge toward 0 when the number of groups tends toward one. Proposition 1 relies on the gross between-group inequalities. A five-term decomposition can be implemented adding the role of the transvariation intensity. We include the economic distance between the z-th and the w-th distributions of the second partition S:

$$
D_{wz} = \frac{d_{wz} - p_{wz}}{d_{wz} + p_{wz}}\ . \tag{25}
$$

Proposition 2. *When the overall population P is separated in two partitions* (*the second partition S along the first partition K*)*, there exists a multilevel Gini decomposition in five components, which brings out the intensity of transvariation between the groups and the sub-groups of the first and the second partition respectively:*

$$\begin{aligned} G = & \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j} G_{j,zz}\, p'_z\, s'_z\right) p_j s_j && (G^{w,S}) \qquad (26) \\ & + \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right) D_{wz}\right) p_j s_j && (G^{nb,SK}) \\ & + \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right)\left(1-D_{wz}\right)\right) p_j s_j && (G^{t,SK}) \\ & + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right) D_{jh} && (G^{nb,K}) \\ & + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right)\left(1-D_{jh}\right). && (G^{t,K}) \end{aligned}$$

(*i*) $G^{w,S}$ *is the contribution of the inequalities along the second partition S within the first partition K;*
(*ii*) $G^{nb,SK}$ *is the net contribution of the inequalities between the sub-populations of the sub-partition S within the partition K* (*the net second-order between-group inequality*)*;*
(*iii*) $G^{t,SK}$ *is the intensity of transvariation between the sub-populations of the sub-partition S within the partition K* (*the second-order between-group inequality of transvariation along S within K*)*;*
(*iv*) $G^{nb,K}$ *is the net contribution of the inequality between the groups along the first partition K* (*the net first-order between-group inequality*)*;*
(*v*) $G^{t,K}$ *is the intensity of transvariation between the groups along the first partition K* (*the first-order between-group inequality of transvariation*).

Proof. Firstly, multiply $G^{B,SK}$ by the economic distance [D_{wz}] of the sub-populations of the second partition *S* and by the ratio of overlap ($1-D_{wz}$). Secondly, multiply $G^{B,K}$ by the economic distance [D_{jh}] of the groups of the first partition *K* and by the ratio of overlap ($1-D_{jh}$). Therefore, the proof of the multilevel decomposition in five elements is completed. ■

Corollary 2. *A generalization of the five-term multilevel Gini decomposition* (26) *based on two group partitions K and S can be proposed for* α *partitions*.

Proof. See Appendix 1. ■

The multilevel Gini decompositions explain the sub-group determinants of the overall inequality, and are of interest when we aim at analysing inequalities and poverty.

III.2. A POOR NON-POOR INEQUALITY ANALYSIS

Suppose two sub-populations (poor and non-poor) within each group of the first partition *K*. Consequently, we obtain a particular case of proposition 2, since the income distributions of the poor and the non-poor sub-groups do not overlap [$D_{wz} = 1$].

Corollary 3. *When each group P_j of the first partition K is partitioned in two non-overlapping sub-groups such as poor and non-poor sub-populations, the multilevel Gini decomposition in five elements becomes a four-element equation as:*

$$
\begin{aligned}
G = & \sum_{j=1}^{k}\left(\sum_{z=1}^{2} G_{j,zz}\, p'_z\, s'_z\right) p_j s_j && (G^{w,S}) && (27)\\
& + \sum_{j=1}^{k}\left(G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right)\right) p_j s_j && (G^{nb,SK}) &&\\
& + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right) D_{jh} && (G^{nb,K}) &&\\
& + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right)\left(1 - D_{jh}\right). && (G^{t,K}) &&
\end{aligned}
$$

(i) $G^{w,S}$ *is the contribution of the inequalities along the poor and the non-poor sub-populations within the first partition K;*
(ii) $G^{nb,Sk}$ *is the net contribution of the inequalities between the poor and the non-poor sub-populations within the partition K;*
(iii) $G^{nb,k}$ *is the net contribution of the inequalities between the groups of the first partition K;*
(iv) $G^{t,k}$ *is the intensity of transvariation between the groups of K.*

