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under multi-element geological uncertainty**

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Multi-product mine scheduling optimization under multi-element geological uncertainty

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Abstract: A new solution approach is developed herein to address the topic of multi-product open-pit mine production scheduling with multi-element uncertainty. The method is based on extending the Bienstock-Zuckerberg (BZ) algorithm to the stochastic optimization. The purpose of the BZ algorithm is to exploit the structure of the problem to solve its linear programming relaxation (LP) efficiently. Numerical results on a real-size instance show that the LP can be solved to optimality in few minutes, while Cplex takes several hours to solve it. Following this with a rounding heuristic based on topological sorting and a second heuristic based on Tabu Search, a feasible integer solution is then obtained within 1–2% of optimality.

Keywords: Open-pit optimization, stochastic programming, Bienstock-Zuckerberg algorithm, multi-element deposit, long-term production planning, Tabu Search

Résumé : Une nouvelle approche est développée dans cet article pour résoudre le problème de la planification stratégique de production d'une mine à ciel-ouvert dans un contexte d'incertitude à éléments multiples. La méthode repose sur l'extension de l'algorithme Bienstock-Zuckerberg (BZ) à l'optimisation stochastique. Le but de l'algorithme BZ est d'exploiter la structure particulière du problème pour résoudre efficacement sa relaxation linéaire. Les résultats numériques, suite à une application sur une instance de taille réelle, montrent que la relaxation linéaire peut être résolue à l'optimal en quelques minutes, alors que Cplex prend plusieurs heures à le résoudre. Ensuite, en appliquant successivement une heuristique d'arrondissement basée sur un ordonnancement topologique et une seconde heuristique basée sur la recherche Tabou, une solution entière réalisable est obtenue avec un saut d'optimalité allant de 1 à 2%.

Mots clés : Optimisation des mines à ciel-ouvert, programmation stochastique, algorithme de Bienstock-Zuckerberg, gisement à éléments multiples, planification stratégique de la production, recherche Tabou

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1 Introduction

The mining industry is a major player in the global economy, providing most of the mineral and metal that is essential to not only construction activities related to economic development and urbanization, but also to both conventional and new high-tech industries, including green technologies and communication. To meet an ever-increasing demand for metal supplies, millions of dollars are invested each year by mining companies in new projects and the development of mining operations. The profitability of these investments goes hand-in-hand with efficiency in mine planning, as all operating and production performance depends on it. It is crucial to develop decision-making tools that aim to maximize the economic value of a mining project and meet customer requirements in terms of metal product tonnage and grade quality to be delivered. For these reasons, mine production scheduling (MPS) has received much attention in the technical literature.

As starting point for optimization, a three-dimensional model of blocks was introduced very early to represent mineral deposits. This block model subdivides the mineral deposit into mining blocks in the form of parallelepipeds and associates with each block a set of pertinent geological attributes, such as metal grades and material types, obtained from the available drill-hole data. At the time that mine planning is undertaken, the actual values of each mining block are never exactly known due to the limited drilling information. Values of pertinent geological attributes are interpolated with a certain degree of accuracy, given the data available. Once the orebody model is created, mine production scheduling involves the determination of which mining blocks should be extracted during each time period, and what is to be done with them once they are extracted, choosing from a given set of possible destinations.

The conventional approaches to solve the MPS use a single deposit model created as a weighted average of local data. They are thus, deterministic in the sense that they consider all the information utilized as precise and well known, ignoring the sources of uncertainty and risks. The first attempts date back to Johnson (1969) who applied the Dantzig-Wolfe decomposition to reduce the resolution of the problem to the resolution of several subproblems of reasonable size. Subsequently, Dagdeleen and Johnson (1986) propose an exact method based on the Lagrangian relaxation of resource constraints. The results were promising, but still not applicable on realistic size instances. To alleviate this problem and make large instances computationally tractable, several methods have been suggested. Heuristic methods for aggregating ore blocks have been proposed by Ramazan (1996) and Whittle (1988, 1999), while Ramazan (2007) has developed the Fundamental Tree Algorithm (FTA), which optimally aggregates blocks based on exploiting the structure of the problem. To reduce the number of required binary variables, Topal (2003) define, for each block, the earliest and the latest period when it can be extracted. Dimitrakopoulos and Ramazan (2008), Cullemine et al. (2011), and Lamghari and Dimitrakopoulos (2014) use the sliding time window by fixing some variables and relaxing the integrity of some others temporarily. Heuristics have been widely proposed. Some works use classical heuristics such as swarm optimization (Khan and Niemann-Delius, 2014) or genetic algorithms (Denby and Schofield, 1994). Others combine heuristics and optimization methods as presented in the paper by Lamghari et al. (2015), which proposes a hybrid method based on linear programming and variable neighborhood local search. Some other efficient approaches relying on decomposition techniques have been proposed. Tachefine (1997) uses in his thesis the Lagrangian relaxation of the resource constraints to be reduced to a maximum closure problem. He innovates by using the bundle method to adjust the Lagrange multipliers, rather than the classical sub-gradient method. Later, Chicoisne et al. (2012) propose a decomposition algorithm to solve, in polynomial time, the LP of the MPS considering one single resource (mining capacity) by a period. Bienstock and Zuckerberg (2009, 2010) develop a new algorithm based on a Lagrangian decomposition to solve also the LP of the problem, but with an arbitrary number of renewable or non-renewable resources. Recently, Munoz et al. (2017) extend the template of the Bienstock-Zuckerberg (BZ) algorithm to handle more problems arising in the context of mine planning.

Nevertheless, there is one critical issue remaining in deterministic optimization methods: ignoring the uncertainty present in several parameters, such as the metal content, and the lack of robustness that it creates. The deterministic approach has the effect of smoothing the representation of the pertinent attributes of the mineral deposit considered (an average of a range of possible values), and thus, misrepresenting the deposit by over or underestimating the actual mining potential (Ravenscroft, 1992; Dowd, 1994, 1997). In addition to the metal content, other parameters are affected by the uncertainty and directly influence the

expected net present value (NPV) of a project. These parameters include the fluctuation of demand on metal products (Sabour and Dimitrakopoulos, 2011; Asad and Dimitrakopoulos, 2013), which consequently affects not only selling prices, but also costs of production, processing, etc. Nevertheless, only the geological uncertainty has really aroused the interest of researchers given its major impact on the profitability of mining projects. Through a sensitivity analysis on the effect of several financial factors (NPV, recovery rate, etc.), Dowd (1994, 1997) demonstrates that geological uncertainty substantially affects the performance of a project and Baker and Giacomo (1998) shows that ignoring it is the main cause of not meeting production forecasts.

