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# Brand imitation: A dynamic-game approach

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**Abstract:** Brand imitation is a common practice that can take different forms, i.e., legal copying, as in the case of clones and knockoffs, or illegal, in the case of counterfeiting. We consider a scenario in which a producer enters the market with a “similar”product to the incumbent’s and we assess the impact of this entry on the incumbent’s strategies and outcomes. A distinctive feature of our model is that it allows for brand dilution, which means that the original brand suffers due to imitation, and for brand enhancement, when the availability of the imitation product actually promotes the original brand.

We characterize and contrast the solutions for the scenario with entry and the benchmark case where no entry occurs, in a fully dynamic context.

**Keywords:** Brand imitation, counterfeiting, pricing strategy, advertising strategy, differential games

**Résumé :** L’imitation d’une marque connue est une pratique courante et peut être légale ou illégale (dans le cas de la contrefaçon). On considère un scénario où une firme entre dans le marché en offrant un produit similaire à celui offert par la firme existante et on évalue l’impact de cette entrée sur les stratégies et gains de cette firme. Un aspect distinctif de ce travail est l’analyse des deux cas, à savoir, la dilution de marque et la diffusion de marque.

On caractérise et contraste les résultats obtenus avec et sans entrée dans un contexte dynamique.

**Mots clés :** Imitation de marque, contrefaçon, stratégie de prix, stratégie de publicité, jeux différentiels

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# 1 Introduction

Introducing a product that copies certain attributes (e.g., logo, design) of a well-established product is common practice. For instance, some private labels launched by retailers in the last three decades or so have had sometimes disturbing similarities to manufacturers' brands. In doing so, retailers want to signal that their products are "the same"as national brands while being affordable.. In the apparel industry, "fast-fashion" producers are able to replicate original styles and designs at high speed, on a large scale, and at low cost (Hemphill & Suk (2009)), and consequently, they sell them at a lower price than the original.

The objective of this paper is to assess the impact of a third party offering a "similar"product on the incumbent's strategies and outcomes. To clearly position our contribution from the outset, we highlight the following characteristics and assumptions of our model:

1. The product of interest is perishable. It is introduced by an incumbent at an initial time 0 and becomes obsolete at time $T$. To give a concrete idea, think of the planning horizon $[0, T]$ as a selling season for a fashion apparel.
2. An entrant has the know-how to copy the incumbent's product and enters the market at an exogenously given date $\epsilon \in (0, T]$. Consequently, we have a monopoly during the time interval $[0, \epsilon)$, and a duopoly during $(\epsilon, T]$, with the two firms offering partially substitutable products.
3. The incumbent invests in advertising to raise its brand's reputation, a variable that influences the current and future market size, and consequently, its demand and profit over time.
4. The entrant does not engage in any advertising activities, and its demand is determined by its price and the incumbent's brand reputation. This assumption is meant to highlight the idea that purchasing, e.g., imitation Nike running shoes is driven by Nike's reputation or goodwill, and of course, by the price of the two competing products.
5. After entry, the evolution of the brand reputation will not only depend on the incumbent's investment in advertising, but also be influenced by the imitator's sales. More specifically, we shall consider two cases: (i) a negative effect (brand dilution) and (ii) a positive effect (brand or reputation enhancement). Brand dilution refers to the loss of reputation by a prominent brand and the devaluation of its exclusive features due to their use by a third party (here, the entrant). In the other case, the producer of the original brand gains in popularity (not necessarily directly through sales) thanks to the unintended free advertising done by the imitator. To illustrate, if one sees a "BlueBerry"or a "BleckBarry"mobile phone, one obviously immediately thinks of "BlackBerry". A T-shirt displaying the slogan "Naik: Just do it" is clearly providing free advertising for Nike.[1] Another example is "Hamossy" liquor offered in some outlets in China whose packaging clones Hennesy, the well-known brand of French cognac. Here, it is the Hennessy's goodwill that is used to sell the alternative brand. These examples can be classified as knockoffs or clones, which are imitation products that are very similar to the original ones, but still use their own brand names. Another case of brand imitation is counterfeiting, which is an illegal activity. Cordell et al. (1996) define counterfeiting as "any unauthorized manufacturing of goods whose special characteristics are protected as intellectual property rights (trademarks, patents and copyrights) constitutes product counterfeiting."Counterfeiters attempt to make products that are, at least at first glance, indistinguishable from the famous genuine ones; otherwise, they would be less attractive to consumers.

Our research questions are as follows:

1. How does the imitator's entry affect the incumbent's strategies?
2. Can the incumbent benefit from being copied?

To answer these questions, we adopt a finite-horizon dynamic game model. We believe this is the only framework that enables us to simultaneously account for the following features: (i) the sequential entry in the market of the two competitors; (ii) the fact that brand reputation can only be built over time and not

[1] Other examples are pictured on https://fr.slideshare.net/sidvinshetty/brand-imitation-15717503, last visited on November 12, 2017.

overnight; (iii) the obsolescence of the product at the end of the selling season; and finally, (iv) the strategic interactions between the incumbent and the entrant.

Despite the prevalence of counterfeiting activities,[2] the literature looking at the strategic interactions between legal firms and pirates is sparse, see, e.g., Banerjee (2003, 2013), Qian (2012), Qian et al. (2014), and Zhang et al. (2012). A central issue in these papers, as in this one, is the impact of counterfeiting on brand reputation and quality; however, their analysis is conducted using a two-period setup (or a static setting).

Our paper follows on Buratto et al. (2016) and Crettez et al. (2017), which are, to the best of our knowledge, the only papers dealing with counterfeiting in a fully dynamic context. We share the following features with these two papers: (i) The differential game is played in two stages, monopoly and duopoly, with the entry date of the imitator being exogenously given. (ii) During the duopoly period, we seek a Nash equilibrium. Here and in Crettez et al., the information structure and the equilibrium are in feedback, whereas they are in open loop in Buratto et al. (iii) The incumbent chooses advertising and price levels, and the entrant chooses its price. In Buratto et al., it is additionally assumed that the entrant advertises the brand. The major, significant difference between this paper and the other two is that here the evolution of the incumbent's brand reputation also depends on the entrant's sales during the duopoly period. This formulation, which allows us to capture brand dilution and reputation boosting, renders the dynamics, and hence the whole decision process, dependent on price. As we will see, the change in the dynamics specification introduced here affects the results considerably. Further, a noteworthy (technical) difference is that our optimization problems are not concave, and therefore, we could not resort to classical existence and uniqueness results, but had to develop new ones. We believe that this is a relevant contribution to the applied differential games literature in general.

Our main results can be summarized as follows. First, counterfeiting also affects the legal firm's strategies before entry occurs and has aftereffects that extend beyond the selling season. Second, when the entrant's sales cause brand dilution, the incumbent can never benefit from being imitated, a result that is different from the case where counterfeiting gives a boost to the brand's goodwill. Note that for the legal firm to benefit from imitation, some conditions must hold, meaning that this result does not hold true for all parametervalues.

