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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Optimal allocation of an opaque product with consumer learning

Amir Farshbaf-Geranmayeh [a]

Georges Zaccour [a,b]

[a] *GERAD, HEC Montréal, Montréal, Canada*

[b] *Chair in Game Theory and Management, GERAD, HEC Montréal, Canada*

amir.farshbaf@gerad.ca
georges.zaccour@gerad.ca

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Abstract: We consider a firm offering an opaque good over one selling season, that is, a product whose full characteristics are only revealed after the consumer completes the transaction. In our model, the consumer can choose between buying either of two transparent goods offered on the market at given high- and low-price, respectively, or paying a price in between to purchase the firm's opaque good. Given that, over time, consumers learn the firm's allocation policy of high- and low-quality product via opaque selling, the seller's objective is to determine the optimal sequence that maximizes its profit over the selling season. We characterize the unique optimal allocation sequence for any parameter constellation.

Keywords: E-commerce, opaque product, learning, consumer belief, optimal control

Résumé : Nous considérons une entreprise offrant un produit opaque, c'est-à-dire un produit dont les caractéristiques complètes ne sont révélées au consommateur qu'après la fin de la transaction. Dans notre modèle, le consommateur peut choisir entre deux produits transparents disponibles sur le marché à un prix élevé ou bas, respectivement, ou l'achat du bien opaque, qui peut être de grande ou basse qualité, à un prix intermédiaire. La firme décide à chaque instant la proportion allouée de chaque qualité et le consommateur apprend cette politique à travers le temps. Le problème de la firme est de déterminer la séquence d'allocation qui maximise son profit intertemporel. On montre que cette séquence est unique pour toute constellation de valeurs de paramètres.

Mots clés : Produit opaque, apprentissage, croyance du consommateur, contrôle optimal

1 Introduction

Nowadays, consumers looking for bargains on airline tickets or hotel rooms are turning to online opaque travel websites such as Priceline and Hotwire. An opaque good is a product whose characteristics or attributes (e.g., the name of the hotel) are disclosed to the buyer only after the transaction has been completed. This form of selling has been widely adopted by service providers in the hospitality industry (e.g., Hotwire.com, gta-travel.com, Wotif.com, booking.com, Lastminute.com) but not limited to this industry. An example is Fukubukuro, a Japanese New Year's Day tradition, where merchants offer to consumers an undefined collection of items placed in "lucky" (or mystery) bags.

In this paper, we consider a firm that offers an opaque product during a selling season. Assuming that the product comes in two types of quality, i.e., high or low, our objective is to answer the following research question: given that consumers learn from past realizations, what is the seller's optimal allocation policy of high- and low-quality products over time?

When engaging in opaque trading, consumers know they will get one of a set of distinct products/services (Fay and Xie, 2008). The elements in the set are typically not fully identified, and consequently, neither the probability of getting a specific one. To credibly hide the identity of the final product the consumer will receive, an opaque offer must involve at least two (horizontally or vertically) differentiated products (Jerath et al., 2009). Most of the literature has considered horizontal product differentiation (Zhang et al., 2014); however, in this research, we retain quality-differentiated (or vertically differentiated) products. The main difference is that, for the same price, consumers always prefer the high-quality product (e.g., an ocean-view room) to the lower-quality one (a parking-view room). To illustrate, consider a car rental firm offering two types of cars, a midsize car at a high price and a compact car at a low price. An opaque offer would be a car at a price between the high and low prices, with the seller assigning the midsize cars to only a fraction of the customers. Price-sensitive consumers will generally prefer not to participate in this type of lottery, but others may be tempted to sacrifice transparency in the hopes of getting their preferred product (e.g., midsize car) at a lower price than the regular list price.

A consumer's purchasing decision process is more complicated when considering an opaque product than when shopping for a transparent one. The reason for this is the difficulty in assessing one's own willingness to pay for a probabilistic product. Here, online product reviews by previous buyers become a valuable information source for consumers to reduce their uncertainty (Papanastasiou and Savva, 2016). For instance, when consumers visit the opaque seller's online platform, they can see the opinions of other consumers who have shared their experiences, and thus learn about the seller's product offering strategies (Huang and Yu, 2014). From this information, and possibly own experience, consumers form a belief about the future policy, that is, about the fraction of requests receiving the high-quality product, and they factor this into their purchasing decisions. In our model, we assume that consumers update their belief using an exponential smoothing function based on the seller's past strategies. This method, which discounts earlier sample information and gives preference to the most recent sample, is popular in the literature and has been empirically validated (Winer, 1986; Greenleaf, 1995).

Using the Internet, consumers can easily find the prices of the transparent products that are part of the opaque offering, e.g., the cost of renting a midsize car or a room with an ocean view at a particular resort. Given the opaque seller's price, consumers decide whether they want to purchase the opaque product whose quality is uncertain or opt for a transparent product. We assume the opaque seller's perceived demand at each instant of time to be a function of customer's belief about the fraction of high-quality products assigned, the opaque price, and product's published price on the transparent market. Here, the seller decides the optimal fraction of customers who will get the high-quality product, which typically has a lower profit margin (Zhang et al., 2014). In this paper, the fraction of high-quality products is the control variable in the opaque seller's finite-horizon dynamic optimization problem.

The rest of the paper is organized as follows. First, we briefly review the literature and position our contribution. Then, we present our model, analysis, and findings. Finally, we conclude with a discussion of our key results and outline directions for future research.

2 Literature review

The literature on opaque selling is fairly recent and is drawing an increasing level of attention from analysts and researchers (see the comprehensive survey in Gönsch (2019)). Previous theoretical studies in revenue management have considered opaque selling as a tool to price discriminate and deal with demand uncertainty. Fay (2008) is the first to study how opaque selling impacts profit and whether introducing an opaque product might affect the prices of non-opaque goods. He finds that introducing an opaque good at a sufficiently large discount can enhance profit but that the discount should not be too large, to avoid cannibalizing existing sales. The author argues that a discounted opaque good allows the seller to segment the market on the basis of buyer's product preference, and to expand its size by reaching previously unserved consumers.

A number of studies have compared opaque selling to other strategies. Fay and Xie (2010) compare opaque selling to an advance selling strategy, which gives consumers the opportunity to buy a product in advance, while they are uncertain about their future consumption. Even though these two selling strategies are implemented via different market mechanisms, both offer consumers a choice involving uncertainty. Jiang (2007) considers heterogeneous consumers and compares opaque and transparent selling to determine the incremental profit and social benefit generated by opaque selling. Using a nested logit model, Anderson and Xie (2012) study optimal pricing in an opaque sales channel. They find that using opaque selling to discriminate between customers is in most cases effective for firms because of consumer heterogeneity. In our model, the price of the opaque product is exogenously given, and consumers decide whether or not to purchase it.

Using a dynamic model, Jerath et al. (2010) compare the benefits of last-minute direct selling and of selling through an opaque intermediary. They show that opaque selling dominates when consumer valuations of the product are high or when the service offerings are relatively homogeneous. Further, as the probability of having a high-demand realization increases, customers compete for products in the first period, and opaque selling becomes increasingly preferred over the other selling strategy. This is because consumers do not benefit from waiting and so prefer to purchase in the first period, which enables the firm to charge higher first-period prices and increase its profits. In our model, we also consider the dynamic aspect of opaque selling and determine the seller's optimal allocation policy, i.e., the fraction of the demand that gets the high-quality product in each period.

Anderson and Xie (2014) study a firm that simultaneously operates both an opaque and a regular, full-information channels. The choice of channel is endogenously determined as a function of prices and channel characteristics (opacity). They show that opaque selling increases the overall demand because it allows the firm to also reach low-valuation consumers who would have been priced out otherwise. They investigate when and how to deploy an opaque selling strategy in concert with regular, full-information pricing, and find that the firm's optimal prices change as a function of opacity, as it manages the tradeoff between demand growth and revenue dilution.

Huang and Yu (2014) consider a case where the homogeneous consumers have limited information about the seller's strategy. They go beyond rational expectations by considering anecdotal reasoning framework and investigate the impact of consumer-bounded rationality on an opaque seller's optimal strategy.

The references cited above (e.g., Jiang (2007), Fay and Xie (2008), Jerath et al. (2010) and Fay et al. (2015)) focus on opaque goods in horizontally differentiated markets. Few studies have dealt with vertically differentiated markets. Zheng et al. (2019) consider opaque selling for two vertically differentiated products and an opaque product, and where consumers receive a high-quality product in a preannounced proportion. Later, they incorporate salient-thinking behavior by consumers and argue that when a seller strategically designs the opaque good and employs the same price for high- and low-quality products, the consumers' attention gets distracted from price, as quality becomes more salient. They study the optimal opaque selling strategy in the presence of salient thinking where quality salience induces consumers to become less price sensitive (i.e., to overvalue the product), while price salience makes consumers undervalue the product. They show that when consumers are rational, the opaque good cannibalizes existing sales and decreases the seller's profit for any level of supply. However, by exploiting consumers' salient-thinking behavior, opaque selling can act as an important tool to improve the seller's profit.

In contrast to Zheng et al. (2019) and most of the papers in the literature, our dynamic model assumes that the fraction of consumers who obtain a high-quality product in each period is not preannounced. Instead, the seller determines it endogenously in a dynamic environment. Further, consumers form their reference price for the opaque product based on their belief regarding the probability of getting the high-quality product and the prices of the transparent products.[1] Therefore, our paper also contributes to the literature where consumers learn about sellers' strategies from their purchasing experiences and social networks. As is common in the literature (e.g., Acuna and Schrater (2008), Wu et al. (2015) and Popescu and Wu (2007)), we use exponential smoothing to update the consumers' belief.