Stochastic simulation (Goovarets, 1997; Boucher and Dimitrakopoulos, 2009) addresses these problems by reproducing the distribution of the grades and spatial statistics of available data and information, and thus, preserves the *in-situ* variability of the deposit. For that, several equiprobable realizations of pertinent attributes of the mineral deposit are simulated. They are then used simultaneously as inputs for the MPS to provide results closer to reality along with risk quantification on production forecasts. Stochastic programming increases the reliability of planning and its chances of achieving production targets through more realistic modeling and risk management. Considering uncertainty also leads to better NPVs (15-30% more on average) than those obtained by deterministic models (Godoy, 2002; Leite and Dimitrakopoulos, 2007, 2014; Albor and Dimitrakopoulos, 2010). This improvement is due in part to the fact that the stochastic programming approach facilitates the elimination of the effects of smoothing while managing the related risk, and, by construction, leads to higher chances of meeting forecasts, higher metal production and higher NPV. In addition, case studies show that stochastic optimal pit limits generate 10-15% additional metal extracted when compared to the traditional optimal limits (Dimitrakopoulos, 2011). This is a direct result of a more efficient blending, as well as a better understanding of the spatial distribution and the relationship between high-quality grade blocks.

All this has led to the stochastic approach's emergence in the last two decades. The optimal scheduling is now the one that maximizes the NPV taking into account the uncertainty described by different equiprobable simulations of the orebody model. Risk management is reflected in terms of deviations from objectives. Different methods have been suggested to solve this problem. Lamghari and Dimitrakopoulos (2012) suggested an efficient metaheuristic solution approach based on a diversified Tabu search while Godoy and Dimitrakopoulos (2004), and Albor and Dimitrakopoulos (2009) use the simulated annealing. Stochastic integer programming (SIP) has also been adopted by several authors for the optimization of mining production planning considering uncertainty (Ramazan and Dimitrakopoulos, 2007, 2013, Leite and Dimitrakopoulos, 2014). One of the applications that particularly interests this current study is proposed by Benndorf and Dimitrakopoulos (2013) who deal with an iron deposit where geological uncertainty affects several elements at once. Also, the authors introduce the objective function a term penalizing an unsmoothed activity. The current trend is towards global optimization models that focus not only on the extraction, but also on processing, consider several metals and minerals, and deal with both blending options and alternative treatments leading to separate products (Whittle, 2009). Goodfellow and Dimitrakopoulos (2016, 2017) and Montiel and Dimitrakopoulos (2015, 2017) have recently proposed a stochastic global optimization model with recourse. They combine three metaheuristics, namely simulated annealing, particle swarm optimization and differential evolutions. Their method includes a policy for choosing the destinations of the blocks once extracted. The results obtained are very promising.

This paper proposes an efficient solution method to solve long-term MSP of an open-pit multi-product mine under multi-element geological uncertainty. The method is based on extending the BZ algorithm (Brenstock and Zuckerberg, 2009) to stochastic optimization. The mathematical formulation is presented in Section 2. Section 3 presents the solution approach, more specifically how the BZ algorithm is adapted to stochastic optimization along with the two heuristics proposed to make the solution integer. An application on a multi-product iron ore deposit is reported as a case study in Section 4. Implementation details, computational results on a multi-product iron ore deposit and some conclusions are drawn in Section 5.

2 Mathematical formulation

Given a 3D block model of an ore body, the relationships among the blocks can be represented through a directed graph $G = (\mathcal{B}, \mathcal{A})$ where the nodes are the mining blocks and an arc (a, b) from a to b means that the block a cannot be extracted before the block b . This category of constraints is termed the slope constraints. They are essential to maintain the stability of the orebody during extraction operations and they imply that a block is accessible only after all its overlying blocks are already extracted. Figure 1 shows a simple 2D block model and its corresponding graph if a 45-degree slope must be respected.

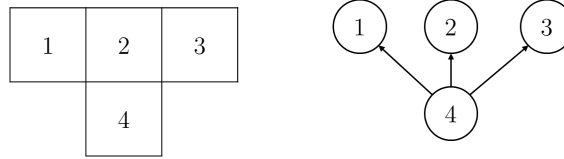


Figure 1: 2D block model and its corresponding graph

Let suppose that the blocks $\mathcal{B}$ should be scheduled over T the lifetime of the mine and sent to one destination (products ore, mills, waste, etc.) among a given set $\mathcal{D} = \{1 \dots D\}$. Let $\mathcal{R} = \{1, \dots, R\}$ be the mineral resources including both ore tonnage and grade targets, and $\mathcal{S} = \{1 \dots S\}$ the set of equally possible scenarios of the unknown orebody. The following notation is also defined to be used in the formulation of the scheduling problem and next sections.

2.1 Variables

- $y_{b,d,t}$ is the activity variable which takes the value 1 if mining block b is extracted and sent to destination d by period t and 0, otherwise.
- $qu_{d,t,s}^r$ and $ql_{d,t,s}^r$ are respectively the upper and lower deviation from production resource r at period t sent to destination d under scenario s .