The rest of the paper is organized as follows: In Section 2, we introduce the model, and in Section 3, we characterize the optimal and equilibrium solutions. A profit comparison is conducted in Section 4. In Section 5, we compare the incumbent's before-entry strategies with and without counterfeiting. Section 6 briefly concludes.

## 2 Model

We consider a selling season defined by the time interval $[0, T]$. The initial date corresponds to the introduction of a new product by an incumbent (player $I$), and $T$ to the date after which the product becomes obsolete because of, e.g., a change of season for fashion apparel, or the arrival of a new version for software. At an intermediate date $\epsilon \in (0, T]$, an entrant (player $E$) enters the market and offers an imitation of the incumbent's product (or brand). Denote by $p_I(t)$ the price of the incumbent's product at time $t \in [0, T]$, and by $p_E(t)$ the price charged by the entrant for its product at $t \in [\epsilon, T]$.

**Remark 1** *As our framework does not depend on whether or not the imitation is legal, we can also refer to the incumbent as the innovator, brand owner, or legal firm, and to the entrant as an imitator, counterfeiter, or competitor.*

Denote by $B(t)$ the incumbent's brand reputation (goodwill or brand equity). In the absence of brand imitation, the demand for the incumbent's brand is given by

$$d_I(t) = \tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t), \qquad t \in [0, T], \tag{1}$$

[2] Indeed, this is a multi-billion-dollar industry. According to Levin (2009), counterfeit goods account for more than \$0.5 trillion each year, and counterfeiting causes a loss of \$200 billion in revenues annually in the US. Also, approximately 2.5 million jobs are lost worldwide due to counterfeiting (Eisend and Schichert-Guler (2006)).

where $\tilde{\alpha}_I$ and $\tilde{\beta}_I$ are strictly positive parameters, and, in the scenario with brand imitation, by

$$d_I(t) = \begin{cases} \tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t), & t \in [0,\epsilon), \\ \alpha_I B(t) - \beta_I p_I(t) + \gamma p_E(t), & t \in [\epsilon, T], \end{cases} \tag{2}$$

$$d_E(t) = \alpha_E B(t) - \beta_E p_E(t) + \gamma p_I(t), \quad t \in [\epsilon, T], \tag{3}$$

where $\alpha_j$ and $\beta_j$ are strictly positive parameters for $j \in \{I, E\} > 0$ and $\gamma \geq 0$ with $\beta_j > \gamma$, that is, each demand is decreasing in own price and increasing in the competitor's price, and the direct-price effect is greater than the cross-price effect.

We note that the demands for the incumbent's product are structurally different with and without brand imitation, that is, $\tilde{\alpha}_I \neq \alpha_I$ and $\tilde{\beta}_I \neq \beta_I$, with $\tilde{\alpha}_I > \alpha_I$ and $\tilde{\beta}_I < \beta_I$. Put differently, setting $p_E(t) = 0$ in the demand in the duopoly period does not yield the demand in the monopoly period. The demand system in (1), (2)–(3) shares some similarities with the one in Crettez et al. (2017): specifically, that the market size depends on the incumbent's brand reputation and that these demand functions are derivable from the maximization of a representative consumer's quadratic utility function. And finally, that the market size for both players is proportional to the incumbent's brand reputation. As alluded to in the introduction, this formulation is meant to highlight the fact that consumer demand for the imitation brand is driven by the reputation of the original/recognized brand.

Although the characterization of the solutions does not require additional assumptions on the demand parameter values, it seems intuitively reasonable to assume that (i) the owner of the brand benefits more from its brand reputation than does the imitator, i.e., $\alpha_I > \alpha_E$; (ii) consumers of the clone brand are more sensitive to price than consumers of the original brand, i.e., $\beta_E > \beta_I$.

The manufacturer can increase the brand reputation by investing in advertising. After the possible entry of the illegal producer, the brand's goodwill will depend on player $I$'s advertising effort and on the demand for the entrant's product. The evolution of brand's reputation is described by the following differential equations:

$$\dot{B}(t) = \begin{cases} ka(t) - \sigma B(t), & t \in [0,\epsilon), \\ ka(t) + \varphi d_E(t) - \sigma B(t), & t \in [\epsilon, T], \end{cases}, \quad B(0) = B_0 > 0, \tag{4}$$

where $a(t)$ is the advertising effort by the incumbent at time $t$, where $k > 0$ is an efficiency (or scaling) parameter and $\varphi$ is a parameter capturing the impact of the entrant's demand on the incumbent's brand reputation, and where $\sigma$ is the decay rate. A positive value of $\varphi$ means that product's reputation benefits from greater availability. Our assumption here is that this availability is actually free advertising for the incumbent's brand. On the other hand, a negative value of $\varphi$ means that the reputation of the original product suffers from imitation, a phenomenon that is referred to as brand dilution.

Following a large literature in differential games (see, e.g., the book by Jørgensen and Zaccour (2004) and the surveys by Huang et al. (2012) and byJørgensen and Zaccour (2014)), we assume that the advertising cost is convex increasing and given by the quadratic function

$$C_I(a) = \frac{\omega}{2} a^2(t),$$

where $\omega$ is a positive parameter.

Setting the unit production cost equal to zero and assuming away discounting,[3] the incumbent's profit optimization program is as follows:

[3]In our finite-horizon game, adding costs and discounting will only have a quantitative impact on the results without altering the qualitative insights.

$$\begin{aligned}\max_{p_I(t),\, a(t)} \Pi_I &= \left[\int_0^{\epsilon} \left(p_I(t)\left(\tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t)\right) - \frac{\omega}{2}a^2(t)\right) dt \right.\\ &\quad \left. + \int_{\epsilon}^{T} \left(p_I(t)\left(\alpha_I B(t) - \beta_I p_I(t) + \gamma p_E(t)\right) - \frac{\omega}{2}a^2(t)\right) dt\right] \\ &\quad + sB(T), \\ &\quad \text{subject to (4)},\end{aligned} \tag{5}$$

where $s$ is a nonnegative parameter, and $sB(T)$ is the salvage value of the brand at $T$, which approximates the potential of future profits that the incumbent can obtain from selling other products under the same brand name.

The entrant's optimization problem is defined as follows:

$$\max_{p_E(t)} \Pi_E = \max_{p_E(t)} \int_{\epsilon}^{T} p_E(t)\left(\alpha_E B(t) - \beta_E p_E(t) + \gamma p_I(t)\right) dt. \tag{6}$$

We assume that the entrant is myopic, that is, it optimizes its stream of profits ignoring the reputation dynamics. Such myopic behavior, which has been assumed in number of contributions in differential games (see, e.g., Benchekroun et al. (2009) and Martín-Herrán et al. (2012)), reflects the idea that the entrant does not care about the incumbent's asset and is only interested in maximizing its current payoff. In such a case, optimizing the above functional, while ignoring the state dynamics, is equivalent to solving the following static problem:

$$\max_{p_E(t)} \pi_E = \max_{p_E(t)} p_E(t)\left(\alpha_E B(t) - \beta_E p_E(t) + \gamma p_I(t)\right), \quad \forall t \in [\epsilon, T].$$

To assess the impact of brand imitation on the incumbent's strategies and outcome, we consider as a benchmark the scenario where no such entry takes place. Here, player $I$ solves the following standard finite-horizon optimal-control problem:

$$\begin{aligned}\max_{p_I(t), a(t)} \Pi_I &= \max_{p_I(t), a(t)} \int_0^{T} \left(p_I(t)\left(\tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t)\right) - \frac{\omega}{2}a^2(t)\right) dt + sB(T), \\ \dot{B}(t) &= ka(t) - \sigma B(t),\ B(0) = B_0.\end{aligned} \tag{7}$$

We shall superscript the optimal price, advertising, and profit values with $M$ (for monopoly).