Proof. The number of sub-populations within each group P_j of the first partition is: $q_j = 2$. Thus, $G^{w,S}$ (*the contribution of the inequalities along the second partition S within the first partition K*) can be reformulated as:

$$G = \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j} G_{j,zz}\, p'_z\, s'_z\right) p_j\, s_j = \sum_{j=1}^{k}\left(\sum_{z=1}^{2} G_{j,zz}\, p'_z\, s'_z\right) p_j s_j\,. \tag{28}$$

As $D_{wz} = 1$, we immediately have:

$$G^{t,SK} = \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right)\left(1 - D_{wz}\right)\right) p_j s_j = 0\,. \tag{29}$$

On the other hand, $G^{nb,SK}$ (*the net second-order between-group inequality*) is:

$$G^{nb,SK} = \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right) D_{wz}\right) p_j s_j \tag{30}$$

$$= \sum_{j=1}^{k} \left(G_{j,wz} \left(p'_w s'_z + p'_z s'_w \right) \right) p_j s_j . \ \blacksquare$$

Example of interpretation. Imagine a spatial decomposition. The first partition is a regional one, whereas the second one deals with poor/non-poor income distributions, after introducing an adequate poverty line. In this case, applying the multilevel Gini decomposition yields four indices: (*i*) $G^{w,S}$ measures the inequalities of the poor and the non-poor sub-populations within each region; (*ii*) $G^{nb,SK}$ gives the net inequalities between the poor and the non-poor sub-populations along each region; (*iii*) $G^{nb,K}$ gauges the net between-group inequalities between each and every pair of regions; and (*iv*) $G^{t,K}$ estimates the intensity of transvariation between regions.

Finally, this kind of multilevel decomposition is useful to analyse poor and non-poor distributions and particularly, their impacts on the overall inequality.

IV. THE MULTI-DECOMPOSITION

Mussard (2004) provided a transparent link between the Gini decomposition by sub-group and the Gini decomposition by factor component[1] in introducing the income source decomposition into the Gini decomposition by sub-group. Let us present the income source method with the well-known formula:

$$\left| x_i - x_r \right| = x_i + x_r - 2\left(x_i \wedge x_r \right), \tag{31}$$

where $\left(x_i \wedge x_r \right)$ selects the minimum value between the incomes x_i and x_r. The Gini ratio can be expressed as follows:

$$G = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} \left(x_i + x_r - 2\left(x_i \wedge x_r \right) \right)}{2\mu n^2} . \tag{32}$$

The incomes are aggregated along q income sources (such as labour income, capital income, child support benefits, taxes, etc.). Then, the i-th individual's income is:

$$x_i = \sum_{m=1}^{q} x_i^m . \tag{33}$$

The overall Gini ratio is now rewritten as:

[1] This technique is also available for the Gini mean difference and the Gini mean ratio [see Mussard (2004)].

$$G = \frac{\sum_{i=1}^{n}\sum_{r=1}^{n}\left(\sum_{m=1}^{q} x_i^m + \sum_{m=1}^{q} x_r^m - 2(x_i \wedge x_r)\right)}{2\mu n^2}. \tag{34}$$

We define the income source decomposition of $2(x_i \wedge x_r)$ by:

$$\sum_{m=1}^{q} 2x_{ir}^{*m} := 2(x_i \wedge x_r). \tag{35}$$

Proposition 3. *Given* (34) *and* (35) *a Gini decomposition by factor component can be expressed as:*

$$G = \sum_{m=1}^{q}\left(\frac{\sum_{i=1}^{n}\sum_{r=1}^{n}\left(x_i^m + x_r^m - 2x_{ir}^{*m}\right)}{2\mu n^2}\right) = \sum_{m=1}^{q}\left(S^m\right), \tag{36}$$

where S^m is the contribution of the m-th source to G.

Proof. It is straightforward. ■

Corollary 4. *If the q income sources are q replications* (*say q i.i.d. variables*), *we have:*

$$S^1 = \ldots = S^m = \ldots = S^q \Rightarrow G = qS^m. \tag{37}$$

Proof. It is straightforward. ■

Corollary 4 indicates that the source contributions are proportional to the Gini mean difference. Consequently, if the factor component distributions are equally distributed, then the source contributions are equal to zero.