3 Model

Consider a retailer offering an opaque good (OG) that could be of low or high quality. During the selling season $[0,T]$, the alternative, constant market prices of the high- and low-quality products are p_H and p_L, respectively. Denote by $\omega(t) \in [0,1]$ the consumer's belief that the seller will assign the high-quality product at time t. Observing the retailer's decision over time, the consumer updates her belief about this probability as follows:

$$\dot{\omega}(t) = \beta(\phi(t) - \omega(t)), \quad \omega(0) = \omega_0 \in (0,1) \text{ given}, \tag{1}$$

where $\phi(t) \in [0,1]$ is the actual allocation of high-quality product in period t, ω_0 is the initial consumer belief and β is the memory parameter ($0 \leq \beta \leq 1$), which can be interpreted as the speed of adjustment of this belief. If β is zero, then this belief is constant and independent of the seller's decision.

Denote by v the consumer's willingness to pay (WTP), which is uniformly distributed over $[0,\overline{U}]$ with a cumulative distribution function $F(\cdot)$. The maximum WTP is given by $\overline{U} = \omega p_H + (1-\omega)p_L$. Indeed, if the price p of the OG is higher than $\omega p_H + (1-\omega)p_L$, then there is no incentive for the consumer to purchase it, and she would opt for the transparent good. If $p = \omega p_H + (1-\omega)p_L$, then the consumer is indifferent between the opaque and transparent goods. If the consumer believes that the OG is sure to be high quality ($\omega = 1$), then she will buy it only if $p \leq p_H$. In the other extreme case, that is, when consumer believes that the OG is sure to be of low quality, then she will buy it only if $p \leq p_L$. Using our assumption that the consumer's WTP is uniformly distributed over $[0, \omega p_H + (1-\omega)p_L]$, the demand function is then formulated as follows:

$$D(\omega(t)) = \max\{0, \gamma(\omega(t)p_H + (1-\omega(t))p_L - p)\},$$

where γ is the market size, which we normalize to one, without any loss of generality. To have a positive demand, the consumer's belief at any $t \in [0,T]$ must be higher than the threshold $\underline{\omega} = \frac{p-p_L}{p_H-p_L}$.

Let c_H and c_L be the seller's cost for the high- and low-quality product, respectively ($c_H > c_L$). To save on parameters, and without any loss of generality, we normalize $c_L = 0$, and set $c_H = c$. We assume $p > c$.

[1] Arslan and Kachani (2011) reviewed the segment of the literature on dynamic pricing models with consumer reference-price effects.

The seller's profit at time t is given by

$$\begin{aligned} J(\omega(t),\phi(t)) &= (p-c)\phi(t)D(\omega(t)) + p(1-\phi(t))D(\omega(t)),\\ &= (p-c\phi(t))D(\omega(t)), \end{aligned}$$

and its intertemporal optimization problem is

$$\max_{\phi(t)} \int_0^T J(\omega(t),\phi(t))dt, \tag{2}$$
$$\text{subject to (1).}$$

To wrap up, we have defined by (2) an optimal control problem with one control variable ($\phi(t) \in [0,1]$) and one state variable ($\omega(t) \in [0,1]$).

4 Optimal policy

To characterize the optimal policy, let us introduce the Hamiltonian

$$\mathcal{H}(\omega(t),\phi(t),\lambda(t)) = J(\omega(t),\phi(t)) + \lambda(t)\beta(\phi(t)-\omega(t)), \tag{3}$$
$$= (p-c\phi(t))(\omega(t)p_H + (1-\omega(t))p_L - p) + \lambda(t)\beta(\phi(t)-\omega(t)), \tag{4}$$

where $\lambda(t)$ is the adjoint variable appended to the state dynamics in (1). The Hamiltonian is linear in the control variable, and hence, the optimal policy is bang-bang.

The first-order optimality conditions are

$$\dot{\omega}(t) = \frac{\partial\mathcal{H}}{\partial\lambda} = \beta(\phi(t)-\omega(t)), \quad \omega(0) = \omega_0, \tag{5}$$
$$\dot{\lambda}(t) = -\frac{\partial\mathcal{H}}{\partial\omega} = \beta\lambda(t) - (p_H - p_L)(p - c\phi(t)), \quad \lambda(T) = 0, \tag{6}$$
$$\frac{\partial\mathcal{H}}{\partial\phi} = \beta\lambda(t) - cD(\omega(t)) \Rightarrow \phi(t) = \begin{cases} 0 & \text{if } \lambda(t) < \frac{cD(\omega(t))}{\beta}, \\ [0,1] & \text{if } \lambda(t) = \frac{cD(\omega(t))}{\beta}, \\ 1 & \text{if } \lambda(t) > \frac{cD(\omega(t))}{\beta}, \end{cases}. \tag{7}$$

The optimality condition in (7) states that the firm will allocate the high-quality product if its total cost (given by $cD(\omega(t))$) is lower than the shadow price $\lambda(t)$ times the speed of adjustment of the consumer's beliefs β.

Introduce the switching function

$$S(t) = \lambda(t) - \frac{cD(\omega(t))}{\beta} = \lambda(t) - \frac{c}{\beta}(\omega(t)(p_H - p_L) + p_L - p).$$

The following proposition establishes that in any optimal sequence, the policy is to set $\phi(t) = 0$ during a terminal interval. This result can be explained by the end-of-horizon effect. Indeed, as there is no future after T, it is then useless to invest in raising the consumer's belief that the allocated product is of high quality.

Proposition 1 *Consider a terminal interval $[\hat{\tau}, T]$, with $0 \leq \hat{\tau} \leq T$. The optimal policy is to set $\phi = 0$ on $[\hat{\tau}, T]$, with $\hat{\tau}$ given by*

$$\hat{\tau} = \max\left(T - \frac{1}{\beta}\ln\frac{(p_H - p_L)p}{(p_H - p_L)(p - c\omega(\hat{\tau})) + c(p - p_L)}, 0\right) \tag{8}$$

where

$$\omega(\hat{\tau}) \leq \frac{p(p_H - p_L) + c(p - p_L)}{2c(p_H - p_L)} \triangleq \overline{\omega}.$$

Proof. See Appendix. □

The proposition tells us when it is optimal to start the terminal interval where $\phi(t) = 0$. Using the result in Proposition 1, we have the following two theorems.

Theorem 1 *Let $[0, \tau_0]$ and $[\tau_0, \tau_1]$ be any two consecutive intervals of time, with $0 < \tau_0 < \tau_1 < T$. Then, if $\phi \neq 0$ on $[0, \tau_0]$ and $\phi = 0$ on $[\tau_0, \tau_1]$, it is infeasible to have $\phi \neq 0$ on $[\tau_1, T]$.*

Proof. See Appendix. □

Theorem 2 *Let $[0, \tau_0]$ and $[\tau_0, \tau_1]$ be any two consecutive intervals of time, with $0 < \tau_0 < \tau_1 < T$. Then, if $\phi \neq 1$ on $[0, \tau_0]$, it is infeasible to have $\phi = 1$ on $[\tau_0, \tau_1]$.*

Proof. See Appendix. □

The implication of the above theorems is that only the following sequences are candidates for optimality:

S1: Set $\phi = 0$ throughout the whole planning horizon;
S2: Start by $\phi = 1$ and end by $\phi = 0$;
S3: Start by $\phi = 1$, switch to $\phi \in (0, 1)$, and end by $\phi = 0$;
S4: Start by $\phi = 0$, switch to $\phi \in (0, 1)$, and end by $\phi = 0$;
S5: Start by $\phi \in (0, 1)$ and end by $\phi = 0$.

In the next five propositions, we determine the conditions under which each of the sequences S1 to S5, is optimal. As they play an important role in the proofs, we recall that $\underline{\omega} = \frac{p-p_L}{p_H-p_L}$ and $\overline{\omega} = \frac{p(p_H-p_L)+c(p-p_L)}{2c(p_H-p_L)}$. Note that $\underline{\omega}$ is the ratio of the difference between the actual price and p_L to the range of prices given by $p_H - p_L$. If the actual price p is close to the high price p_H, then $\underline{\omega}$ would be close to one. Further, we have $\overline{\omega} = \frac{1}{2}\left(\frac{p}{c} + \underline{\omega}\right)$, that is, half of the price markup plus half of $\underline{\omega}$.

Proposition 2 *Sequence S1. If $T \leq \overline{T}^{S_1}(\omega_0)$, where*

$$
\begin{aligned}
\overline{T}^{S_1}(\omega_0) &= \begin{cases} \overline{T}_1^{S_1}(\omega_0), & \text{if } \underline{\omega} \leq \omega_0 \leq \min(\overline{\omega}, 1), \\ \\ \overline{T}_2^{S_1}(\omega_0), & \text{if } \min(\overline{\omega}, 1) < \omega_0 \leq 1, \end{cases} \\
\overline{T}_1^{S_1}(\omega_0) &= \frac{1}{\beta} \ln\left(\frac{(p_H - p_L)\, p}{(p_H - p_L)(p - c\omega_0) + c\,(p - p_L)} \right), \\
\overline{T}_2^{S_1}(\omega_0) &= \frac{1}{\beta} \ln\left(\frac{4cp\omega_0 (p_H - p_L)^2}{(p(p_H - p_L) + c(p - p_L))^2} \right)
\end{aligned}
$$

then, the optimal solution is to set $\phi^{S_1}(t) = 0$ for all $t \in [0, T]$.