In the scenario where the imitator enters at time $\epsilon < T$, the two firms play a finite-horizon differential game on the time interval $[\epsilon, T]$, with the incumbent maximizing

$$\begin{aligned}\Pi_{I2} &= \int_{\epsilon}^{T} \left(p_I(t)\left(\alpha_I B(t) - \beta_I p_I(t) + \gamma p_E(t)\right) - \frac{\omega}{2}a^2(t)\right) dt + sB(T), \\ &\quad \text{subject to (4) and } B(\epsilon) \text{ given},\end{aligned}$$

and the entrant's maximization problem being given by (6). A Nash equilibrium will be sought and the equilibrium solution will be superscripted with $D$ (for duopoly). Starting from position $(\epsilon, B(\epsilon))$, the incumbent's Nash equilibrium outcome is given by the value function $W_I(\epsilon, B(\epsilon))$, which plays the role of a salvage value in its first-period optimization problem given by

$$\Pi_{I1} = \left[\int_0^{\epsilon} \left(p_I(t)\left(\tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t)\right) - \frac{\omega}{2}a^2(t)\right) dt\right] + W_I(\epsilon, B(\epsilon)).$$

Comparing the outcomes of the two scenarios will allow us to measure the impact of brand imitation on the incumbent's profit. We shall first study the case where there is entry, and next use this solution to compute the incumbent's optimal price and advertising for the problem without entry. This is done by letting $\epsilon$ go to $T$ and choosing the appropriate parameter values. We henceforth skip the time argument when no ambiguity may arise.

# 3 Duopoly equilibrium

The incumbent's optimization problem is in two stages: between 0 and $\epsilon$, it is an optimal-control problem; between $\epsilon$ and $T$, a duopoly game is played and a Nash equilibrium is sought. To obtain a subgame-perfect Nash equilibrium (SPNE) in the two-stage problem, we first solve the second stage with $B(\epsilon)$ as an initial value of the brand's reputation.

## 3.1 The second-stage equilibrium

Denote by $S_j$ the strategy of player $j \in \{I, E\}$. We suppose that the duopoly game is played with feedback-information structure, that is, each player selects the control action according to the rule $u_j(t) = S_j(t, B(t))$, where

$$u_I(t) = (p_{I2}(t), a_2(t)) \in \mathbb{R}_+^2 \quad \text{and} \quad u_E(t) = (p_E(t)) \in \mathbb{R}_+.$$

This means that firm $j \in \{I, E\}$ observes the state $(t, B(t))$ of the system and then chooses its action as prescribed by the decision rule $S_j$.

**Definition 1** *A pair $\left(S_I^N, S_E^N\right)$ of functions $S_j : [\epsilon, T] \times \mathbb{R}_+ \longrightarrow R^{m_j}$, $j \in \{I, E\}$ is a feedback-Nash equilibrium if*

$$\begin{array}{rcl} \Pi_{I2}^D(S_I^N, S_E^N) & \geq & \Pi_{I2}^D(S_I, S_E^N), \quad \forall S_I \in \mathbb{R}_+^2, \\ \Pi_E^D(S_I^N, S_E^N) & \geq & \Pi_E^D(S_I^N, S_E), \quad \forall S_E \in \mathbb{R}_+. \end{array}$$

To characterize a feedback-Nash equilibrium, denote by $W_I(t, B(t)) : [\epsilon, T] \times \mathbb{R}_+ \to \mathbb{R}$ the incumbent's value function. To save on notation, let

$$\begin{array}{rcl} \triangle_E & = & \dfrac{2\beta_I\alpha_E + \gamma\alpha_I}{4\beta_E\beta_I - \gamma^2} > 0, \ \ \triangle_I = \dfrac{2\beta_E\alpha_I + \gamma\alpha_E}{4\beta_E\beta_I - \gamma^2} > 0, \\ \Omega & = & \left(\varphi\beta_E\triangle_E + \triangle_I \dfrac{\varphi\gamma^3}{4\beta_E\beta_I - \gamma^2} - \sigma\right), \\ \Gamma & = & 4\beta_I\triangle_I^2\left(\dfrac{k^2}{2\omega} + \dfrac{\varphi^2\gamma^4\beta_E}{(4\beta_E\beta_I - \gamma^2)^2}\right) > 0. \end{array}$$

The following proposition gives the equilibrium solution of the duopoly game.[4]

**Proposition 1** *Assume that*

$$\Psi = \Omega^2 - \Gamma \geq 0.$$

*For $t \in [\epsilon, T]$, there exists a unique feedback-Nash equilibrium, given by*

$$p_I^D(t, B(t)) = \triangle_I B(t) + \frac{\partial W_I(t, B)}{\partial B}\frac{2\varphi\gamma\beta_E}{4\beta_E\beta_I - \gamma^2}, \tag{8}$$

$$p_E^D(t, B(t)) = \triangle_E B(t) + \frac{\partial W_I(t, B)}{\partial B}\frac{\varphi\gamma^2}{4\beta_E\beta_I - \gamma^2}, \tag{9}$$

$$a^D(t, B(t)) = \frac{k}{\omega}\left(2x(t)B(t) + y(t)\right), \tag{10}$$

*where the value function $W_I$ is given by*

$$W_I(t, B(t)) = x(t)B^2(t) + y(t)B(t) + z(t), \tag{11}$$

*with*

[4] See Haurie et al. (2012) for details on determining a feedback-Nash equilibrium in differential games.

$$x(t) = \frac{\beta_I \triangle_I^2 \left(e^{2(T-t)\sqrt{\Psi}} - 1\right)}{\left(1 - e^{2(T-t)\sqrt{\Psi}}\right)\Omega + \left(1 + e^{2(T-t)\sqrt{\Psi}}\right)\sqrt{\Psi}},$$

$$y(t) = se^{\int_t^T \left(\frac{\Gamma}{\beta_I \triangle_I^2} x(i) + \Omega\right) di},$$

$$z(t) = \int_t^T \frac{\Gamma}{4\beta_I \triangle_I^2} y^2(i) di.$$

*The brand's reputation trajectory is given by the solution of the differential equation*

$$\dot{B}^D(t) = \left(2x(t)\frac{k^2}{\omega} + \varphi\beta_E\triangle_E - \sigma\right) B(t) + y(t)\frac{k^2}{\omega} + \frac{\partial W_I(t,B)}{\partial B}\frac{\varphi^2\gamma^2\beta_E}{4\beta_E\beta_I - \gamma^2}, \quad B(\epsilon) = B_\epsilon.$$