As the Gini index associated with one group [G_{jj}], or with two groups [G_{jh}] has the same structure than the overall Gini ratio, we prove that these indices are factor decomposable:

$$G_{jj} = \sum_{m=1}^{q}\left(\frac{\sum_{i=1}^{n_j}\sum_{r=1}^{n_j}\left(x_{ij}^m + x_{rj}^m - 2x_{irj}^{*m}\right)}{2\mu_j n_j^2}\right) = \sum_{m=1}^{q}\left(S_{jj}^m\right), \tag{38}$$

$$G_{jh} = \sum_{m=1}^{q}\left(\frac{\sum_{i=1}^{n_j}\sum_{r=1}^{n_h}\left(x_{ij}^m + x_{rh}^m - 2x_{irjh}^{*m}\right)}{(\mu_j + \mu_h)\, n_j\, n_h}\right) = \sum_{m=1}^{q}\left(S_{jh}^m\right), \tag{39}$$

where x_{irj}^{*m} is the m-th source of the minimum value between x_{ij} and x_{rj} , and where x_{irjh}^{*m} is the m-th source of the minimum value between x_{ij} and x_{rh}.

Therefore, the introduction of these source decompositions into the sub-group decomposition yields a synthesis decomposition, the so-called multi-decomposition.

Proposition 4. *The Gini index is two-term multi-decomposable within an exact structure.*

Proof. Substituting (38) and (39) in Dagum's Gini decomposition in two elements (8), we find:

$$\begin{aligned} G = & \sum_{j=1}^{k}\sum_{m=1}^{q}\left(S_{jj}^{m}\right) p_j\, s_j && (G_w) \qquad (40)\\ & + \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q} G_{jh}^{m}\left(p_j\, s_h + p_h\, s_j\right) . && (G_{gb}) \end{aligned}$$

This decomposition in two dimensions (group and income source) entails the estimation of the combinations "group j / source m" and "between groups j and h / source m". Furthermore, this multi-decomposition is exact since all the elements are decomposable without redundant terms (except the denominator):

$$G = \sum_{m=1}^{q}\left(\frac{\sum_{j=1}^{k}\left(\sum_{i=1}^{n_j}\sum_{r=1}^{n_j}\left(x_{ij}^{m} + x_{rj}^{m} - 2x_{irj}^{*m}\right)\right)}{2\mu n^2}\right) + \sum_{m=1}^{q}\left(\frac{2\sum_{j=2}^{k}\sum_{h=1}^{j-1}\left(\sum_{i=1}^{n_j}\sum_{r=1}^{n_h}\left(x_{ij}^{m} + x_{rh}^{m} - 2x_{irjh}^{*m}\right)\right)}{2\mu n^2}\right) . \ \blacksquare \qquad (41)$$

Proposition 5. *The Gini index is three-term multi-decomposable within an exact structure.*

Proof. Firstly, decompose the economic distance between the j-th and the h-th groups by factor component:

$$D_{jh} = \sum_{m=1}^{q} D_{jh}^{m} ,\forall \mu_j > \mu_h \text{ where,} \qquad (42)$$

$$D_{jh}^{m} = \left(\frac{\left(\sum_{x_{ij}>x_{rh}}\left(x_{ij}^{m} + x_{rh}^{m} - 2x_{irjh}^{*m}\right)\right) - \left(\sum_{x_{ij}<x_{rh}}\left(x_{ij}^{m} + x_{rh}^{m} - 2x_{irjh}^{*m}\right)\right)}{\sum_{i=1}^{n_j}\sum_{r=1}^{n_h}\left|x_{ij} - x_{rh}\right|}\right) . \qquad (43)$$

Secondly, decompose the $(1-D_{jh})$ statistics, say the overlap ratio:

$$1-D_{jh}=P_{jh}=\sum_{m=1}^{q}\frac{2\left(\sum\limits_{x_{ij}<x_{rh}}\left(x_{ij}^{m}+x_{rh}^{m}-2x_{irjh}^{*m}\right)\right)}{\sum\limits_{i=1}^{n_j}\sum\limits_{r=1}^{n_h}\left|x_{ij}-x_{rh}\right|}=\sum_{m=1}^{q}P_{jh}^{m}\,,\ \forall\mu_j>\mu_h\,, \tag{44}$$