Proof. See Appendix. □

When the planning horizon is sufficiently short, leaving little time for consumers to learn the seller's policy, it is in the seller's best interest to always assign the low-quality product, because it yields a higher revenue margin than the alternative, high-quality product. Note that the upper bound on T for the feasibility of S1, i.e., $\overline{T}^{S1}(\omega_0)$, is increasing in $\omega(0)$, which is intuitive. Indeed, if the consumer initially believes there is a high probability that the retailer will assign the high-quality good, then there is no need for the retailer to invest in boosting this belief. The result in this proposition extends the result in Proposition 1 to the whole planning horizon.

Figure 1 provides a visual representation of the results of Proposition 2. In the left-hand-side panels, we show the trajectories of $\lambda(t)$ and $\frac{cD(\omega(t))}{\beta}$. As expected in this scenario, the gap $S\left(t\right)=\lambda(t)-\frac{cD(\omega(t))}{\beta}$ is negative for all $t\in[0,T]$ and is increasing in the initial belief $\omega\left(0\right)$. In the right-hand-side panels, we illustrate the dependence between $\omega\left(0\right)$ and the maximum duration of the planning horizon for which this sequence can materialize. As stated above, the larger is $\omega\left(0\right)$, the longer the planning horizon in which it is optimal to set $\phi^{S_1}=0$ on $[0,T]$.

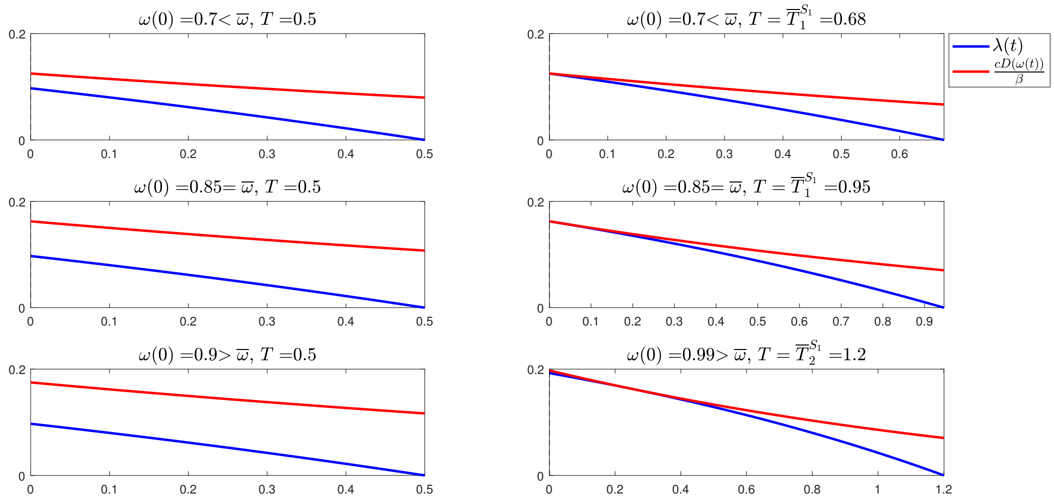


Figure 1: Numerical examples for S1: $p_H=0.9$, $p_L=0.15$, $c=0.2$, $p=0.3\,\beta=0.6$.

Proposition 3 *Sequence S2. If* $\underline{\omega}\leq\omega_0<\min(\overline{\omega},1)$ *and* $\overline{T}_1^{S_1}(\omega_0)<T\leq\overline{T}^{S_2}(\omega_0)$*, where*

$$\overline{T}^{S_2}(\omega_0)=\begin{cases}\overline{T}_1^{S_2}(\omega_0), & \textit{if } \min(\overline{\omega},1)=\overline{\omega}\\ \infty, & \textit{if } \min(\overline{\omega},1)=1\end{cases},$$

and

$$\overline{T}_1^{S_2}(\omega_0)=\frac{1}{\beta}\ln\left(\frac{4cp\left(p_H-p_L\right)^2\left(1-\omega_0\right)}{c\left(p_H-p\right)+\left(p_H-p_L\right)\left(p-c\right)\left(\left(p-p_L\right)c+p\left(p_H-p_L\right)\right)}\right),$$

then, the optimal solution is to set

$$\phi^{S_2}\left(t\right)=\begin{cases}1, & \textit{for } t\in[0,\hat{\tau}]\\ 0, & \textit{for } t\in[\hat{\tau},T]\end{cases},$$

where

$$\hat{\tau}=\frac{1}{\beta}\ln\left(\frac{\left(p\left(p_H-p_L\right)-c\left(p_H-p\right)\right)+\sqrt{\left(p\left(p_H-p_L\right)-c\left(p_H-p\right)\right)^2+4\left(p_H-p_L\right)^2pe^{-\beta T}c\left(1-\omega_0\right)}}{2\left(p_H-p_L\right)pe^{-\beta T}}\right).$$

Proof. See Appendix. □

The above proposition gives the conditions under which it is optimal for the seller to start by allocating the high-quality product till an intermediary date $\hat{\tau}$, and next switch to the low-quality product till the end of the planning horizon. Intuitively speaking, the seller is first investing in raising the consumer's belief that the probability of getting the high-quality product is sufficiently high to opt for the opaque product, and next the seller surfs on this belief till T. Note that $\hat{\tau}$ depends on all the model's parameters and especially the terminal date T.

Figure 2 illustrates the result for different values of T, p, and the upper bound $\overline{\omega}$. For a given initial belief ω_0, the larger T and the longer (proportionally speaking) is the duration of the first phase where $\phi^{S_2}(t) = 1$. Further, we know from Proposition 1 that at the time of the switch to $\phi(t) = 0$, $\omega(\hat{\tau}) \leq \overline{\omega}$. Therefore, the upper bound of T for which S2 is the optimal sequence is the sum of two numbers, namely, the time to raise the belief from ω_0 to $\overline{\omega}$ plus the duration of the terminal phase given in both Proposition 1 and Proposition 2. In the two top panels of Figure 2, the upper bound $\overline{\omega}$ is equal to 0.85, whereas in the two bottom panels, it is equal to 1.17, which implies that there is no upper bound for T (This situation occurs when $\frac{c(2p_H - p_L)}{p_H - p_L + c} \leq p$). Therefore, for a sufficiently high price, the firm aims to increase the consumer's belief that it will assign the high-quality product. The bottom-right panel shows that for a sufficiently high T, the seller pushes the belief to one and keeps it there until a short time before the end of the horizon.

Remark 1 *To make a link between S1 and S2, we notice that* $\overline{T}_1^{S_2}(\omega_0) = \overline{T}_1^{S_1}(\overline{\omega}) + \frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$, *which is intuitive. The time required to reach* $\omega(\hat{\tau})$ *from* ω_0 *is* $\frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\omega(\hat{\tau})}\right)$, *which is increasing in* $\omega(\hat{\tau})$. *From Proposition 1, we know that* $\omega(\hat{\tau})$ *must be lower than* $\overline{\omega}$ *at the switching point* $\hat{\tau}$. *Therefore, the maximum time for having* $\phi = 1$ *is* $\frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$. *On the other hand, the time of the last phase, that is* $T - \hat{\tau}$, *is equal to* $\overline{T}_1^{S_1}(\omega(\tau))$, *which is increasing in* $\omega(\tau)$. *We can conclude that the upper bound for* T *for which S1 is feasible, is equal to* $\overline{T}_1^{S_1}(\overline{\omega}) + \frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$. *Since* $\omega(\hat{\tau}) \leq \overline{\omega}$, *it is clear that S1 is feasible only if* $\omega_0 \leq \overline{\omega}$.

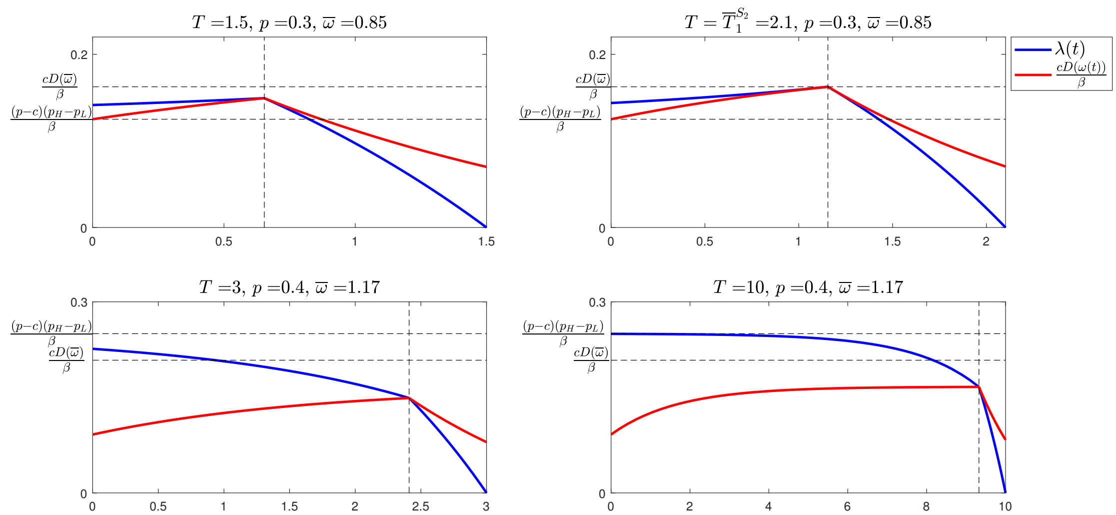


Figure 2: Numerical examples for S2: $p_H = 0.9$, $p_L = 0.15$, $c = 0.2$, $\beta = 0.6, \omega_0 = 0.7$.