**Proof.** See Appendix. □

The results in the above proposition call for the following comments:

1. The proposition assumes an interior solution, which holds true. To show this, we need to prove that $x(t)$ and $y(t)$ are nonnegative for all $t \in [\epsilon, T]$. To show that $x(t)$ is nonnegative for all $t \in [\epsilon, T]$, it suffices to compute the time derivative of $x(t)$ to obtain

$$\dot{x}(t) = -\frac{2\beta_I \triangle_I^2 \Psi e^{2(T-t)\sqrt{\Psi}}(1 + e^{2(T-t)\sqrt{\Psi}})}{\left(\Omega + \sqrt{\Psi} - e^{2(T-t)\sqrt{\Psi}}\left(\Omega - \sqrt{\Psi}\right)\right)^2} < 0,$$

   and to observe that $x(T) = 0$. Consequently, $y(t)$ is strictly positive for all $t \in [\epsilon, T]$, and so is $z(t)$. In turn, this implies that the value function and its derivative are positive for all $t$. Therefore, the prices and advertising are strictly positive for all $t$, with $a^D(T, B(T)) = \frac{ks}{\omega}$.
2. The incumbent's value function is quadratic, which is reminiscent of the linear-quadratic structure of the differential game. Player $I$'s total equilibrium profit on $[\epsilon, T]$ is given by

$$W_I(\epsilon, B(\epsilon)) = x(\epsilon)B^2(\epsilon) + y(\epsilon)B(\epsilon) + z(\epsilon).$$

   As $x(t)$ and $y(t)$ are positive for all $t$, and in particular at $\epsilon$, we have that the larger is the brand reputation at entry date, the larger the profit that the incumbent obtains during the duopoly period, irrespective of the sign of $\varphi$.

## 3.2 The first-stage optimal solution

The first-stage optimization problem of the incumbent is defined by

$$\max_{p_I(t),\, a(t)} \Pi_I = \max_{p_I(t),\, a(t)} \int_0^\epsilon \left(p_I(t)\left(\tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t)\right) - \frac{\omega}{2}a^2(t)\right) dt + W_I(\epsilon, B(\epsilon)),$$

subject to the reputation dynamics

$$\dot{B}(t) = ka(t) - \sigma B(t), \; B(0) = B_0,$$

and where $W_I(\epsilon, B(\epsilon))$ is the salvage value giving the profit in the duopoly period as a function of the reputation achieved at the starting date of that period.

The above optimization problem is a standard finite-horizon dynamic-optimization problem that can be solved equivalently using either the maximum principle or the Hamilton-Jacobi-Belmann approach. We shall follow the first approach as it happens to be easier to implement in our case.

Introduce the Hamiltonian

$$H\left(p_I(t), a(t), B(t), \eta(t)\right) = p_I(t)\left(\tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t)\right) - \frac{\omega}{2}a^2(t) + \eta(t)\left(ka(t) - \sigma B(t)\right), \tag{12}$$

where $\eta(t)$ is the adjoint variable appended to the state dynamics. Before stating our results, we wish to highlight that the problem at hand is not concave. Indeed, it can be easily checked that the Hessian matrix, which is given by

$$\begin{pmatrix} \frac{\partial^2 H}{\partial p_I^2} & \frac{\partial^2 H}{\partial p_I \partial a} & \frac{\partial^2 H}{\partial p_I \partial B} \\ \frac{\partial^2 H}{\partial a \partial p_I} & \frac{\partial^2 H}{\partial a^2} & \frac{\partial^2 H}{\partial a \partial B} \\ \frac{\partial^2 H}{\partial B \partial p_I} & \frac{\partial^2 H}{\partial B \partial a} & \frac{\partial^2 H}{\partial B^2} \end{pmatrix} = \begin{pmatrix} -2\tilde{\beta}_I & 0 & \tilde{\alpha}_I \\ 0 & -\omega & 0 \\ \tilde{\alpha}_I & 0 & 0 \end{pmatrix},$$

is not negative definite. The following proposition summarizes our results.

**Proposition 2** *Let $t \in [0, \epsilon)$. Assume that the incumbent's price and advertising are upper-bounded as follows:*

$$\begin{aligned} p_I(t, B(t)) &< \overline{p} \text{ for all } t \text{ and } B(t), \\ a(t, B(t)) &< \bar{a} \text{ for all } t \text{ and } B(t), \end{aligned}$$

*with*

$$\overline{p} > \frac{\tilde{\alpha}_I}{2\tilde{\beta}_I}\left(B_0 + \frac{\bar{a}k}{\sigma}\right) \tag{13}$$

*Then, there is a threshold $\underline{\omega}(\varphi)$ such that for any $\omega > \underline{\omega}(\varphi)$ the interior solution is given by*

$$p_I^D(B(t)) = \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I}, \tag{14}$$

$$a^D(t) = \frac{k\left(2\tilde{\beta}_I \eta(\epsilon)\left((\sigma+v)e^{v(\epsilon+t)} - (\sigma-v)e^{v(\epsilon-t)}\right) + \tilde{\alpha}_I^2 B_0\left(e^{v(2\epsilon-t)} - e^{vt}\right)\right)}{2\tilde{\beta}_I \omega\left((\sigma+v)e^{2v\epsilon} - (\sigma-v)\right)}, \tag{15}$$

$$B^D(t) = \frac{2\tilde{\beta}_I \eta(\epsilon)(\sigma^2 - v^2)\left(e^{v(\epsilon+t)} - e^{v(\epsilon-t)}\right) - \tilde{\alpha}_I^2 B_0\left((\sigma-v)e^{vt} - e^{v(2\epsilon-t)}(\sigma+v)\right)}{\tilde{\alpha}_I^2\left((\sigma+v)e^{2v\epsilon} - (\sigma-v)\right)}, \tag{16}$$

*where*

$$v = \sqrt{\sigma^2 - \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}\frac{k^2}{\omega}}.$$

**Proof.** See Appendix. □

The following proposition gives the results in the no-entry benchmark scenario.