where P_{jh}^{m} is the contribution of the m-th source to P_{jh}. This overlap ratio tends toward one when the j-th and the h-th distributions overlap, and tends toward zero when the distributions do not overlap. Hence, substituting (43) and (44) in the sub-group decomposition (15) gives:

$$\begin{aligned} G=&\sum_{j=1}^{k}\sum_{m=1}^{q}\left(S_{jj}^{m}\right)p_j\ s_j && (G_w)\\ &+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}D_{jh}^{m}\left(p_j s_h+\ p_h s_j\right)G_{jh} && (G_{nb})\\ &+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}P_{jh}^{m}\left(p_j s_h+\ p_h s_j\right)G_{jh}\ . && (G_t)\end{aligned} \tag{45}$$

This three-term multi-decomposition is exact since there are no redundant terms:

$$\begin{aligned} G=&\sum_{m=1}^{q}\left(\frac{\sum\limits_{j=1}^{k}\left(\sum\limits_{i=1}^{n_j}\sum\limits_{r=1}^{n_j}\left(x_{ij}^{m}+x_{rj}^{m}-2x_{irj}^{*m}\right)\right)}{2\mu n^2}\right) && (G_w)\\ &+\sum_{m=1}^{q}\frac{2\sum\limits_{j=2}^{k}\sum\limits_{h=1}^{j-1}\left\{\left(\sum\limits_{x_{ij}>x_{rh}}\left(x_{ij}^{m}+x_{rh}^{m}-2x_{irjh}^{*m}\right)\right)-\left(\sum\limits_{x_{ij}<x_{rh}}\left(x_{ij}^{m}+x_{rh}^{m}-2x_{irjh}^{*m}\right)\right)\right\}}{2\mu n^2} && (G_{nb})\\ &+\sum_{m=1}^{q}\frac{4\sum\limits_{j=2}^{k}\sum\limits_{h=1}^{j-1}\left(\sum\limits_{x_{ij}<x_{rh}}\left(x_{ij}^{m}+x_{rh}^{m}-2x_{irjh}^{*m}\right)\right)}{2\mu n^2}\,. && (G_t)\end{aligned} \tag{46}$$

It is possible to compute exactly the contribution of each income source to the within-group Gini index [G_w], to the net between-group Gini index [G_{nb}], and to the intensity of transvariation [G_t]. Precisely, we have: (*i*) the contribution of the m-th income source of the j-th group to the total Gini ratio; (*ii*) the contribution of the m-th income source of the net inequalities between the j-th and the h-th groups to G; and (*iii*) the contribution of the m-th income source of the intensity of transvariation between the j-th and the h-th groups to G. ■

Corollary 5. *The multi-decomposition and the multilevel Gini decomposition* (26) *are combined such as:*

$$G = \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j}\sum_{m=1}^{q} S_{j,zz}^{m} p'_z s'_z\right) p_j s_j \qquad (G^{w,S}) \qquad (47)$$

$$+ \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1}\sum_{m=1}^{q} D_{j,wz}^{m} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right)\right) p_j s_j \qquad (G^{nb,SK})$$

$$+ \sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1}\sum_{m=1}^{q} P_{j,wz}^{m} G_{j,wz}\left(p'_w s'_z + p'_z s'_w\right)\right) p_j s_j \qquad (G^{t,SK})$$

$$+ \sum_{j=2}^{k}\sum_{h=1}^{j-1}\sum_{m=1}^{q} D_{jh}^{m} G_{jh}\left(p_j s_h + p_h s_j\right) \qquad (G^{nb,K})$$

$$+ \sum_{j=2}^{k}\sum_{h=1}^{j-1}\sum_{m=1}^{q} P_{jh}^{m} G_{jh}\left(p_j s_h + p_h s_j\right). \qquad (G^{t,K})$$

Proof. On the one hand, we decompose the Gini coefficient of the z-th sub-group of P_j of the first partition $G_{j,zz}$ by income source. We obtain the contribution of each source $S_{j,zz}^{m}$ to $G_{j,zz}$. On the other hand, we decompose the economic distance between the groups of the first and the second partition in order to obtain a mixture of decomposition that relies on multilevel and multi-decomposition techniques. Then, all the sub-group determinants of the overall inequality are explained by the influence of each income source.[2] ■

V. ECONOMETRIC MODELISATION

In the preceding section, the inequalities are both generated by sub-group and factor component. Analysing only income sources limits considerably the number of dimensions, say attributes. To extend the multi-decomposition in a multidimensional context, we linearly model incomes by q explanatory variables [see Morduch and Sicular (2002)].