2.2 Parameters

- $p_{b,d,t,s}$ is the discounted profit obtained, if block b is sent to destination d in period t under scenario s .
- $g_{b,s}^r$ is the grade of block b for element r under scenario s .
- $Gmax_{d,t}^r$ and $Gmin_{d,t}^r$ are respectively the expected maximum and minimum grades for resource r sent to destination d at period t .
- cu_t^r and cl_t^r are respectively the discounted unit cost of upper and lower deviation from $Gmax_{d,t}^r$ and $Gmin_{d,t}^r$.
- Without loss of generality, all blocks have the same tonnage equal to Q .
- M_{max} is the maximum mining capacity per period.
- Γ_b^+ and Γ_b^- are respectively the successors and the predecessors of block b defined by: $\Gamma_b^+ = \{j \in \mathcal{B} | (j, b) \in \mathcal{A}\}$ and $\Gamma_b^- = \{j \in \mathcal{B} | (b, j) \in \mathcal{A}\}$

Using this notation, the objective function can be formulated as a two-stage stochastic integer programming model (Birge and Louveaux, 1997) as follows:

2.3 Objective function

$$\max \quad \frac{1}{S} \sum_{b=1}^B \sum_{d=1}^D \sum_{t=1}^T \left(\sum_{s=1}^S \tilde{p}_{b,d,t,s} y_{b,d,t} \right) - \frac{1}{S} \sum_{d=1}^D \sum_{t=1}^T \sum_{s=1}^S \sum_{r=1}^R (cu_t^r qu_{d,t,s}^r + cl_t^r ql_{d,t,s}^r)$$

where:

$$\tilde{p}_{b,d,t,s} = \begin{cases} p_{b,d,t,s} - p_{p,(d+1),t,s} & \forall b \in \mathcal{B}, \quad d < D, \quad t = 1 \dots T, \quad s = 1 \dots S \\ p_{b,D,t,s} - p_{b,1,t+1,s} & \forall b \in \mathcal{B}, \quad t < T, \quad s = 1 \dots S \\ p_{b,D,T,s} & \forall b \in \mathcal{B}, \quad s = 1 \dots S \end{cases}$$

The objective function is a combination of two parts. Each one represents a different goal. The first part indicates that discounted operating income is to be maximized considering all the scenarios. The second part manages the deviations from production targets over the simulated orebody models. This part ensures the consideration and the management of uncertainty and variability since the optimization is done over S equally probable scenarios. Note that the corresponding cost parameters determine the weight of the two parts in the objective function.

2.4 Constraints

The reserve constraint assures that each block b cannot be extracted and processed more than once overall periods and is defined by:

$$y_{b,D,t} \leq y_{b,1,t+1} \quad \forall b \in \mathcal{B}, \quad t = 1 \dots T-1 \quad (1)$$

$$y_{b,(d-1),t} \leq y_{b,d,t} \quad \forall b \in \mathcal{B}, \quad d = 2 \dots D, \quad t = 1 \dots T \quad (2)$$

Before a block i can be extracted, all the overlaying blocks j must be extracted in the same period or earlier:

$$y_{i,D,t} \leq y_{j,D,t} \quad \forall t = 1 \dots T, \quad (i, j) \in \mathcal{A} \quad (3)$$

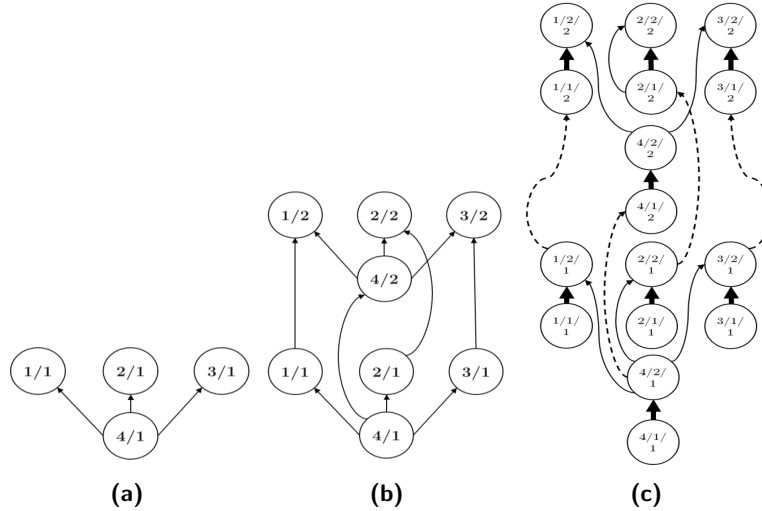


Figure 2: Figure 2a represents the original graph G containing only the slope constraints, 2b is obtained by duplicating T times the graph G and adding the inter-period arcs. 2c represents the graph $\bar{G}$ and it is derived from 2b by duplicating each node D times and adding the inter-destination arcs

Figure 2 shows the steps to introduce, respectively, the notion of multi-period and multi-destinations scheduling into the graph $G = (\mathcal{B}, \mathcal{A})$ to obtain a new directed graph $\bar{G} = (\bar{\mathcal{B}}, \bar{\mathcal{A}})$ where for each period t and destination d , a block b is represented by a vertex $(b/d/t)$. $\bar{G}$ also represents the constraints (1), (2) and (3) by three different categories of arcs: the inter-period arcs for the constraints (1) represented by dashed lines, the inter-destination arcs for the constraints (2) represented by thick lines and finally, the slope-constraint arcs for the constraints (3) represented by thin lines.

In what follows, a new variable $\tilde{y}_{bdt}$ is used to simplify the constraints expressions and defined by:

$$\tilde{y}_{b,d,t} = \begin{cases} y_{b,d,t} - y_{p,(d-1),t} & \forall b \in \mathcal{B}, \quad d = 2 \dots D, \quad t = 1 \dots T, \quad s = 1 \dots S \\ y_{b,1,t} - p_{b,D,t-1} & \forall b \in \mathcal{B}, \quad t = 2 \dots T, \quad s = 1 \dots S \\ y_{b,1,1} & \forall b \in \mathcal{B}, \quad s = 1 \dots S \end{cases}$$

Grade deviations $qu_{d,t,s}^r$ and $ql_{d,t,s}^r$ for each destination d , period t , element r and under scenario s are defined by blending constraints formulated in inequations (4) and (5) for the upper and lower deviations

respectively and for each $(s, r) \in \mathcal{S} \times \mathfrak{R}$:

$$\sum_{b=1}^B (g_{bs}^r - Gmax_{d,t}^r) \cdot O_{bs} \cdot \tilde{y}_{b,d,t} - qu_{d,t,s}^r \leq 0 \quad \forall t = 1 \dots T, d = 1 \dots D \quad (4)$$

$$\sum_{b=1}^B (g_{bs}^r - Gmin_{d,t}^r) \cdot O_{bs} \cdot \tilde{y}_{b,d,t} + ql_{d,t,s}^r \geq 0 \quad \forall t = 1 \dots T, d = 1 \dots D \quad (5)$$