**Proposition 3** *The value function of the incumbent in the no-entry scenario is given by*

$$V_I(t, B(t)) = x_M(t)B^2(t) + y_M(t)B(t) + z_M(t),$$

*where*

$$\begin{aligned} x_M(t) &= \frac{\tilde{\alpha}_I^2}{4\tilde{\beta}}\left(\frac{(e^{2v(T-t)} - 1)}{v - \sigma + (v+\sigma)e^{2v(T-t)}}\right) \geq 0, \\ y_M(t) &= s e^{\int_t^T \left(2x(i)\frac{k^2}{\omega} - \sigma\right)di} > 0, \\ z_M(t) &= \int_t^T \frac{k^2}{2\omega} y_I^2(i) di > 0, \end{aligned}$$

*and the optimal solution is given by*

$$p_I^M(B(t)) = \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I}, \tag{17}$$

$$a^{M}(t) = \frac{k\left(2\tilde{\beta}_I s\left((\sigma+v)e^{v(T+t)}-(\sigma-v)\,e^{v(T-t)}\right)+\tilde{\alpha}_I^2 B_0\left(e^{v(2T-t)}-e^{vt}\right)\right)}{2\tilde{\beta}_I\omega\left((\sigma+v)\,e^{2vT}-(\sigma-v)\right)}, \tag{18}$$

$$B^{M}(t) = \frac{2\tilde{\beta}_I s(\sigma^2-v^2)\left(e^{v(T+t)}-e^{v(T-t)}\right)-\tilde{\alpha}_I^2 B_0\left((\sigma-v)e^{vt}-e^{v(2T-t)}(\sigma+v)\right)}{\tilde{\alpha}_I^2\left((\sigma+v)\,e^{2vT}-(\sigma-v)\right)}. \tag{19}$$

**Proof.** See Appendix. □

Given the linear-quadratic structure of the incumbent's optimization problem, the above result is not surprising. Note that the three coefficients of the value function are strictly positive for all $t$, with the exception of at $T$, where $x_M(T)=0$. Further, the price is proportional to the brand's reputation value and is stationary, that is, it does not depend explicitly on time. The advertising effort is strictly positive at all $t\in[0,T]$. The evolution over time of the advertising effort is given by

$$\dot{a}^{M}(t) = \frac{k\nu\left(2\tilde{\beta}_I s\left((\sigma+v)e^{v(T+t)}+(\sigma-v)\,e^{v(T-t)}\right)-\tilde{\alpha}_I^2 B_0\left(e^{v(2T-t)}+e^{vt}\right)\right)}{2\tilde{\beta}_I\omega\left((\sigma+v)\,e^{2vT}-(\sigma-v)\right)},$$

with its sign depending on the parameter values. If the incumbent does not value its brand reputation at the terminal date, that is, $s=0$, then $\dot{a}^{M}(t)$ is negative and advertising is monotonically decreasing over time.

**Remark 2** *Comparing (14) and (17) shows that the pricing policies during the time interval $[0,\epsilon)$ are the same. This does not mean that the trajectories are the same, because the reputation stock will evolve differently in the two scenarios, that is, $B^{M}(t)\neq B^{D}(t)$ for $t\in(0,\epsilon)$.*

**Remark 3** *The incumbent's optimization problem in the absence of entry is also not concave. Indeed, the Hamiltonian is given by*

$$H\left(p_I(t),a(t),B(t),\lambda(t)\right)=p_I(t)\left(\alpha_I B(t)-\beta_I p_I(t)\right)-\frac{\omega}{2}a^2(t)+\lambda(t)\left(ka(t)-\sigma B(t)\right),$$

*where $\lambda(t)$ is the adjoint variable appended to the state dynamics. The Hessian matrix reads as follows:*

$$\begin{pmatrix} \frac{\partial^2 H}{\partial p_I^2} & \frac{\partial^2 H}{\partial p_I\partial a} & \frac{\partial^2 H}{\partial p_I\partial B}\\ \frac{\partial^2 H}{\partial a\partial p_I} & \frac{\partial^2 H}{\partial a^2} & \frac{\partial^2 H}{\partial a\partial B}\\ \frac{\partial^2 H}{\partial B\partial p_I} & \frac{\partial^2 H}{\partial B\partial a} & \frac{\partial^2 H}{\partial B^2}\end{pmatrix}=\begin{pmatrix}-2\beta_I & 0 & \alpha_I\\ 0 & -\omega & 0\\ \alpha_I & 0 & 0\end{pmatrix},$$

*and is not negative definite. The existence proof given for the scenario with entry remains valid.*

# 4 Profit comparison

In this section, we compare the solutions obtained in the two scenarios. We shall first focus on the incumbent's profits, and next, compare the strategies implemented with and without entry. As the results vary with the sign of $\varphi$, which is the parameter that captures the impact of the entrant's sales on the incumbent's brand reputation, we shall deal in turn with the case of brand dilution ($\varphi\leq 0$) and the case of free advertising ($\varphi>0$).

## 4.1 Brand dilution

Suppose that $\varphi\leq 0$, and denote by $r_I^M(t)$ and $r_I^D(t)$ the instantaneous revenue in the monopoly and duopoly scenario, respectively. These revenues are given by

$$r_I^M(t) = p_I^M(B(t))\,d_I^M(t)=\frac{\tilde{\alpha}_I^2}{4\tilde{\beta}_I}B^2(t)$$

$$r_I^D(t) = p_I^D(B(t))\,d_I^D(t)=\frac{\beta_I\left(2\beta_E\alpha_I+\alpha_E\gamma\right)^2}{\left(4\beta_E\beta_I-\gamma^2\right)^2}B^2(t).$$

**Lemma 1** *For all $t \in [\epsilon, T]$, we have $r_I^M(t) > r_I^D(t)$.*

**Proof.** See Lemma 1 in Crettez et al. (2017). □

Although the result in the above lemma is intuitive, its proof is not immediate and requires that we reformulate the second-stage equilibrium problem as a quantity game à la Cournot, and next, use the classical result in industrial organization stating that quantities are lower under Cournot competition than under Bertrand (price) competition. From this result follows the inequality

$$W_I(\epsilon, B(\epsilon)) < V_I(\epsilon, B(\epsilon)), \quad \text{for any } B(\epsilon),$$

that is, the total profit that the incumbent realizes from date $\epsilon$ onward is lower in the scenario with entry than in the case without entry. The intuition behind why Lemma 1 implies the above inequality lies in the fact that, in the absence of a

competitor, the incumbent can always choose the same path for $a_I(t)$ that it would have chosen in the presence of a competitor.

The following proposition compares the incumbent's profits in the two scenarios over the whole planning horizon.

**Proposition 4** *If $\varphi \leq 0$, then $\Pi_I^D < \Pi_I^M$.*

**Proof.** See Appendix. □

The proposition shows that if the demand for the entrant's product damages the incumbent's brand reputation, then the incumbent's total profit is lower than what it would achieve in the absence of entry. This result is intuitive because entry is causing two negative effects, namely, brand dilution and price competition.

## 4.2 Brand enhancement

Now, we consider the case where the entrant's sales influence the brand's reputation positively.