Proposition 6. *When incomes (x_i) are linearly regressed on q socio-economic variables X's (for example economic conditions and social policy indicators: health, education level, etc.) as follows:*

$$x_i = \sum_{m=1}^{q} \alpha_m X_i^m + \varepsilon_i\,, \qquad (48)$$

where $X_i^1 = 1$ and where ε_i is the random error term, then G_w, G_{nb}, and G_t are determined by the q attributes:

$$\hat{G} = \sum_{j=1}^{k}\sum_{m=1}^{q}\left(\hat{S}_{jj}^{m}\right)\hat{p}_j\,\hat{s}_j \qquad (\hat{G}^{w}) \qquad (49)$$

[2] We can prove that this decomposition is exact and can be generalized to α group partitions.

$$+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}\hat{D}_{jh}^{m}\left(\hat{p}_j\,\hat{s}_h+\hat{p}_h\,\hat{s}_j\right)\hat{G}_{jh} \qquad (\hat{G}^{nb})$$

$$+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}\hat{P}_{jh}^{m}\left(\hat{p}_j\,\hat{s}_h+\hat{p}_h\,\hat{s}_j\right)\hat{G}_{jh}\;, \qquad (\hat{G}^{t})$$

where the variables with '^' denote the parametric modelisation (48).

Proof. As x_i is linearly disaggregated, we find exactly the same result as in the non-parametric case[3]. ■

This enlarges considerably the multi-decomposition (46) since the q variables are not standard income sources. Therefore, using the t-Student tests, we isolate the significant variables that tend to increase the overall inequality as well as the following estimated Ginis: $\hat{G}^{w}$, $\hat{G}^{nb}$ and $\hat{G}^{t}$.

Corollary 6. *Equation* (50) *can be merged with the multilevel decomposition* (26) *in two partitions* (*or* α *partitions*) *such as:*

$$\hat{G}=\sum_{j=1}^{k}\left(\sum_{z=1}^{q_j}\sum_{m=1}^{q}\hat{S}_{j,zz}^{m}\hat{p}'_z\,\hat{s}'_z\right)\hat{p}_j\hat{s}_j \qquad (\hat{G}^{w,S}) \qquad (50)$$

$$+\sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1}\sum_{m=1}^{q}\hat{D}_{j,wz}^{m}\,\hat{G}_{j,wz}\left(\hat{p}'_w\,\hat{s}'_z+\hat{p}'_z\,\hat{s}'_w\right)\right)\hat{p}_j\hat{s}_j \qquad (\hat{G}^{nb,SK})$$

$$+\sum_{j=1}^{k}\left(\sum_{z=2}^{q_j}\sum_{w=1}^{z-1}\sum_{m=1}^{q}\hat{P}_{j,wz}^{m}\,\hat{G}_{j,wz}\left(\hat{p}'_w\,\hat{s}'_z+\hat{p}'_z\,\hat{s}'_w\right)\right)\hat{p}_j\hat{s}_j \qquad (\hat{G}^{t,SK})$$

$$+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}\hat{D}_{jh}^{m}\hat{G}_{jh}\left(\hat{p}_j\hat{s}_h+\hat{p}_h\hat{s}_j\right) \qquad (\hat{G}^{nb,K})$$

$$+\sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1}\sum_{m=1}^{q}\hat{P}_{jh}^{m}\hat{G}_{jh}\left(\hat{p}_j\hat{s}_h+\hat{p}_h\hat{s}_j\right). \qquad (\hat{G}^{t,K})$$

Proof. It is straightforward. ■

The practical difficulty of this model relies on the quality of adjustment. If the overall variance is enough explained by the regression model, one can logically expect that the different attributes make clear the evolution of the inequalities within groups and between groups (of particular partitions).