Note here that there are three different expressions for each type of constraint due to the definition of the decision variables “sent by” and not “sent at”. Also, the O_{bs} allows to consider only the ore blocks. Similarly, ore tonnage deviations $qu_{d,t,s}^{ore}$ and $ql_{d,t,s}^{ore}$ are defined in inequations (6) and (7):

$$\sum_{b=1}^B O_{bs} \cdot \tilde{y}_{b,d,t} - qu_{d,t,s}^{ore} \leq Gmax_{d,t}^{ore} \quad \forall t = 1 \dots T, d = 1 \dots D, s = 1 \dots S \quad (6)$$

$$\sum_{b=1}^B O_{bs} \cdot \tilde{y}_{b,d,t} + ql_{d,t,s}^{ore} \geq Gmin_{d,t}^{ore} \quad \forall t = 1 \dots T, d = 1 \dots D, s = 1 \dots S \quad (7)$$

Then, the constraints (8) limit the total tonnage of handled material at period t subject to maximum mining capacity:

$$Q \times \sum_{b=1}^B y_{b,D,1} \leq M_{max} \quad (8a)$$

$$Q \times \sum_{b=1}^B (y_{b,D,t} - y_{b,D,t-1}) \leq M_{max} \quad \forall t = 2 \dots T \quad (8b)$$

Finally, the integrity constraints (9) and the non-negativity constraints (10):

$$y_{b,d,t} \in \{0, 1\} \quad \forall b \in \mathcal{B}, d \in \mathfrak{D}, t = 1 \dots T \quad (9)$$

$$qu_{d,t,s}^r, ql_{d,t,s}^r \geq 0 \quad \forall d \in \mathfrak{D}, t = 1 \dots T, s \in \mathcal{S}, r \in \mathfrak{R} \quad (10)$$

3 Solution approach

The proposed methodology consists of three steps. The first step is an adaptation of the algorithm initially introduced in Bienstock and Zuckerberg (2009). In Section 3.1, a description of how this algorithm was extended to the stochastic optimization is presented. The second one is a rounding heuristic to the fractional solution obtained in the first step. This heuristic described in Section 3.2 takes up ideas from the TopoSort Heuristic proposed by Chicoisne et al. (2012), but some changes are introduced to handle the stochastic aspect. The third step is to apply a TabuSearch iteratively to improve the quality of the solution obtained by the rounding heuristic.

3.1 Solving the linear relaxation

The formulation described in Section 2 can be represented in a more compact form as follows where q represents the deviation variables, and c the discounted unit cost of deviation:

$$\begin{aligned} Z &= \max p^t y + c^t q \\ \text{s.t. } y_i &\leq y_j \quad \forall (i, j) \in \bar{\mathcal{A}} \\ My + Hq &\leq d \\ y &\in \{0, 1\}^n \end{aligned}$$

This formulation is more general than the General Precedence Constrained Problem (GPCP) presented in Bienstock and Zuckerberg (2011), and that doesn't consider the variables q . Munoz et al. (2017) proved the correctness of the BZ algorithm even under this form with these extra variables. The authors suggested that these variables could be used for several modeling purposes (stockpiles, variable capacities, etc.). The idea proposed in this paper is to use the q variables to model the deviations from production targets, and thus, extend it to the stochastic formulation.

Recall that the BZ algorithm efficiency lies in three main ideas. First, the algorithm takes advantage of the problem structure once the side constraints $My + Hq \leq d$ are relaxed. Indeed, the resulted problem is the well-known maximum closure problem which can be solved quite efficiently using appropriate algorithms such as the Pseudoflow Algorithm proposed by Hochbaum (2008). Second, for this class of problem, the integrality gap remains small in practice. Thus, the optimal solution of the linear programming relaxation (LP) obtained by the BZ algorithm represents a tight upper bound for the integer problem all the more so that traditional methods often fail to solve even the linear relaxations of real size instances. Third, unlike traditional Lagrangian relaxation methods that require many iterations and converge very slowly with no satisfying accuracy, the BZ approach uses the dual information to reduce efficiently the size of the LP. In fact, Bienstock and Zuckerberg (2010) proved that the optimal solution of any instance of GPCP attains no more than $l+2$ distinct values with l being the number of linearly independent rows in the matrix D , this number remains relatively small for this class of problem. Thus, in each iteration k , the original feasible region of the problem can be restricted to the linear space spanned by a generator matrix G^k that forces variables corresponding to the non-zero entries of each column to be equal. This resembles an aggregation operation in the sense that it leads to a significant reduction of variables, but it differs in preserving the individual properties of the blocks. G^k is updated at each iteration after solving the maximum closure problem to generate G^{k+1} , this procedure is called "refining". This procedure and the BZ extended version algorithm are formally described in Algorithm 1.

Algorithm 1 BZ extended version algorithm

Input: a feasible linear programming problem of the form:

$$(P_1) \quad \begin{cases} \max & p^t y + c^t q \\ \text{s.t.} & Ay \leq b \\ & My + Hq \leq d \\ & 0 \leq y \leq 1 \end{cases}$$

where constraints $Ay \leq b$ represent the precedence constraints and $My + Hq \leq d$ the side constraints.

Next, S_k will design a partition of the nodes $\mathcal{N}$ at iteration k where $|S_k|$ is the number of sets that make up the partition S_k . Each element S_k^h of the partition is a set of nodes where $1 \leq h \leq |S_k|$.

1. $\lambda_0 \leftarrow 0$, $S_0^1 \leftarrow \mathcal{N}$ and $S_0 \leftarrow \{S_0^1\}$, $k \leftarrow 1$
2. Solve the problem $L(P_1, \lambda_{k-1})$ which is the Lagrangian relaxation of P_1 with multipliers equal to λ_{k-1} and having the form:

$$L(P_1, \lambda_{k-1}) \quad \begin{cases} \max & p^t y + c^t q + \lambda_{k-1}(d - (My + Hq)) \\ \text{s.t.} & Ay \leq b \\ & 0 \leq y \leq 1 \end{cases}$$

Note that the variables q are free here. Thus they can be ignored, and the problem can be treated as a classic maximum flow problem.