**Proposition 5** *Under the assumption that both solutions exist, there exists $\check{\varphi} > 0$ such that, for all $\varphi \geq \check{\varphi}$, we have the following results:*

1. *At the entry date, we have*
$$V_I(\epsilon, B^M(\epsilon)) < W_I(\epsilon, B^M(\epsilon)).$$
2. *Throughout the planning horizon $[0, T]$, the incumbent's total profit with entry is higher than its profit without entry, that is,*
$$\Pi_I^D > \Pi_I^M.$$

**Proof.** See Appendix. □

The proposition characterizes the conditions under which the incumbent benefits from the presence of an imitator. In both results, these conditions say that if the free advertising effect, which is due to sales by the imitator, is sufficiently high, then the incumbent's payoff during the duopoly period $[\epsilon, T]$ and its profit during the whole planning horizon are larger with entry than without. In result 1, the comparison between the payoffs-to-go is done at $B^M(\epsilon)$, that is, at the reputation level achieved by time of entry in the scenario without entry. One way of interpreting this comparison is as follows: It is as if the incumbent did not foresee entry from the outset and determined its pricing and advertising strategies as if it would be alone in the market during the whole planning horizon. Then, the monopolist is surprised by the arrival of the competitor at $\epsilon$, the time at which $B^M(\epsilon)$ is a given number that cannot be changed. In result 2, we show that when the monopolist fully accounts for entry, its total payoff with entry dominates its total payoff without entry under the condition stated in the proposition, that is, when $\varphi$ is sufficiently large.

To shed an additional light on the $\varphi$ on the incumbent's profit, recall that when $\varphi$ is negative (brand dilution), the entrant is damaging the brand's reputation and also putting a competitive pressure on the incumbent. When $\varphi$ is positive, the presence of the imitator has one positive effect (promoting the incumbent's brand for free) and one negative effect (price competition). The above proposition tells us when the positive effect overrides the negative effect of entry.

In the Appendix, it is shown that $\check{\varphi}$ is determined by the following condition:

$$se^{\left(\check{\varphi}\beta_E\triangle_E+\triangle_I\frac{\check{\varphi}\gamma^3}{4\beta_E\beta_I-\gamma^2}-\sigma\right)(T-\epsilon)}>\frac{V_I(\epsilon,B^M(\epsilon))}{B^M(\epsilon)}.$$

Straightforward manipulations allow us to rewrite this inequality as

$$\frac{\partial W\left(t,B\left(t\right)\right)}{\partial B}\left|_{B(\epsilon)=0},\varphi=\check{\varphi}>\frac{V_I(\epsilon,B^M(\epsilon))}{B^M(\epsilon)},\right.$$

that is, the marginal value of the payoff-to-go at $\epsilon$ with entry, evaluated at $B(\epsilon)=0$ and $\varphi=\check{\varphi}$, is larger than the per unit of reputation of the payoff-to-go without entry. Of course this condition is demanding, but it is only a sufficient condition.

# 5 Comparison of pre-entry strategies

In this section, we focus on the incumbent's pricing and advertising policies during the interval $[0,\epsilon]$, with the objective of assessing how entry affects the incumbent before it actually occurs. The following proposition compares the pricing, advertising, and reputation trajectories obtained in the two scenarios. Recall that $\lambda^M(\epsilon)$, respectively $\eta^D(\epsilon)$, is the shadow price of brand reputation at date $\epsilon$ in the absence, respectively the presence, of the entrant.

**Proposition 6** *For all $t\in[0,\epsilon]$, the differences in reputation, advertising, and price between the two scenarios are given by*

$$\begin{aligned}
B^M(t)-B^D(t) &= \frac{2\tilde{\beta}_I(\sigma^2-v^2)\left(e^{v(\epsilon+t)}-e^{v(\epsilon-t)}\right)}{\tilde{\alpha}_I^2\left((\sigma+v)\,e^{2v\epsilon}-(\sigma-v)\right)}\left(\lambda^M(\epsilon)-\eta^D(\epsilon)\right),\\
a^M(t)-a^D(t) &= \frac{2k\tilde{\beta}_I\left((\sigma+v)e^{v(\epsilon+t)}-(\sigma-v)\,e^{v(\epsilon-t)}\right)}{2\tilde{\beta}_I\omega\left((\sigma+v)\,e^{2v\epsilon}-(\sigma-v)\right)}\left(\lambda^M(\epsilon)-\eta^D(\epsilon)\right),\\
p_I^M(t)-p_I^D(t) &= \frac{(\sigma^2-v^2)\left(e^{v(\epsilon+t)}-e^{v(\epsilon-t)}\right)}{\tilde{\alpha}_I\left((\sigma+v)\,e^{2v\epsilon}-(\sigma-v)\right)}\left(\lambda^M(\epsilon)-\eta^D(\epsilon)\right),
\end{aligned}$$

*and therefore:*

$$\begin{aligned}
sign\left(B^M(t)-B^D(t)\right) &= sign\left(a^M(t)-a^D(t)\right)=sign\left(p_I^M(t)-p_I^D(t)\right)\\
&= sign\left(\lambda^M(\epsilon)-\eta^D(\epsilon)\right).
\end{aligned}$$

**Proof.** Straightforward computations lead to the results. □

The above proposition bears few interesting messages. First, it shows that comparing the incumbent's decisions during the whole time interval $[0,\epsilon]$ boils down to just comparing two numbers, i.e., $\lambda^M(\epsilon)$ and $\eta^D(\epsilon)$. The implication is that the two price trajectories never cross each other. This implies that during the pre-entry period $[0,\epsilon]$, the price in the no-entry scenario ($p_I^M(t)$) is either always higher than or equal to, or always lower than or equal to, the price in the entry scenario ($p_I^D(t)$). The same statement can be made for advertising and reputation.

Second, the evolution over time of these differences also depends on the sign of $\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right)$. Indeed, it suffices to differentiate these differences with respect to time to obtain

$$\begin{aligned}
\dot{B}^M(t) - \dot{B}^D(t) &= \frac{2\tilde{\beta}_I(\sigma^2 - v^2)v\left(e^{v(\epsilon+t)} + e^{v(\epsilon-t)}\right)}{\tilde{\alpha}_I^2\left((\sigma+v)e^{2v\epsilon} - (\sigma - v)\right)}\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right), \\
\dot{a}^M(t) - \dot{a}^D(t) &= \frac{2k\tilde{\beta}_I v\left((\sigma+v)e^{v(\epsilon+t)} + (\sigma - v)e^{v(\epsilon-t)}\right)}{2\tilde{\beta}_I\omega\left((\sigma+v)e^{2v\epsilon} - (\sigma - v)\right)}\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right), \\
\dot{p}_I^M(t) - \dot{p}_I^D(t) &= \frac{(\sigma^2 - v^2)v\left(e^{v(\epsilon+t)} + e^{v(\epsilon-t)}\right)}{\tilde{\alpha}_I\left((\sigma+v)e^{2v\epsilon} - (\sigma - v)\right)}\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right).
\end{aligned}$$

Clearly, all coefficients of $\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right)$ are positive, and hence, the result. Finally, at the initial instant of time, we have $p_I^M(0) = p_I^D(0)$, which is a direct consequence of the fact that $B^M(0) = B^D(0) = B_0$. However, the two advertising levels are not the same at the initial date, with the difference between them being given by

$$a^M(0) - a^D(0) = \frac{2kve^{v\epsilon}}{\omega\left((\sigma+v)e^{2v\epsilon} - (\sigma - v)\right)}\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right).$$

Unfortunately, we could not sign the difference $\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right)$, which is a very long and highly non-linear expression of all parameter values. However, we can provide a sufficient condition showing that there exist parameter values for which this difference is negative.