VI. CONCLUSION

The Gini index belongs to the class of decomposable measures based on interpersonal comparisons [see Pyatt (1976)]. This principle is crucial since individuals examine

[3] As a restriction, we also have the contribution of the intercept and those of the error term.

perpetually such comparisons. It is not satisfied by the generalized entropy inequality measures [see Dagum (1998)].

Even if the generalized entropy is sub-group decomposable as well as in a multilevel background [see Salas (2002)], it does not fulfil the property of multi-decomposition [see Mussard (2004)] that merges the sub-group and the income source decompositions. Furthermore, the Gini decomposition exhibits a third element that is not a residual term since it corresponds to the intensity of transvariation [see Gini (1916) and Dagum (1959, 1960, 1961)], which captures the deprivation of the sub-groups with higher mean incomes. Furthermore, when the overall population is divided into many sub-partitions, the multilevel Gini decomposition offers the intensity of transvariation between the groups of each partition [$G^{t,SK}$ and $G^{t,\alpha K}$], the net between-group Gini within each partition [$G^{b,SK}$ and $G^{nb,\alpha K}$], and the Gini within the groups of each partition [$G^{w,K}$ and $G^{w,\alpha}$].

Finally, both the multi-decompositions and the parametric multi-decompositions precise, within an exact structure, the components of the overall inequality, which can help decision makers to contemplate socio-economic policies of redistribution.

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APPENDIX 1.

Proof of Corollary 2. We use the same approach as corollary 1:

$$G = \sum_{j=1}^{k}\left(\overbrace{\sum_{z=1}^{q_j} \ldots \Big(\sum_{z=1}^{q_j^{\alpha}}}^{\alpha-1 \text{ times}} G_{j,z}^{\alpha} p_{j,z}^{\alpha} s_{j,z}^{\alpha}\Big) \ \ldots \ p_z s_z \right) p_j s_j \qquad (G^{w,\alpha})$$

$$+ \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right) \qquad (G^{nb,K})$$

$$+ \sum_{j=2}^{k}\sum_{h=1}^{j\text{-}1} G_{jh}\left(p_j s_h + p_h s_j\right)\left(1 - D_{jh}\right) \qquad (G^{t,K})$$

$$+ \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j^2}\sum_{w=1}^{z-1} G_{j,wz}^{2}\varpi_{j,wz}^{2} D_{j,wz}^{2}\ p_z^2 s_z^2 \right) p_j s_j \qquad (G^{nb,2K})$$

$$\vdots \qquad\qquad \vdots$$

$$+ \sum_{j=1}^{k}\left(\overbrace{\sum_{z=1}^{q_j} \ldots \Big(\sum_{z=1}^{q_j^{\alpha-1}}}^{\alpha-2 \text{ times}} \Big(\sum_{z=1}^{q_j^{\alpha}}\sum_{w=1}^{z-1} G_{j,wz}^{\alpha} D_{j,wz}^{2} \varpi_{j,wz}^{\alpha}\Big) p_{j,z}^{\alpha-1} s_{j,z}^{\alpha-1}\Big) \ \ldots \ p_z^2 s_z^2 \right) p_j s_j \qquad (G^{nb,\alpha K})$$

$$+ \sum_{j=1}^{k}\left(\sum_{z=1}^{q_j^2}\sum_{w=1}^{z-1} G_{j,wz}^{2}\varpi_{j,wz}^{2}(1 - D_{j,wz}^{2})\ p_z^2 s_z^2 \right) p_j s_j \qquad (G^{t,2K})$$

$$\vdots \qquad\qquad \vdots$$

$$+ \sum_{j=1}^{k}\left(\overbrace{\sum_{z=1}^{q_j} \ldots \Big(\sum_{z=1}^{q_j^{\alpha-1}}}^{\alpha-2 \text{ times}} \Big(\sum_{z=1}^{q_j^{\alpha}}\sum_{w=1}^{z-1} G_{j,wz}^{\alpha}(1 - D_{j,wz}^{2}) \varpi_{j,wz}^{\alpha}\Big) p_{j,z}^{\alpha-1} s_{j,z}^{\alpha-1}\Big) \ \ldots \ p_z^2 s_z^2 \right) p_j s_j \ , \qquad (G^{t,\alpha K})$$

where $D_{j,wz}^{\alpha}$ stands for the economic distance between the w-th and the z-th sub-populations of the α-th partition.