As the next two propositions share some common ground, we postpone our comments till after they both have been presented.

Proposition 4 *Sequence S3. If* $\underline{\omega} \leq \omega_0 < \overline{\omega} \leq 1$ *and* $T > \overline{T}_1^{S_2}(\omega_0)$, *then the optimal policy is to set*

$$\phi^{S_3} = \begin{cases} 1, & \textit{for } t \in \left[0, \tau_1^{S_3}\right] \\ \overline{\omega}, & \textit{for } t \in \left[\tau_1^{S_3}, \tau_2^{S_3}\right] \\ 0, & \textit{for } t \in \left[\tau_2^{S_3}, T\right] \end{cases},$$

where

$$\begin{aligned}
\tau_1^{S_3} &= \frac{1}{\beta}\ln\left(\frac{2c\,(p_H-p_L)\,(1-\omega_0)}{c\,(p_H-p)-(p-c)\,(p_H-p_L)}\right)=\frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right),\\
\tau_2^{S_3} &= T-\frac{1}{\beta}\ln\left(\frac{2p(p_H-p_L)}{p(p_H-p_L)+c\,(p_L-p)}\right)=T-\overline{T}_1^{S_1}(\overline{\omega}).
\end{aligned}$$

Proof. See Appendix. □

Proposition 5 *Sequence S4. If $\omega_0>\overline{\omega}$ and $T>\overline{T}_2^{S_1}(\omega_0)$, then the optimal policy is to set*

$$\phi^{S_4}=\begin{cases}0, & \text{for } t\in\left[0,\tau_1^{S_4}\right]\\ \overline{\omega}, & \text{for } t\in\left[\tau_1^{S_4},\tau_2^{S_4}\right]\\ 0, & \text{for } t\in\left[\tau_2^{S_4},T\right]\end{cases},$$

where

$$\begin{aligned}
\tau_1^{S_4} &= \frac{1}{\beta}\ln\left(\frac{2c(p_H-p_L)\omega_0}{p(p_H-p_L)+c\,(p-p_L)}\right)=\frac{1}{\beta}\ln\left(\frac{\omega_0}{\overline{\omega}}\right),\\
\tau_2^{S_4} &= T-\frac{1}{\beta}\ln\left(\frac{2p(p_H-p_L)}{p(p_H-p_L)+c\,(p_L-p)}\right)=\tau_2^{S_3}.
\end{aligned}$$

Proof. See Appendix. □

Propositions 4 and 5 show that if the terminal date is sufficiently distant, then it is optimal to have three phases (or two switches), with the second and third phases involving the same policy in both propositions. The last switch occurs at the same moment in both propositions $\left(\tau_2^{S_4}=\tau_2^{S_3}\right)$ and therefore the terminal phase has the same duration, $T-\tau_2^{S_3}$ in both cases. The main difference between the two propositions lies in the policy implemented during the first phase. In Proposition 4, the assumption is that the initial belief is of intermediate value, that is, where $\underline{\omega}\le\omega_0<\overline{\omega}\le 1$, and the seller starts by setting $\phi=1$ to drive the belief to $\overline{\omega}$. During the second phase, the belief remains constant. When T is large, as it is the case in the right-hand-side panels of Figures 3 and 4, the second phase is the longest. Here the optimal policy is to reach $\overline{\omega}$ as quickly as possible and to stay there for as long as possible before switching to $\phi=0$, a move driven by the transversality condition $\lambda(T)=0$. In Proposition 5, the assumption is that the initial belief that the seller allocates the high-quality product is very high. Consequently, the seller can afford to reduce it a bit by implementing $\phi=0$. As we have already alluded to, when the initial belief is low, the seller needs to invest in raising consumer belief to make the probabilistic good attractive, and next, the seller exploits this high level of belief by allocating the low-cost (here, zero), low-quality product during the final time interval. However, if the initial belief is very high, than the seller can skip this “investment”phase.

Proposition 6 *Sequence S5. If $\omega_0=\overline{\omega}<1$ and $T>\overline{T}_1^{S_1}(\omega_0)$, then the optimal policy is to set*

$$\phi^{S_5}=\begin{cases}\overline{\omega} & \text{for } t\in\left[0,\tau_1^{S_5}\right]\\ 0 & \text{for } t\in\left[\tau_1^{S_5},T\right]\end{cases},$$

where

$$\tau_1^{S_5}=T-\frac{1}{\beta}\ln\left(\frac{2p(p_H-p_L)}{p(p_H-p_L)+c\,(p_L-p)}\right)=\tau_2^{S_3}=\tau_2^{S_4}.$$

Proof. See Appendix. □

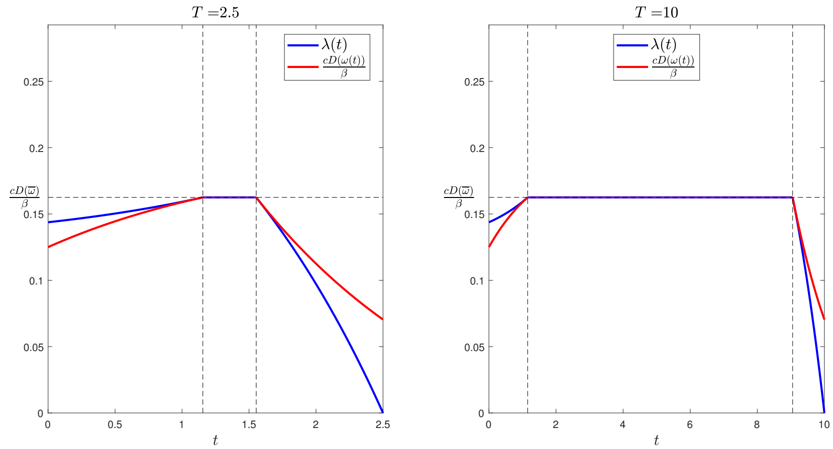


Figure 3: Numerical examples for S3: $p_H = 0.9$, $p_L = 0.15$, $c = 0.2$, $p = 0.3\,\beta = 0.6$, $\omega_0 = 0.7, \varpi = 0.85$.

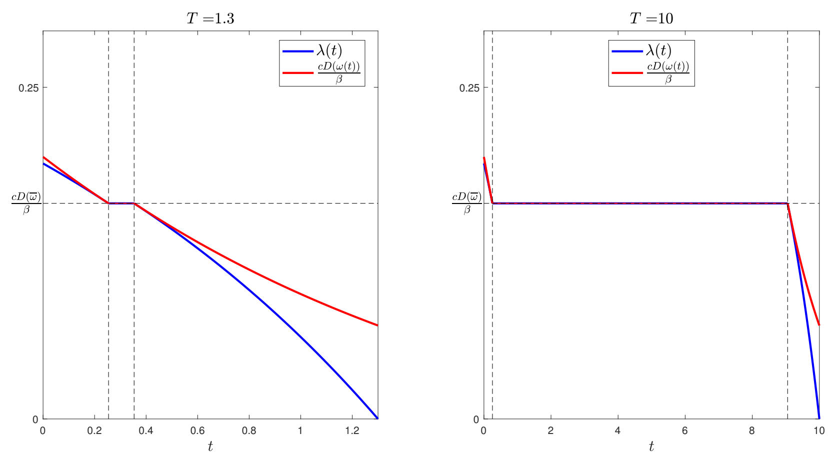


Figure 4: Numerical examples for S4: $p_H = 0.9$, $p_L = 0.15$, $c = 0.2$, $p = 0.3\,\beta = 0.6$, $\omega_0 = 0.99, \varpi = 0.85$.

In Proposition 4, the initial belief is of intermediate value and the policy consists in implementing $\phi^{S_3} = 1$ in a first phase to increase it to ϖ. In Proposition 5, the initial belief is too high, and it is optimal to implement $\phi^{S_4} = 0$ in a first phase to decrease it to ϖ. The above proposition deals with the very special case, where the initial belief is exactly equal to ϖ. If this is so, then the first phase in both Propositions 4 and 5 can be skipped.

Table 1 gives an overview of the results.

Figure 6 illustrates the control- and state-variable trajectories for different parameter values. In each row, all parameter values are the same except for the value of T. Each set of three figures in a row shows how the optimal policy changes with an increase in T.

Row 1: The three figures correspond to sequences S1, S2, and S3, respectively. When T is small, the optimal policy is to set $\phi(t) = 0$ for all $t \in [0, T]$. Increasing T to 1.5 and 5 leads to one and two switches, respectively.

Row 2: Panel 6(d) illustrates sequence S1, and panels 6(e)-(f) both correspond to sequence S5, that is, we have one switch from $\phi(t) = \omega(t)$ to $\phi(t) = 0$. The difference between 6(e) and 6(f) lies is in the duration of the two phases. In 6(f), T is larger than in 6(e), which dictates waiting (proportionally) longer before switching to $\phi(t) = 0$.