3. Let y^k be the optimal solution of $L(P_1, \lambda_{k-1})$. Define the two sets: $I^k = \{i \in \mathcal{N} \mid y_i^k = 1\}$ and $O^k = \{i \in \mathcal{N} \mid y_i^k = 0\}$
 4. Define the new partition S_k which is the union of all nonempty sets in $\{S_{k-1}^h \cap I^k \mid 1 \leq h \leq |S_{k-1}|\} \cup \{S_{k-1}^h \cap O^k \mid 1 \leq h \leq |S_{k-1}|\}$. If $k > 1$ and the partition didn't change, **STOP** and return x^{k-1} as optimal solution of P_1 .
 5. Solve the problem P_2^k which is P_1 plus the additional clustering constraints $y_i = y_j$ for each pair (i, j) in S_k^h for $1 \leq h \leq |S_k|$.
 6. Let x^k the optimal solution of P_2^k , and λ_k the optimal dual variables corresponding to the side constraints $Dy + Hq \leq d$. If the dual variables λ_k didn't change **STOP** and return x^k as optimal solution of P_1 .
 7. $k \leftarrow k + 1$ and go to step 2 after applying *coarsification** if needed.
-

*** Remark:** to avoid an intractable growth in the size of the matrix, the coarsification of S_k is applied by merging the sets S_k^h with common values in x^k only when the size of S_k exceeds $l+2$ (where l is the number of linearly independent rows in D).

3.2 Rounding heuristic

The rounding heuristic, as its name implies, makes the optimal solution of the LP integer. It is an adaptation of the TopoSort Heuristic proposed in Chicoisne et al. (2012). Some changes are introduced to handle the stochastic aspect and discussed later in this section. This heuristic takes as input a topological ordering of the blocks. This ordering defines a feasible extraction sequence in the sense that it satisfies the slope constraints. This way, a block appears in the sequence after all its predecessors. Because there are many possible topological orderings, associating a “weight” to each block leads to finer orderings. Thus, blocks with lower weights will be scheduled earlier in the sequence. In this paper, the same weight function proposed by Chicoisne et al. (2012) is retained. It uses the LP solution. For each $b \in \mathcal{B}$, the weight is calculated as follows:

$$w_b = x_{b,D,1} + \sum_{t=2}^T t \times (x_{b,D,t} - x_{b,D,t-1}) + (T+1) \times (1 - x_{b,D,T}).$$

The weight w_b can be interpreted as the probability that block b will be extracted in time t . Blocks not extracted by time T will be assumed to be extracted in time $T+1$. Then, let $G = (\mathcal{B}, \mathcal{A})$ be the graph defined in Section 2. Given this graph and a weight vector $\mathbf{w}$, the blocks are classified according to the topological sorting to obtain a feasible extraction sequence $\{b_1, b_2, \dots, b_n\}$ where b_i represents a block member. This procedure is described in Algorithm 2. To summarize, a block b_i will appear before b_j in the sequence, if it satisfies either 1) b_i is a predecessor of b_j or 2) $w_i < w_j$.

Algorithm 2 TopoSorting (Tsort($G, \mathbf{w}$))

Inputs: An acyclic directed graph $G = (\mathcal{B}, \mathcal{A})$, $\mathcal{N}$ a set of nodes, a weight vector $\mathbf{w}$, and $\forall b \in \mathcal{N}$, a set Ω_b^- is defined by $\Omega_b^- = \{j \in \mathcal{N} \mid (b, j) \in \mathcal{A}\}$:

$i \leftarrow 1, \mathcal{N} \leftarrow \mathcal{B}, n \leftarrow |\mathcal{N}|$

while $i < n$ **do**

$b \leftarrow \underset{b \in \mathcal{N}}{\operatorname{argmax}} \{w_b \mid \Omega_b^- = \emptyset\}$

$\mathcal{N} \leftarrow \mathcal{N} \setminus \{b\}$

$b_i \leftarrow b$ and $i \leftarrow i + 1$

end while

return topological ordering $b_1, b_2, \dots, b_n$ in G that is weighted with respect to $\mathbf{w}$.

Once the topological ordering of the blocks is obtained, a rounding heuristic (RH) is applied. Unlike the heuristic proposed in Chicoisne et al. (2012) where the model is deterministic, and there are only hard constraints that represent upper bounds to satisfy, the stochastic model allows the violation of some constraints by penalizing the objective function and makes them soft. Also, the model contains both upper and lower bounds for each target. To address these differences, instead of regarding the remaining resources, to decide whether or not a block can be mined at a specific period and sent to one particular destination, the solution value after scheduling this block is calculated. If it doesn't lead to a better solution, the extraction is refused at this specific period and destination, and other destinations and later periods are checked. Only the best move is then selected. Furthermore, the heuristic takes advantages of the LP solution by fixing the blocks that were entirely extracted in the same period. These blocks are sent to the destinations for which the fractional values are the highest. Only the extraction periods and the destinations of the remaining blocks (i.e., those that have not been extracted or not been completely extracted in one single period) can be changed.

More specifically, to apply this heuristic, first, the blocks that should be fixed (conditions referred above) are identified then, the fractional solution of the LP is simply rounded, and the new solution value is computed. Next, based on the topological sorting, all the blocks, except for those that have been fixed, are moved (i.e. advanced, postponed, or simply have their destinations changed) one by one if it leads to a better solution. A

block can be scheduled at any period and any destination as long as that occurs between the earliest period of extraction (after all its predecessors) and the lastest one (before all its successors). The move that leads to the best solution is always selected. The heuristic ends when all the blocks are scheduled and their times of extraction $\{t_1, t_2 \dots t_{|\mathcal{B}|}\}$ are defined. This procedure is summarized in Algorithm 3.