**Proposition 7** *There exists $\hat{\varphi} > 0$ such that, for all $\varphi > \hat{\varphi}$, we have*

$$\frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)) < \frac{\partial W_I}{\partial B}(\epsilon, B), \quad \forall B > 0.$$

*In particular, we have*

$$\frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)) < \frac{\partial W_I}{\partial B}(\epsilon, B^M(\epsilon)) \text{ and } \frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)) < \frac{\partial W_I}{\partial B}(\epsilon, B^D(\epsilon)).$$

**Proof.** See Appendix. □

To interpret the above proposition, it is useful to recall that when the value function is differentiable, which is clearly the case here, we have

$$\lambda^M(\epsilon) = \frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)), \tag{20}$$

$$\eta^D(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B^D(\epsilon)), \tag{21}$$

and that the adjoint variable appended to the state dynamics is equal to the derivative of the value function. Consequently, we have

$$\frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)) = \lambda^M(\epsilon) < \eta^D(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B^D(\epsilon)).$$

Combining this result with the one in Proposition 6 leads to

$$\begin{aligned}
sign\left(B^M(t) - B^D(t)\right) &= sign\left(a^M(t) - a^D(t)\right) = sign\left(p_I^M(t) - p_I^D(t)\right) \\
&= sign\left(\lambda^M(\epsilon) - \eta^D(\epsilon)\right) < 0.
\end{aligned}$$

That is, under the assumption of the above proposition, and during the period $[0, \epsilon)$, we have that (i) the incumbent prices at a higher level in the entry scenario than in the no-entry scenario; (ii) the incumbent advertises less in the no-entry scenario than in the entry scenario; and (iii) the brand reputation is lower in the no-entry scenario than in the entry scenario. Item (iii) is a direct consequence of item (ii). Recalling that the price during the monopoly period is proportional to the brand reputation stock, then (i) follows from (iii). Consequently, the driving force of all the results here is the fact that $a^M(t) < a^D(t)$. The intuition behind this inequality is as follows: when $\varphi$ is positive, the brand reputation becomes a public good to which both players contribute: the incumbent through its advertising and the entrant through its sales. If $\varphi$ is large enough, then the incumbent finds it optimal to increase its advertising during the monopoly period, when it foresees the arrival of the entrant later on, at $\epsilon$, such that it can benefit even more from the free promotion done by the imitator after entry.

# 6 Concluding remarks

In this paper, we extended the work in Buratto et al. (2016) and Crettez et al. (2017) on counterfeiting, by considering a situation in which the imitator's sales affect the brand dynamics. As one might expect, when this impact is negative, the incumbent suffers from such an entry. However, if a greater product availability promotes the brand, then the incumbent will welcome being copied if some conditions are satisfied. Further, we obtained that entry does not only affect the incumbent's strategies during the duopoly period but also before entry.

Two extensions of this work can be pursued. First, letting the entry date be stochastic or dependant on the incumbent's decisions. Second, we can consider a setup where the imitator has a specific brand, and sales of both players depend on both brands' goodwill.

# Appendix: Proofs

# Proof of Proposition 1

**Proof.** Recall the change of notation introduced before, namely:

$$\begin{aligned}
\triangle_E &= \frac{2\beta_I\alpha_E+\gamma\alpha_I}{4\beta_E\beta_I-\gamma^2}>0,\ \ \triangle_I=\frac{2\beta_E\alpha_I+\gamma\alpha_E}{4\beta_E\beta_I-\gamma^2}>0,\\
\Omega &= \left(\varphi\beta_E\triangle_E+\triangle_I\frac{\varphi\gamma^3}{4\beta_E\beta_I-\gamma^2}-\sigma\right),\\
\Gamma &= 4\beta_I\triangle_I^2\left(\frac{k^2}{2\omega}+\frac{\varphi^2\gamma^4\beta_E}{(4\beta_E\beta_I-\gamma^2)^2}\right)>0,\\
\Psi &= \Omega^2-\Gamma.
\end{aligned}$$

Let $p_I(t,R)$ be given, then the entrant's problem is a static maximization problem, and the first-order optimality condition is

$$\alpha_E B+\gamma p_I(t,B)-2\beta_E p_E(t,B)=0. \tag{22}$$

Given $p_E(t,B)$, the incumbent's problem is a dynamic optimization problem.

Denote by $W_I$ the incumbent's value function. For all $(t,B)\in[\epsilon,T]\times\mathbb{R}_+$, the Hamilton-Jacobi-Bellman (HJB) equation reads as follows:

$$-\frac{\partial W_I(t,B)}{\partial t}=\max_{p_I,a}\{p_I\left(\alpha_I B-\beta_I p_I+\gamma p_E(t,B)\right)-\frac{\omega}{2}a^2+\frac{\partial W_I(t,B)}{\partial B}\left(ka+\varphi d_E(t,B)-\sigma B\right)\},$$

with the terminal condition $W_I(T,B)=sB$. Assuming an interior solution and maximizing the right-hand side of the HJB equation yields

$$\alpha_I B+\gamma p_E(t,B)-2\beta_I p_I(t,B)+\varphi\gamma\frac{\partial W_I(t,B)}{\partial B}=0. \tag{23}$$

$$a(t,B)=\frac{k}{\omega}\frac{\partial W_I(t,B)}{\partial B}. \tag{24}$$

Using (22) and (23), we get the following pricing policies:

$$p_E=\frac{\gamma\alpha_I+2\beta_I\alpha_E}{4\beta_E\beta_I-\gamma^2}B+\frac{\partial W_I(t,B)}{\partial B}\frac{\varphi\gamma^2}{4\beta_E\beta_I-\gamma^2}=\triangle_E B+\frac{\partial W_I(t,B)}{\partial B}\frac{\varphi\gamma^2}{4\beta_E\beta_I-\gamma^2},$$

$$p_I = \frac{\gamma\alpha_E + 2\beta_E\alpha_I}{4\beta_E\beta_I - \gamma^2}B + \frac{\partial W_I(t,B)}{\partial B}\frac{2\varphi\gamma\beta_E}{4\beta_E\beta_I - \gamma^2} = \triangle_I B + \frac{\partial W_I(t,B)}{\partial B}\frac{2\varphi\gamma\beta_E}{4\beta_E\beta_I - \gamma^2}.$$

Consequently, the demand of the entrant is

$$d_E = \beta_E \triangle_E B + \frac{\partial W_I(t,B)}{\partial B}\frac{\varphi\gamma^2\beta_E}{4\beta_E\beta_I - \gamma^2}.$$

Substituting for the prices and advertising in the HJB equation, we obtain

$$-\frac{\partial W_I(t,B)}{\partial t} = \beta_I \triangle_I^2 B^2 + \frac{\partial W_I(t,B)}{\partial B}\Omega B + \frac{\Gamma}{4\beta_I\triangle_I^2}\left(\frac{\partial W_I(t,B)}{\partial B}\right)^2.$$

Given the linear-quadratic structure of the game, we make the informed guess that the value function is of the form

$$W_I(t,B) = x(t)B^2 + y(t)B + z(t),$$

with $W_I(T,B) = sB$ for all nonnegative $B$.