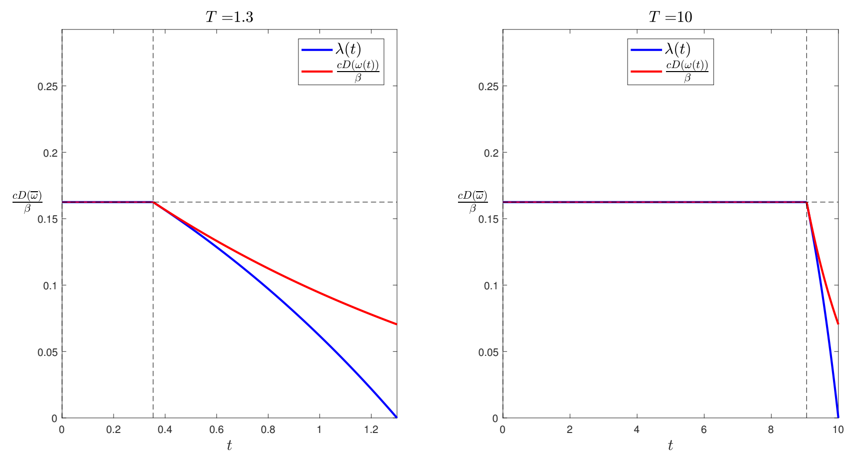


Figure 5: Numerical examples for S5: $p_H = 0.9$, $p_L = 0.15$, $c = 0.2$, $p = 0.3\,\beta = 0.6$, $\omega_0 = \overline{\omega} = 0.85$.

Table 1: Summary of results.

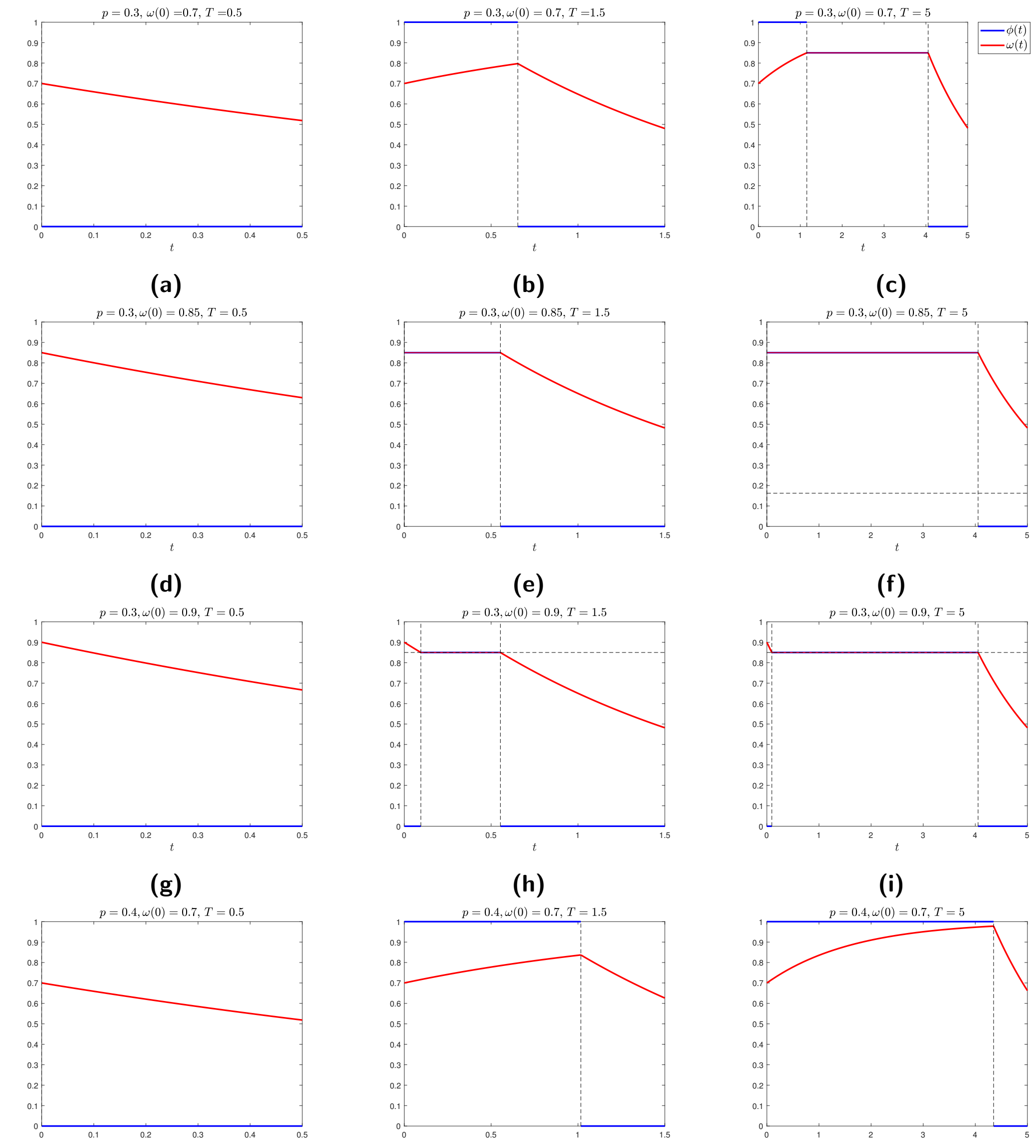
<table>
<tr><th></th><th>if</th><th>$0 \leq T \leq \overline{T}_2^{S_1}(\omega_0)$</th><th>$\overline{T}_2^{S_1}(\omega_0) < T \leq \overline{T}_1^{S_1}(\omega_0)$</th><th>$\overline{T}_1^{S_1}(\omega_0) < T \leq \overline{T}_1^{S_2}(\omega_0)$</th><th>$\overline{T}_1^{S_2}(\omega_0) < T$</th></tr>
<tr><td rowspan="3">$p < \dfrac{c(2p_H - p_L)}{p_H - p_L + c}$
$(\overline{\omega} < 1)$</td><td>$\underline{\omega} \leq \omega_0 < \overline{\omega}$</td><td colspan="2">S1</td><td>S2</td><td>S3</td></tr>
<tr><td>$\omega_0 = \overline{\omega}$</td><td colspan="2">S1</td><td colspan="2">S5</td></tr>
<tr><td>$\overline{\omega} < \omega_0 \leq 1$</td><td>S1</td><td colspan="3">S4</td></tr>
<tr><td>$p \geq \dfrac{c(2p_H - p_L)}{p_H - p_L + c}$
$(\overline{\omega} \geq 1)$</td><td>$\underline{\omega} \leq \omega_0 \leq 1$</td><td colspan="2">S1</td><td colspan="2">S2</td></tr>
</table>

Row 3: Compared to the panels in the second row, the initial belief here is higher. Whereas this change does not affect (qualitatively) the results when T is small, it triggers an additional switch in the cases where T is increased to 1.5 and 5. Here, the seller can afford to start by setting $\phi(t) = 0$ for a while before switching to $\phi(t) = \overline{\omega}$ and next $\phi(t) = 0$. The intuition is that the initial belief is so high that the seller can surf on it for some time before investing in raising it.

Row 4: Again, when T is sufficiently small, we have $\phi(t) = 0$ for all $t \in [0, T]$. Comparing panel 6(k) to panel 6(b) shows that increasing the price from 0.3 to 0.4 only has a quantitative impact. That is, for $T = 1.5$, the number of switches is the same, but the regime $\phi(t) = 1$ is in place for a longer period of time. Now, comparing 6(l) to 6(c) shows that a higher price also leads to a qualitative change, that is, the phase where $\phi(t) = \omega(t)$ in 6(c) is skipped in 6(l). In short, a higher price (meaning a higher margin) allows the seller to allocate the high-quality product for a longer part of the planning horizon.

5 Concluding remarks

To the best of our knowledge, this paper is the first to deal with opaque selling in a fully dynamic environment while accounting for consumer's learning. In selling a vertically differentiated opaque product, there is an incentive for the seller to always assign the low-quality product, because it yields a higher revenue margin than the high-quality product. On the other hand, in a repeated purchasing environment, consumers can learn about the history of a retailer's policy, form their belief, and adjust their purchasing decisions accordingly. This naturally raises a question of how a opaque product seller should assign the high-quality product to the incoming demand in response to such learning behavior.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

Figure 6: Numerical examples: $p_H = 0.9$, $p_L = 0.15$, $c = 0.2$, $\beta = 0.6$.

To answer this question, we fully characterized the conditions under which each of the possible optimal sequences can emerge. The general findings of this paper, based on the length of the planning horizon, can be summarized as follows:

- **Sufficiently long planning horizon:** We obtained an optimal level for the fraction of high-quality product which, in the second phase, it is optimal to keep the consumers' belief on that level. The optimal level depends on the price of the opaque product. If the price is sufficiently high, the optimal level would be one, which means that the seller must assign high quality product for all requested demand. At the first phase, the seller must drive their belief to the optimal level. So, the optimal policy of the seller in this phase depends on the consumers' initial belief. The optimal policy in the last phase, which is the same for all the sequences, is to assign just the low-quality product.
- **Sufficiently short planning horizon:** In this case, if the initial belief of the consumers is lower that the optimal level, there is no sufficient time to drive their belief to that level and

apply the three phase sequence. Therefore, in this case, there is a two phase sequence which is to increase their belief as well as demand by assigning the high-quality product at the first phase and assigning the low-quality product to increase the revenue margin at the second phase.

- **Very Short planning horizon:** In this case, since there is a little time for consumers to learn the seller's policy, it is the seller's best interest to always assign the low-quality product to yield a higher revenue margin.

As in any modeling work, some assumptions were made to focus on what is considered to be the main issues. Relaxing these assumptions is surely of intellectual and managerial interest. First, one could extend the model to an infinite planning horizon to capture a situation where there is no obvious reason to retain any given finite date for the optimization problem. Second, the price of the opaque good itself could be a control variable. Finally, whereas competition is captured in a passive way in this work through the prices of the regular product, considering active competition would be interesting but challenging in terms of characterization of the different pricing sequences.