Algorithm 3 RH (Rounding Heuristic)

Inputs: An acyclic directed graph $G = (\mathcal{B}, \mathcal{A})$, a weight w_b for each block $b \in \mathcal{B}$, T periods, R resources, X an initial integer solution which is the result of rounding the LP solution with $X \oplus (b, t, d)$ a new solution obtained from X by moving the extraction of block b to period t and to destination d , $\mathcal{F} = \{b \in \mathcal{B} \mid \exists t \leq T, x_{bDt} - x_{bDt-1} = 1\}$ the set of blocks that are considered fixed, and finally $\{t_1, t_2 \dots t_{|\mathcal{B}|}\}$ are initialized to the extraction time determined by X .

```

 $i \leftarrow 1$ 
 $\{b_1, \dots, b_n\} \leftarrow \text{Tsort}(G, w)$ 
while  $i < n$  do
    Let  $u = b_i$  and let  $i \leftarrow i + 1$ 
    if  $u \notin \mathcal{F}$  then
         $t_{\text{earliest}} \leftarrow \max \{t_j \mid j \in \Gamma_u^-\}$ 
         $t_{\text{latest}} \leftarrow \min \{t_j \mid j \in \Gamma_u^+\}$ 
         $t_u \leftarrow \text{argmax} \{ \text{value}(X \oplus (u, t, d)) \mid t_{\text{earliest}} \leq t \leq t_{\text{latest}} \}$ 
         $X \leftarrow X \oplus (u, t, d)$ 
    end if
end while
return  $t_b \forall b \in \mathcal{B}$ 

```

3.3 Tabu search

A Tabu Search is then applied considering successively periods $p = 1, \dots, T$ to improve the solution obtained by the Rounding Heuristic. At iteration p , blocks scheduled in earlier periods are considered fixed, and only the remaining blocks can be moved. This approach uses the cost structure in the mine planning problems by preferring the extraction of a profitable block as soon as possible, rather than delaying its extraction at future periods. Next, this heuristic is described more specifically.

At the p^{th} iteration, let S^p be the set of blocks $b \in \mathcal{B}$ scheduled in periods $t_b \leq p$ (i.e. $b \in S^p \Leftrightarrow \sum_{d=1}^D y_{bdp} = 1$) defined by the current solution X^p , and let $\overline{S}^{p-1}$ be the set of blocks $b \in \mathcal{B}$ scheduled and fixed in periods $t_b \leq p-1$ and obtained after applying the Tabu Search the $p-1$ previous iterations. Also, the two sets B_p^- and B_p^+ are defined as follows: $B_p^- = \{i \in \mathcal{B} \mid t_i = p \text{ and } j \notin S^p \ \forall j \in \Gamma_i^+\}$ and $B_p^+ = \{i \in \mathcal{B} \mid t_i = p+1 \text{ and } j \in S^p \ \forall j \in \Gamma_i^-\}$

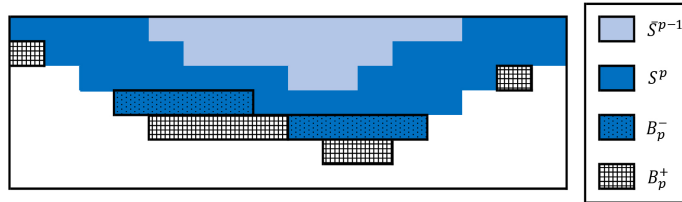


Figure 3: Current solution and corresponding neighborhood illustrated by B_p^- (blocks can be removed from S^p) and B_p^+ (block can be added to S^p)

The first set represents the blocks located on the “inner” border of S^p . They can be removed from S^p (i.e., their extraction can be delayed to a later period or cancelled). The second set represents the blocks located on the “outer” border of S^p . They can be added to S^p (i.e., pushing forward their extraction or including new ones if they weren’t scheduled in the current solution) without violating the precedence constraints. Figure 3 schematizes these two sets B_p^- and B_p^+ .

The neighborhood of the solution X^p is represented by all the solutions generated either by removing a block from B_p^- or by adding a block from B_p^+ . Algorithm 4 shows how to find all these neighbors given an initial solution X and a period p . In this algorithm, $X \oplus (i, t, d)$ represents the solution obtained by moving a block i from t_i to t and from d_i to d .

Algorithm 4 $\text{getNeighbors}(X, p)$: building the neighborhood of a given solution**Inputs:** A feasible solution X , a period p $\text{neighbors} \leftarrow \emptyset$ **for** (block i in $\mathcal{B}$) **do** $\text{test} \leftarrow \text{true}$ **if** ($t_i = p$) **then** $t_{\min} \leftarrow \min \{t_j \mid j \in \Gamma_i^+\}$ **if** ($t_{\min} > t_i$) **then****for** ($d = 1 \dots D$) **do** $\text{neighbors} \leftarrow \text{neighbors} \cup (X \oplus (i, p+1, d))$ $\triangleright$ (i.e. postpone the extraction to $p+1$)**end for****end if****end if****if** ($t_i = p+1$) **then** $t_{\max} \leftarrow \max \{t_j \mid j \in \Gamma_i^-\}$ **if** ($t_{\max} \leq p$) **then****for** ($d = 1 \dots D$) **do** $\text{neighbors} \leftarrow \text{neighbors} \cup (X \oplus (i, p, d))$ $\triangleright$ (i.e. advancing the extraction to period p)**end for****end if****end if****end for****return** neighbors

Given the neighbors, the best solution among them is selected even if it leads to a worsening move and it is automatically added to what it is denoted as Tabu-list. This list regroups all the temporary prohibited moves. Thus, cycling and coming back to previously-visited solutions are avoided. If the Tabu-list reaches its maximum size maxTabuSize , some elements are no longer considered Tabu following the rule of FIFO (First In First Out). However, exceptionally, a Tabu move can be accepted if it generates a better solution than the best feasible solution known so far. The algorithm will continue searching for an optimal solution until nitermax successive non-improving iterations are reached. The best solution found so far is then returned and used as an initial solution for the next period. Algorithm 5 summarizes how the Tabu Search is applied. Note that the value of a solution X will be noted Z_X .