Appendix: Proofs of propositions

A Proof of Theorem 1

Suppose such a sequence exists. Then, we must have

$$S(\tau_0) = \lambda(\tau_0) - \frac{c}{\beta}\left(\omega(\tau_0)\left(p_H - p_L\right) + p_L - p\right) = 0, \tag{9}$$

$$S(t) = \lambda(t) - \frac{c}{\beta}\left(\omega(t)\left(p_H - p_L\right) + p_L - p\right) < 0 \text{ and } \phi = 0 \text{ for } t \in (\tau_0, \tau_1), \tag{10}$$

$$S(\tau_1) = \lambda(\tau_1) - \frac{c}{\beta}\left(\omega(\tau_1)\left(p_H - p_L\right) + p_L - p\right) = 0. \tag{11}$$

On $t \in [\tau_0, \tau_1]$, the state dynamics is given by $\dot{\omega}(t) = -\beta\omega(t)$, with $\dot{\omega} \leq 0$ and $\ddot{\omega}(t) = -\beta\dot{\omega}(t) \geq 0$ which implies that $\omega(t)$ is convex decreasing on $t \in [\tau_0, \tau_1]$. Further, since the demand is linear in $\omega(t)$, its evolution over time is the same as $\omega(t)$.

The evolution of the costate on $t \in [\tau_0, \tau_1]$ is characterized as

$$\dot{\lambda}(t) = \beta\lambda(t) - p(p_H - p_L),$$

with

$$\dot{\lambda}(t) \text{ is } \begin{cases} \geq 0 & \text{if } \lambda(t) \geq \frac{p(p_H - p_L)}{\beta}, \\ \leq 0 & \text{if } \lambda(t) \leq \frac{p(p_H - p_L)}{\beta}, \end{cases}$$

and

$$\ddot{\lambda}(t) = \beta\dot{\lambda}(t) \Rightarrow sign\ \ddot{\lambda}(t) = sign\ \dot{\lambda}(t).$$

Considering (10), on $t \in (\tau_0, \tau_1)$ it must hold that $\lambda(t) < \frac{cD(\omega(t))}{\beta}$, that is,

$$\begin{aligned} \lambda(t) &< \frac{c}{\beta}\left(\omega(t)\left(p_H - p_L\right) + p_L - p\right), \\ &\leq \frac{c}{\beta}\left(\left(p_H - p_L\right) + p_L - p\right) \text{ (using } \omega_{\tau_0} \in [0,1]), \\ &= \frac{c}{\beta}\left(p_H - p\right) \leq \frac{p(p_H - p)}{\beta} \text{ (using } c < p). \end{aligned}$$

Therefore, the case $\lambda(t) \geq \frac{p(p_H - p_L)}{\beta}$ can be ruled out and we are left with $\lambda(t) \leq \frac{p(p_H - p_L)}{\beta}$, which implies that $\lambda(t)$ is concave decreasing on $[\tau_0, \tau_1]$.

Therefore, on (τ_0, τ_1), $\omega(t)$ (and consequently $\frac{CD(\omega(t))}{\beta}$) is convex decreasing and $\lambda(t)$ is concave decreasing. Considering the fact that $S(\tau_0) = 0$ and $\dot{S}(t) < 0$ for $t \in (\tau_0, \tau_1)$ (see (9) and (10)), $S(t)$ cannot be equal to zero at any time between (τ_0 and T). Therefore, there is no switching point before T and the proof is complete.

B Proof of Theorem 2

Suppose such a sequence exists. From Proposition 1, we know that the policy in the terminal interval must be $\phi = 0$. Therefore, if the policy is switched to $\phi = 1$ at time $t = \tau_0$, there will be at least one other switches until T. Assume that the first switch after τ_0 occurs at $t = \tau_1 \in (\tau_0, T)$. Then, we must have

$$\begin{aligned} S(\tau_0) &= \lambda(\tau_0) - \frac{c}{\beta}(\omega(\tau_0)(p_H - p_L) + p_L - p) = 0, && (12)\\ S(t) &= \lambda(t) - \frac{c}{\beta}(\omega(t)(p_H - p_L) + p_L - p) > 0 \text{ and } \phi = 1 \text{ for } t \in (\tau_0, \tau_1), && (13)\\ S(\tau_1) &= \lambda(\tau_1) - \frac{c}{\beta}(\omega(\tau_1)(p_H - p_L) + p_L - p) = 0. && (14) \end{aligned}$$

On $t \in [\tau_0, \tau_1]$, the state dynamics is given by $\dot{\omega}(t) = \beta(1-\omega(t))$, with $\dot{\omega} \geq 0$ and $\ddot{\omega}(t) = -\beta\dot{\omega}(t) \leq 0$ which implies that $\omega(t)$ is concave increasing on $t \in [\tau_0, \tau_1]$. Further, since the demand is linear in $\omega(t)$, its evolution over time is the same as $\omega(t)$.

The evolution of the costate on $t \in [\tau_0, \tau_1]$ is characterized as follows:

$$\dot{\lambda}(t) = \beta\lambda(t) - (p_H - p_L)(p - c),$$

with

$$\dot{\lambda}(t) \text{ is } \begin{cases} \geq 0 & \text{if } \lambda(t) \geq \frac{(p_H - p_L)(p-c)}{\beta}, \\ \leq 0 & \text{if } \lambda(t) \leq \frac{(p_H - p_L)(p-c)}{\beta}, \end{cases}$$

and

$$\ddot{\lambda}(t) = \beta\dot{\lambda}(t) \Rightarrow sign\ \ddot{\lambda}(t) = sign\ \dot{\lambda}(t).$$

Therefore, on $[\tau_0, \tau_1]$, $\omega(t)$ (and consequently $\frac{CD(\omega(t))}{\beta}$) is concave increasing and $\lambda(t)$ is either convex increasing or concave decreasing. Since $S(\tau_0) = 0$, if $\lambda(t)$ is concave decreasing, then $S(t)$ cannot be positive on $[\tau_0, \tau_1]$, and therefore ϕ cannot be equal to one. If $\lambda(t)$ is convex increasing, then, considering the fact that $S(\tau_0) = 0$ and $S(t) > 0$ on (τ_0, τ_1), policy switching can not happen at an arbitrary point τ_1, which is necessary for the feasibility of the sequence (see (12),(13), and (14)). The proof is complete.

C Proof of Proposition 1

Given that $\lambda(t) = 0$, the switching function at T is given by

$$S(T) = \lambda(T) - \frac{cD(\omega(T))}{\beta} = -\frac{c}{\beta}(\omega(T)p_H + p_L(1 - \omega(T)) - p) \leq 0.$$

Consequently, at T, the optimal policy is to set $\phi = 0$ if the demand $D(\omega(T))$ is strictly positive, that is, $p < \omega(T)p_H + p_L(1 - \omega(T))$. If the demand is zero, then ϕ can take any value in $[0, 1]$.

Suppose $D(\omega(T)) > 0$. Then, it is optimal to finish with $\phi = 0$ on a certain interval $[\hat{\tau}, T]$, with $\hat{\tau}$ to be determined. On $[\hat{\tau}, T]$, the state and costate dynamics are

$$\begin{aligned} \dot{\omega}(t) &= -\beta\omega(t), \\ \dot{\lambda}(t) &= \beta\lambda(t) - (p_H - p_L)p, \quad \lambda(T) = 0. \end{aligned}$$

Solving the two differential equations, we get

$$\omega(t) = e^{-\beta(t-\hat{\tau})}\omega(\hat{\tau}), \tag{15}$$

$$\lambda(t) = \frac{1}{\beta}(p_H - p_L)\,p\left(1 - e^{-\beta(T-t)}\right), \tag{16}$$

and consequently

$$\frac{c}{\beta}D(\omega(t)) = \frac{c}{\beta}\left(e^{-\beta(t-\hat{\tau})}\omega(\hat{\tau})(p_H - p_L) + p_L - p\right), \tag{17}$$

$$\frac{c}{\beta}D(\omega(\hat{\tau})) = \frac{c}{\beta}\left(\omega(\hat{\tau})(p_H - p_L) + p_L - p\right), \tag{18}$$

$$\lambda(\hat{\tau}) = \frac{1}{\beta}(p_H - p_L)\,p\left(1 - e^{-\beta(T-\hat{\tau})}\right). \tag{19}$$

On an interval $[\hat{\tau}, T]$, the two following conditions must be satisfied:

1. At $\hat{\tau}$, the switching function must be equal to zero, that is, $S(\hat{\tau}) = 0$;
2. The switching function must be negative for the rest of the horizon, that is, $S(t) < 0, \forall t \in (\hat{\tau}, T]$.

Let us start with the first condition:

$$\begin{aligned} S(\hat{\tau}) &= \lambda(\hat{\tau}) - \frac{c}{\beta}D(\omega(\hat{\tau})) = 0 \\ &\Leftrightarrow \hat{\tau} = T - \frac{1}{\beta}\ln\frac{(p_H - p_L)\,p}{(p_H - p_L)(p - c\omega(\hat{\tau})) + c(p - p_L)} \qquad (20) \\ &\triangleq T - \frac{1}{\beta}\ln\Gamma. \qquad (21) \end{aligned}$$

Clearly, $\Gamma \geq 1$, and therefore, we have the following possible cases:

- $T - \frac{1}{\beta}\ln\Gamma > 0$, then $\hat{\tau} \in (0, T]$;
- $T - \frac{1}{\beta}\ln\Gamma \leq 0$, then $\hat{\tau} = 0$.

For the second condition, that is, $S(t) < 0, \forall t \in (\hat{\tau}, T]$, it suffices to ensure that there is no other intersection between the threshold and $\lambda(t)$ on interval $(\tau, T]$. To do so, we insert $\hat{\tau}$ from Equation (20) into Equations (16) and (17) to obtain another solution than $\hat{\tau}$ that makes $\lambda(t) - \dfrac{cD(\omega(t))}{\beta}$ equal to zero. This solution, denoted by ν, is given by

$$\hat{\nu} = T - \frac{1}{\beta}\ln\left(\frac{p}{c\omega(\hat{\tau})}\right) \tag{22}$$

Clearly, to have $\hat{\tau} \geq \hat{\nu}$, $\omega(\hat{\tau})$ must be less than $\frac{p(p_H-p_L)+c(p-p_L)}{2c(p_H-p_L)}$. In other words, for $\omega(\hat{\tau}) > \frac{p(p_H-p_L)+c(p-p_L)}{2c(p_H-p_L)}$, the derivative of $S(t)$ with respect to t will be positive at point $t = \hat{\tau}$. Since $S(\hat{\tau}) = 0$, $S(t)$ will be positive on interval $[\hat{\tau}, \hat{\nu}]$, which is in conflict with the optimality condition for $\phi = 0$.

D Proof of Proposition 2

From Proposition 1, for $\omega(\hat{\tau}) \leq \overline{\omega}$, the switching point is given by

$$\hat{\tau} = \max\left(T - \frac{1}{\beta}\ln\frac{(p_H - p_L)\,p}{(p_H - p_L)(p - c\omega(\hat{\tau})) + c(p - p_L)}, 0\right).$$

Setting $\hat{\tau} = 0$, the condition $T + \frac{1}{\beta} \ln \Gamma \leq 0$ becomes

$$T \leq \frac{1}{\beta} \ln \left(\frac{(p_H - p_L)\, p}{(p_H - p_L)\,(p - c\omega(\hat{\tau})) + c\,(p - p_L)} \right) \triangleq \hat{T}_1^{S_1}(\omega_0).$$

For the optimality of sequence 1, that is, $\phi = 0$ on $[0,T]$, $S(t) \leq 0, \forall t \in [0,T]$. From Proposition 1, we know that if $\omega_0 > \overline{\omega}$, $S(0)$ cannot be equal to zero. In other words, for $\omega_0 > \overline{\omega}$, if $T = \frac{1}{\beta} \ln \frac{(p_H - p_L)p}{(p_H - p_L)(p - c\omega_0) + c(p - p_L)}$, $S(0)$ will be equal to zero, which is infeasible due to positive derivative of $S(t)$ w.r.t. t at $t = 0$. Therefore, $S(0)$ is strictly less than zero in this case. From the proof of Theorem 1, we know that for $\phi = 0$, $\frac{cD(\omega(t))}{\beta}$ is convex decreasing and $\lambda(t)$ is concave decreasing w.r.t t. Therefore, the extreme case for which sequence one is optimal for $\omega_0 > \overline{\omega}$ happens when the two roots of $S(t) = 0$ are equal and are between 0 and T. Considering equation (16) and setting $\hat{\tau}$ in (17), $S(t)$ on interval $[0,T]$ is obtained as

$$S(t) = \frac{(p_H - p_L)\left(p(1 - e^{-\beta(T-t)}) - cw_0 e^{-\beta t}\right) + c(p - p_L)}{\beta}.$$

For $T = \frac{1}{\beta} \ln \frac{4cp\omega_0 (p_H - p_L)^2}{(p(p_H - p_L) + c(p - p_L))^2}$, the two roots of $S(t)$ are equal. Therefore, we can conclude that for $\omega_0 > \overline{\omega}$, if $T \leq \frac{1}{\beta} \ln \frac{4cp\omega_0 (p_H - p_L)^2}{(p(p_H - p_L) + c(p - p_L))^2}$, $S(t) \leq 0, \forall t \in [0,T]$ and $\phi = 0$ is the optimal policy on the interval (sequence $S1$).

The trajectory of $\lambda(t)$ in $S1$ is as (16), and the trajectory of $\omega(t)$ can be obtained by setting $\hat{\tau} = 0$ in (17).

E Proof of Proposition 3

If a switching date $\hat{\tau} \in (0,T)$ exists, then we have: (i) $\phi = 1$ on $[0, \hat{\tau}]$; (ii) $\phi = 0$ on $[\hat{\tau}, T]$; (iii) $S(\hat{\tau}) = 0$. We have to derive the conditions under which

$$\hat{\tau} = T - \frac{1}{\beta} \ln \left(\frac{(p_H - p_L)\, p}{(p_H - p_L)\,(p - c\omega(\hat{\tau})) + c\,(p - p_L)} \right), \tag{23}$$

is interior. Since $\frac{(p_H - p_L)p}{(p_H - p_L)(p - c\omega(\hat{\tau})) + c(p - p_L)} > 1$, we know that $\hat{\tau} < T$. It remains to check when $\hat{\tau} > 0$.

First, we need to determine $\hat{\tau}$ as function of only the model's parameters. Recall that on $[0, \hat{\tau}]$, the state dynamics are given by

$$\dot{\omega}(t) = \beta(1 - \omega(t)), \qquad \omega(0) = \omega_0.$$

Solving, we get

$$\begin{aligned} \omega(t) &= 1 - e^{-\beta t}(1 - \omega_0), && (24)\\ \omega(\hat{\tau}) &= 1 - e^{-\beta \hat{\tau}}(1 - \omega_0). && (25) \end{aligned}$$

Substituting in (23), we get

$$\hat{\tau} = T - \frac{1}{\beta} \ln \left(\frac{(p_H - p_L)\, p}{p\,(p_H - p_L) - c\,(p_H - p) + ce^{-\beta \hat{\tau}}\left((p_H - p_L) - \omega_0\,(p_H - p_L)\right)} \right).$$

By solving the above equation, $\hat{\tau}$ can be obtained as

$$\hat{\tau} = \frac{1}{\beta} \ln \left(\frac{(p\,(p_H - p_L) - c\,(p_H - p)) + \sqrt{(p\,(p_H - p_L) - c\,(p_H - p))^2 + 4\,(p_H - p_L)^2\, pe^{-\beta T} c\,(1 - \omega_0)}}{2\,(p_H - p_L)\, pe^{-\beta T}} \right). \tag{26}$$

Now, $\hat{\tau} \geq 0$ is equivalent to $T \geq \overline{T}_1^{S_1}(\omega_0)$.

$S(t)$ must be non-negative on the interval $[0, \hat{\tau}]$. To check this condition, the trajectory for $\lambda(t)$ on the interval must be determined. To get $\lambda(t)$, the following equations considering (25) must be solved:

$$\dot{\lambda}(t) = \beta\lambda(t) - (p-c)(p_H - p_L), \tag{27}$$

$$S(\hat{\tau}) = \lambda(\hat{\tau}) - \frac{c}{\beta} D(\omega(\hat{\tau})) = 0. \tag{28}$$

We get $\lambda(t)$ for $t \in [0, \hat{\tau}]$ as

$$\lambda(t) = \frac{1}{\beta}\left(\left((2p_H - p - p_L)c - p(p_H - p_L)\right) e^{\beta(t+\hat{\tau})} + \Big((p-c)e^{(2\beta\hat{\tau})} - ce^{\beta t}(1-w_0)\Big)(p_H - p_L)\right) e^{-2\beta\hat{\tau}}. \tag{29}$$

By inserting (26) into (24) and (29), $S(t)$ for $t \in [0, \hat{\tau}]$ can be obtained. To satisfy $S(t) \leq 0$ on the interval $[0, \hat{\tau}]$, the second root of $S(t) = 0$ must be greater than $\hat{\tau}$, which leads to the following condition:

$$T \leq \frac{1}{\beta} \ln\left(\frac{4cp(p_H - p_L)^2(1-w_0)}{\left(c(p_H - p) - (p_H - p_L)(p-c)\right)\left((p-p_L)c + p(p_H - p_L)\right)}\right) = \frac{1}{\beta}\ln\Phi \triangleq \overline{T}_1^{S_2}(w_0).$$

If $\overline{\omega} > 1$, $\Phi < 0$. So, in this case, there will be no upper bound for T. For $\overline{T}_1^{S_1}(w_0) \leq \overline{T}_1^{S_2}(w_0)$, $\omega_0 \leq \overline{\omega} \leq 1$, which is intuitive.

We can show that $\overline{T}_1^{S_2}(\omega_0) = \overline{T}_1^{S_1}(\overline{\omega}) + \frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$, which is intuitive. Based on (25), the time required to reach from ω_0 to $\omega(\hat{\tau})$ is $\frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\omega(\hat{\tau})}\right)$, which is increasing in $\omega(\hat{\tau})$. From Proposition 1 we know that at the switching point, $\omega(\hat{\tau})$ must be lower than $\overline{\omega}$. Therefore, the maximum time for setting $\phi = 1$ is $\frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$. On the other hand, the time of the last phase, that is, $T - \hat{\tau}$, equals to $\overline{T}_1^{S_1}(\omega(\tau))$, which is increasing in $\omega(\tau)$. Therefore, since $\omega(\tau) = \overline{\omega}$, we can conclude that the upper bound for T, for which S1 is feasible, equals to $\overline{T}_1^{S_1}(\overline{\omega}) + \frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right)$. Since $\omega(\hat{\tau}) \leq \overline{\omega}$, it is clear that S1 is feasible only if $\omega_0 \leq \overline{\omega}$.