Algorithm 5 $\text{TabuSearch}(X_0, p)$: Improving the initial solution X_0 considering all blocks scheduled before p as fixed**Inputs:** A feasible solution X_0 , a period p , nitermax the maximum number of iterations without improving the solution, and maxTabuSize the size of the Tabu list. $s\text{Best} \leftarrow X_0, \text{bestCandidate} \leftarrow X_0, \text{tabuList} \leftarrow \emptyset, \text{niter} \leftarrow 0$ **while** ($\text{niter} < \text{nitermax}$) **do** $s\text{Neighborhood} \leftarrow \text{getNeighbors}(\text{bestCandidate}, p)$ $\text{bestCandidate} \leftarrow s\text{Neighborhood}[1]$ **for** ($s\text{Candidate}$ in $s\text{Neighborhood}$) **do****if** ($((s\text{Candidate} \notin \text{tabuList}) \text{ and } (Z_{s\text{Candidate}} > Z_{\text{bestCandidate}})) \text{ or } ((s\text{Candidate} \in \text{tabuList}) \text{ and } (Z_{s\text{Candidate}} > Z_{\text{Best}})))$) **then** $\text{bestCandidate} \leftarrow s\text{Candidate}$ **end if****end for****if** $Z_{\text{bestCandidate}} > Z_{s\text{Best}}$ **then** $s\text{Best} \leftarrow \text{bestCandidate}$ $\text{niter} \leftarrow 0$ **end if** $\text{tabuList} \leftarrow \text{tabuList} \cup \text{bestCandidate}$ **if** ($|\text{tabuList}| > \text{maxTabuSize}$) **then** $\text{tabuList} \leftarrow \text{tabuList} \setminus \text{tabuList}[1]$ $\triangleright$ the first element of the list was considered tabu for maxTabuSize iterations**end if** $\text{niter} \leftarrow \text{niter} + 1$ **end while****return** $s\text{Best}$

4 Numerical results

The numerical tests presented in this section have two objectives: (a) compare the performance of the extended version of the BZ algorithm to Cplex linear programming algorithms, (b) discuss the quality of the solutions provided by the two heuristics proposed herein and (c) evaluate the results in terms of risk profiles of produced grades per period.

4.1 The case study

In this paper, the algorithm is applied to a two-product iron ore deposit in West Australia. The main objective of this deposit is to meet, at lowest costs, the customer requirements in terms of quality and quantity of the different geochemical elements for the two products the mine provides. The critical elements considered here are iron content (Fe), silica content (SiO₂), alumina content (Al₂O₃), phosphorus content (P), and the loss on ignition (LOI). Table 1 summarizes the ore tonnage and grade limits for the two products over the lifetime of the mine.

Table 1: Ore tonnage and grade constraints

Production requirements for iron ore deposit over 5 years						
One mine with 2 products						
Product 1						
Year	Ore Tonnage (wt)	Fe (%)	P (%)	SiO ₂ (%)	Al ₂ O ₃ (%)	LOI (%)
1	6 000 000	56.0-59.2	0.021-0.038	4.1-5.20	0.9-1.03	9.5-11.0
2	4 000 000	56.0-59.2	0.021-0.038	4.1-5.20	0.9-1.03	9.5-11.0
3	4 000 000	56.0-59.2	0.021-0.038	4.1-5.20	0.9-1.03	9.5-11.0
4	5 300 000	56.0-59.2	0.021-0.038	4.1-5.20	0.9-1.03	9.5-11.0
5	2 500 000	56.0-59.2	0.021-0.038	4.1-5.20	0.9-1.03	9.5-11.0
Product 2						
Year	Ore Tonnage (wt)	Fe (%)	P (%)	SiO ₂ (%)	Al ₂ O ₃ (%)	LOI (%)
1	8 000 000	57.7-60.0	0.032-0.038	4.2-5.2	0.88-1.05	9.5-11.0
2	6 000 000	57.7-60.0	0.032-0.038	4.2-5.2	0.88-1.05	9.5-11.0
3	6 000 000	57.7-60.0	0.032-0.038	4.2-5.2	0.88-1.05	9.5-11.0
4	3 300 000	57.7-60.0	0.032-0.038	4.2-5.2	0.88-1.05	9.5-11.0
5	4 200 000	57.7-60.0	0.032-0.038	4.2-5.2	0.88-1.05	9.5-11.0

The economic parameters including unit costs, unit revenues and discount factor are summarized in Table 2. Penalties for deviating from the production targets are also included in Table 2. Several combinations were tested to determine the penalty costs presented here. The idea was to find a compromise between a very strict selectivity imposed by high penalties and a more permissive one that would affect the products' homogeneity. To depict the *in-situ* variability and uncertainty of these parameters in the scheduling, 10 stochastically simulated orebody models are considered. They are obtained from the computationally joint direct block simulation approach proposed by Boucher and Dimitrakopoulos (2012). All the simulated models have 33,168 blocks. Each block contains the total attributes tonnage, ore tonnage as well as the total content of each element Fe, P, SiO₂, Al₂O₃, and LOI. Furthermore. Also, no Fe cut-off grade is fixed, the solver is free to consider a block as ore or waste.

4.2 Implementation and results

All the algorithms were developed in the C++ programming language, and all the computational tests were carried out on an Intel Core i5 computer (2.3 GHz) with 8.00 GB of RAM running under Windows 10. The extended version of the BZ algorithm, the Rounding Heuristic, and the Tabu Search will be referred as ExBZ, RH and TS, respectively. Before presenting the results, the different parameter values are first explained.

ExBZ. To solve maximum-flow subproblems, the pseudoflow algorithm presented in Hochbaum (2008) is used, and the LP is solved using Cplex with the default settings.

Table 2: Economic parameters for long-term production scheduling

	Product 1	Product 2
Price of recovered metal	26\$/t	30\$/t
Processing costs	5\$/t	6\$/t
Mining costs		5\$/t
Waste costs		5\$/t
Economic discount rate		10 %
Penalty for Fe deviations		10\$/t
Penalty for SiO ₂ deviations		10\$/t
Penalty for P deviations		10\$/t
Penalty for Al ₂ O ₃ deviations		10\$/t
Penalty for LOI deviations		1\$/t
Penalty for Ore target deviations		25\$/t

TS. For the tabu search, there are two major parameters. They were defined based on preliminary tests:

- The maximum size of the TabuList or in other terms, the number of iterations during which a move remains tabu or forbidden is fixed at 1000 iterations which is, in average, nearly equal to $0.6N$ where N is the number of blocks that can be moved at each iteration or, in other words, the size of the neighborhood at each iteration.
- Stop criterion: Maximum number of successive non-improving iterations: $0.3N$.

In what follows, the objective function values will always be divided by the upper bound obtained with Cplex. This allows having an idea about the quality of the solutions and their proximity to the optimal value of the linear relaxation. Table 3 shows the running times and the objective function values using Cplex, ExBZ and the heuristics RH and TS. ExBZ manages to solve the LP relaxation within 44 minutes while Cplex takes more than 4 hours to do it. The time required to run the RH is negligible. However, the value obtained with this heuristic is relatively poor (within 15–20% of optimality). The table also shows that TS does a very good job in terms of improving the solution obtained by RH. After running for 17 minutes, the solution value goes from 20% to less than 1,2% of optimality.