F Proof of Proposition 4

In S3, we have two switching moments, which we denote τ_1 and τ_2, with the policy being as follows:

$$\phi = \begin{cases} 1 & \text{for } t \in [0, \tau_1] \\ \tilde{\phi} \in (0,1) & \text{for } t \in [\tau_1, \tau_2] \\ 0 & \text{for } t \in [\tau_2, T] \end{cases}.$$

The corresponding state and costate dynamics are given by

$$\dot{\omega}(t) = \begin{cases} \beta(1-\omega(t)), \quad \omega(0) = \omega_0, & \text{for } t \in [0, \tau_1] \\ \beta(\tilde{\phi} - \omega(t)), & \text{for } t \in [\tau_1, \tau_2] \\ -\beta\omega(t), & \text{for } t \in [\tau_2, T] \end{cases},$$

$$\dot{\lambda}(t) = \begin{cases} \beta\lambda(t) - (p_H - p_L)(p - c), & \text{for } t \in [0, \tau_1] \\ \beta\lambda(t) - (p_H - p_L)(p - c\tilde{\phi}), & \text{for } t \in [\tau_1, \tau_2] \\ \beta\lambda(t) - (p_H - p_L)p, \quad \lambda(T) = 0, & \text{for } t \in [\tau_2, T] \end{cases},$$

and the switching function by

$$S(t) = \begin{cases} \lambda(t) - \frac{c}{\beta}\left(\omega(t)\left(p_H - p_L\right) + p_L - p\right) > 0, & \text{for } t \in [0, \tau_1] \\ \lambda(t) - \frac{c}{\beta}\left(\omega(t)\left(p_H - p_L\right) + p_L - p\right) = 0, & \text{for } t \in [\tau_1, \tau_2] \\ \lambda(t) - \frac{c}{\beta}\left(\omega(t)\left(p_H - p_L\right) + p_L - p\right) < 0, & \text{for } t \in [\tau_2, T] \end{cases}.$$

In S3, we must have

$$S(\tau_1) = S(\tau_2) = 0 \text{ and } S(t) = 0 \text{ for } t \in [\tau_1, \tau_2],$$

and consequently $\dot{S}(t) = 0$ for $t \in [\tau_1, \tau_2]$. Further, on this interval, the evolution of the state variable is characterized as follows:

$$\dot{\omega}(t) \text{ is } \begin{cases} > 0 & \text{if } \tilde{\phi} > \omega(t), \\ 0 & \text{if } \tilde{\phi} = \omega(t), \\ < 0 & \text{if } \tilde{\phi} < \omega(t). \end{cases}$$

$$\ddot{\omega}(t) = -\beta\dot{\omega}(t) \Rightarrow sign\ \ddot{\omega}(t) = -sign\ \dot{\omega}(t),$$

that is, $\omega(t)$ is either concave increasing, constant, or convex decreasing. The evolution of the costate is characterized as follows:

$$\dot{\lambda}(t) \text{ is } \begin{cases} > 0 & \text{if } \lambda(t) > \frac{(p_H - p_L)(p - c\tilde{\phi})}{\beta}, \\ 0 & \text{if } \lambda(t) = \frac{(p_H - p_L)(p - c\tilde{\phi})}{\beta}, \\ < 0 & \text{if } \lambda(t) < \frac{(p_H - p_L)(p - c\tilde{\phi})}{\beta}, \end{cases}$$

$$\ddot{\lambda}(t) = \beta\dot{\lambda}(t) \Rightarrow sign\ \ddot{\lambda}(t) = sign\ \dot{\lambda}(t),$$

which implies that $\lambda(t)$ is either convex, increasing, constant or concave decreasing. The only possibility, in order to have $S(\tau_1) = S(\tau_2) = 0$ and $\dot{S}(t) = 0$ on $[\tau_1, \tau_2]$, is to have a constant $\omega(t)$ and $\lambda(t)$, that is, $\dot{\lambda}(t) = \dot{\omega}(t) = 0$. Therefore, on $[\tau_1, \tau_2]$ we have

$$\tilde{\phi} = \omega(t), \tag{30}$$

$$\lambda(t) = \frac{(p_H - p_L)(p - c\tilde{\phi})}{\beta}, \tag{31}$$

$$\lambda(t) = \frac{cD(\omega(t))}{\beta}. \tag{32}$$

Solving the equations, for interval $[\tau_1, \tau_2]$ we get

$$\lambda(t) = \frac{p(p_H - p_L) - c(p - p_L)}{2\beta}, \tag{33}$$

$$\omega(t) = \phi(t) = \frac{p(p_H - p_L) + c(p - p_L)}{2c(p_H - p_L)} = \overline{\omega}, \tag{34}$$

Since on $[0, \tau_1]$, ϕ equals one, $\omega(t)$ on the interval starts from ω_0 and increases to reach $\overline{\omega}$ at $t = \tau_1$. Therefore, for the feasibility of this sequence, $\omega_0 \leq \overline{\omega} \leq 1$. Considering $\omega(t)$ on interval $[0, \tau_1]$ in (25), we have

$$\omega(\hat{\tau}_1) = 1 - (1 - \omega_0)\, e^{-\beta\hat{\tau}_1} = \overline{\omega} \tag{35}$$

$$\Leftrightarrow \hat{\tau}_1 = \frac{1}{\beta}\ln\left(\frac{1-w_0}{1-\overline{\omega}}\right) = \frac{1}{\beta}\ln\left(\frac{2c\,(p_H-p_L)\,(1-\omega_0)}{c\,(p_H-p)-(p-c)\,(p_H-p_L)}\right). \quad (36)$$

To obtain $\hat{\tau}_2$, we simply set $\omega(\hat{\tau}) = \overline{\omega}$ in (8) in Proposition 1, which gives

$$\tau_2 = T - \frac{1}{\beta}\ln\left(\frac{2p(p_H-p_L)}{p(p_H-p_L)+c\,(p_L-p)}\right) = T - \overline{T}_1^{S_1}(\overline{\omega}).$$

Clearly, for the optimality of S3, T must be greater than $\tau_1 + T - \tau_2$, that is, the duration of the first and last phases in this sequence. From the proof of Proposition 3, we know that $\tau_1 + T - \tau_2 = \frac{1}{\beta}\ln\left(\frac{1-\omega_0}{1-\overline{\omega}}\right) + \overline{T}_1^{S_1}(\overline{\omega}) = \overline{T}_1^{S_2}(\omega_0)$.

G Proof of Proposition 5

Denote by τ_1 and τ_2 the switching moments. The policy is then as follows:

$$\phi = \begin{cases} 0 & \text{for } t \in [0,\tau_1] \\ \tilde{\phi} \in (0,1) & \text{for } t \in [\tau_1,\tau_2] \\ 0 & \text{for } t \in [\tau_2,T] \end{cases}.$$

Similarly to the proof of Proposition 4, for interval $[\tau_1,\tau_2]$, $\lambda(t)$ and $\omega(t)$ are as (33) and (34), respectively.

Since on $[0,\tau_1]$, ϕ equals zero, $\omega(t)$ on the interval starts from ω_0 and decreases to reach $\overline{\omega}$ at $t=\tau_1$. Therefore, for the feasibility of this sequence, $\omega_0 > \overline{\omega}$. Considering the trajectory of ω on $[0,\tau_1]$, that is, $\omega(t) = \omega_0 e^{-\beta t}$, we have

$$\omega(\hat{\tau}_1) = \omega_0 e^{-\beta\hat{\tau}_1} = \overline{\omega} \quad (37)$$

$$\Leftrightarrow \hat{\tau}_1 = \frac{1}{\beta}\ln\left(\frac{w_0}{\overline{\omega}}\right) = \frac{1}{\beta}\ln\left(\frac{2c(p_H-p_L)\omega_0}{p(p_H-p_L)+c\,(p-p_L)}\right). \quad (38)$$

Since, similarly to S3, $\omega(\tau_2) = \overline{\omega}$, to obtain $\hat{\tau}_2$, we simply set $\omega(\hat{\tau}) = \overline{\omega}$ in (8) in Proposition 1, which gives $\tau_2 = \tau_2^{S_3}$.

Clearly, for the optimality of S4, τ_2 must be greater than τ_1, which gives

$$T > \frac{1}{\beta}\ln\left(\frac{4cp(p_H-p_L)^2(1-w_0)}{\left(c(p_H-p)-(p_H-p_L)(p-c)\right)\left((p-p_L)c+p(p_H-p_L)\right)}\right) = \overline{T}_1^{S_2}(\omega_0).$$

H Proof of Proposition 6

Denote by τ_1 the switching moments. The policy is then as follows:

$$\phi = \begin{cases} \tilde{\phi} \in (0,1) & \text{for } t \in [0,\tau_1] \\ 0 & \text{for } t \in [\tau_1,T] \end{cases}.$$

From the proof of Propositions 4 and 5, we can conclude that for interval $[0,\tau_1]$, $\lambda(t)$ and $\omega(t)$ are as (33) and (34), respectively.

It is clear that for the feasibility of this sequence, ω_0 must be equal to $\overline{\omega}$. Similarly to S3 and S4, to obtain $\hat{\tau}_1$, we simply set $\omega(\hat{\tau}) = \overline{\omega}$ in (8) in Proposition 1, which gives $\tau_2 = \tau_2^{S_3} = \tau_2^{S_4}$.

Clearly, for the optimality of S5, τ_1 must be greater than zero, which gives $T > \overline{T}_1^{S_1}(\omega_0)$.

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