Table 3: Running time and solution quality using Cplex, ExBZ, RH, TS

	Cplex	ExBZ	RH	TS
Time	4h20m	44m	< 10s	17m
Gap %	0	0	15–20	< 1, 2

4.3 Risk profiles

The results are now evaluated regarding risk profiles of produced grades for the critical elements Fe, SiO₂, Al₂O₃, P , and LOI. Figure 4 shows the grades for each period, each possible scenario for both products. A point outside the bounds reflects a deviation from production targets.

The risk profiles of Fe, P and LOI content lead to the conclusion that there is no risk of deviating from production targets. However, it is more critical with SiO₂ and Al₂O₃ that failed to meet the targets for at least one period and one simulation. The deviations of SiO₂ remain very limited. The most important deviation is observed at Al₂O₃ grades of Product 1 with a chance of 30% at period 1 and 50% at period 5. Higher penalties were tested to decrease this risk, but that improved the result only marginally. A high in-situ variability and uncertainty of the element can explain it, and blending cannot bypass this issue.

5 Conclusions

In this paper, a method based on a stochastic adaptation of the BZ algorithm, a rounding heuristic and a Tabu search has been proposed to solve a real case of an open-pit mine production scheduling problem with multi-destination and under multi-element geological uncertainty. The first heuristic used aims to round the fractional solution obtained by the BZ algorithm adaptation. It is a greedy heuristic based on topological

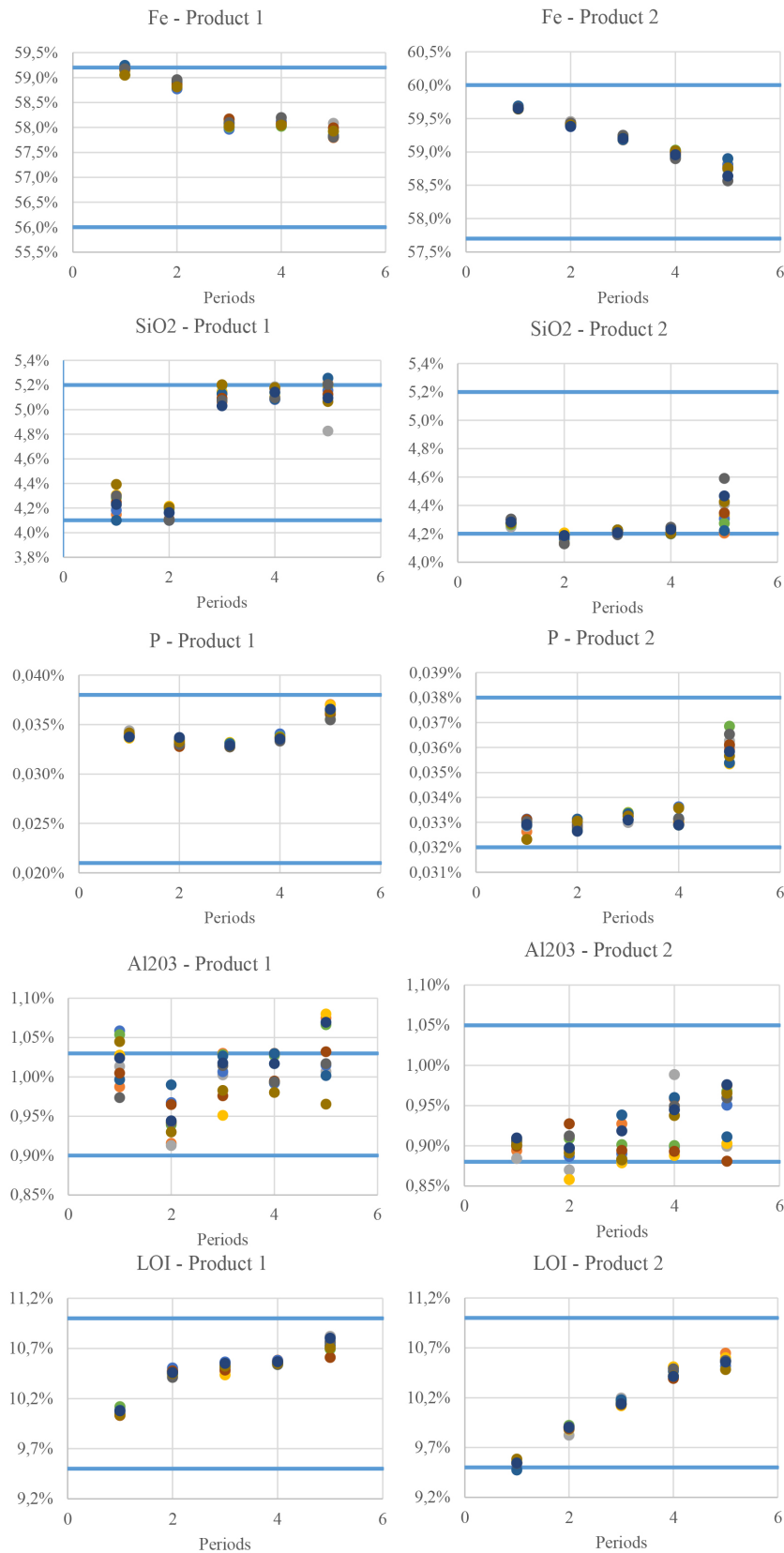


Figure 4: Results of stochastic scheduling in terms of risk profiles for Fe, SiO₂, Al₂O₃, P, and LOI

sorting techniques. The second one is a Tabu search that improves the quality of the solution. Results show that the method performs very well when compared to the upper bound provided by Cplex. Indeed, the gap between the solution obtained and the upper bound is smaller than 1.2%. Also, the running time is reasonable ($\cong 1$ hour) and even much lower than that obtained by Cplex to solve only the linear relaxation of the problem. Also, results demonstrate how the stochastic approach leads to control the risk of deviating from production targets.

A natural next step would consist of testing the method on other instances to verify its robustness. An additional natural improvement includes finding finer decision-making criteria to accept or reject a block in the rounding heuristic and, thus, have better solutions to provide the Tabu search with. Another interesting work will consist of adding mid-short-term constraints for consideration.

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