The following expressions will be useful later on:

$$\frac{\partial W_I(t,B)}{\partial t} = \dot{x}(t)B^2 + \dot{y}(t)B + \dot{z}(t),$$
$$\left(\frac{\partial W_I(t,B)}{\partial B}\right)^2 = 4x(t)^2B^2 + y(t)^2 + 4x(t)y(t)B.$$

Plugging these expressions into the HJB equation, we get

$$\begin{aligned} -\left(\dot{x}(t)B^2 + \dot{y}(t)B + \dot{z}(t)\right) &= \beta_I\triangle_I^2B^2 + \left(2x(t)B + y(t)\right)\Omega B \\ &\quad + \frac{\Gamma}{4\beta_I\triangle_I^2}\left(4x(t)^2B^2 + y(t)^2 + 4x(t)y(t)B\right), \end{aligned}$$

for all $t$ and $B$. By identification, we obtain

$$-\dot{x}(t) = \beta_I\triangle_I^2 + 2x(t)\Omega + \frac{\Gamma}{\beta_I\triangle_I^2}x(t)^2 \tag{25}$$

$$-\dot{y}(t) = \frac{\Gamma}{\beta_I\triangle_I^2}x(t)y(t) + \Omega y(t), \tag{26}$$

$$-\dot{z}(t) = \frac{\Gamma}{4\beta_I\triangle_I^2}y(t)^2. \tag{27}$$

Moreover, from the condition $W_l(T,B) = x(T)B^2 + y(T)B + sz(T) = sB$ for all nonnegative $B$, we have $x(T) = 0$, $y(T) = s$ and $z(T) = 0$.

We solve the above system recursively. Let us start with Equation (25), which is a Ricatti equation. Consider the differential equation:

$$\dot{\mathfrak{z}}(t) = 2\iota(t)\mathfrak{z}(t) + \kappa(t) - \rho(t)\mathfrak{z}(t)^2, \tag{28}$$

where $a(t)$, $b(t)$ and $c(t)$ are continuous functions of $t$. Let us now consider the system

$$\dot{m} = \iota(t)m + \kappa(t)n, \tag{29}$$

$$\dot{n} = \rho(t)m - \iota(t)n. \tag{30}$$

We can check that if $m(t)$ and $n(t)$ are a solution to the above system (with $t \in (t_0, t_1)$), $n(t) \neq 0$, then $\mathfrak{z}(t) = m(t)/n(t)$ is a solution to Equation (28). In our problem, we have $\kappa(t) = -\beta_I\triangle_I^2$, $\rho(t) = \frac{\Gamma}{\beta_I\triangle_I^2}$, $\iota(t) = -\Omega$. Therefore, we want to solve the following differential system:

$$\begin{pmatrix}\dot{m}\\ \dot{n}\end{pmatrix} = \begin{pmatrix}-\Omega & -\beta_I\triangle_I^2 \\ \frac{\Gamma}{\beta_I\triangle_I^2} & \Omega\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix} \tag{31}$$

with $m(T) = 0$, and $n(T) = 1$.

The Eigenvalues of the above matrix are solutions to the equation

$$u^2 = \Psi,$$

that is,

$$u_1 = \sqrt{\Psi} \text{ and } \quad u_2 = -u_1.$$

The corresponding eigenvectors are respectively

$$\begin{pmatrix} 1 \\ m_2^1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\left(\Omega + \frac{\sqrt{\Psi}}{\beta_I \triangle_I^2}\right) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ m_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\left(\Omega - \frac{\sqrt{\Psi}}{\beta_I \triangle_I^2}\right) \end{pmatrix}.$$

Now, the solution of the differential system (31) can be written as

$$\begin{pmatrix} m(t) \\ n(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ m_2^1 \end{pmatrix} e^{\sqrt{\Psi}t} + c_2 \begin{pmatrix} 1 \\ m_2^2 \end{pmatrix} e^{-\sqrt{\Psi}t}$$

To find $c_1$ and $c_2$ we solve the following system:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{\sqrt{\Psi}T} & e^{-\sqrt{\Psi}T} \\ m_2^1 e^{\sqrt{\Psi}T} & m_2^2 e^{-\sqrt{\Psi}T} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

We find that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{m_2^2 - m_2^1} \begin{pmatrix} -e^{-\sqrt{\Psi}T} \\ e^{\sqrt{\Psi}T} \end{pmatrix},$$

where $m_2^2 - m_2^1 = \frac{2\sqrt{\Psi}}{\beta_I \triangle_I^2}$. Therefore,

$$x(t) = \frac{c_1 e^{\sqrt{\Psi}t} + c_2 e^{-\sqrt{\Psi}t}}{c_1 m_2^1 e^{\sqrt{\Psi}t} + c_2 m_2^2 e^{-\sqrt{\Psi}t}} = \frac{\beta_I \triangle_I^2 \left(e^{2\sqrt{\Psi}(T-t)} - 1\right)}{(1 - e^{2\sqrt{\Psi}(T-t)})\Omega + (1 + e^{2\sqrt{\Psi}(T-t)})\sqrt{\Psi}}.$$

Now, consider the differential equation in $y(t)$. We readily see that

$$y(t) = se^{\int_t^T \left(\frac{\Gamma}{\beta_I \triangle_I^2} x(i) + \Omega\right) di}.$$

Notice that $y(t) > 0$ for all $t$.

We finally obtain

$$z(t) = \int_t^T \frac{\Gamma}{4\beta_l \triangle_I^2} y^2(i) di,$$

which is always nonnegative.

Now, $a(t) = \frac{k}{\omega}(2x(t)B(t) + y(t))$ and

$$\begin{aligned} \dot{B}(t) &= ka(t) + \varphi d_E(t) - \sigma B(t), && (32) \\ &= \left(\frac{k^2}{\omega} 2x(t) + \varphi \beta_E \triangle_E - \sigma\right) B(t) + \frac{k^2}{\omega} y(t) + \frac{\partial W_I(t,B)}{\partial B} \frac{\varphi^2 \gamma^2 \beta_E}{4\beta_E \beta_I - \gamma^2}. && (33) \end{aligned}$$

We now show that (9) and $a(t) = \frac{k}{\omega} \frac{\partial W_I(t,R)}{\partial B}$ define a Nash equilibrium in the second-stage counterfeiting scenario. Given $p_I(t, B)$ in (8), we have that $p_E(t, B)$ in (9) solves the counterfeiter's problem since its payoff is concave with respect to $p_E$. Given $p_E(t, B)$ in (9), we have seen that $p_I(t, B)$ and $a(t, B)$ solve the legal firm's problem. Hence, the result. □

# Proof of Proposition 2

**Proof.** Proposition 1 ensures the existence of a feedback-Nash equilibrium in the second stage. The same proposition yields the expression of the value function associated to the incumbent's problem in the second stage. Now, assume for a while that there exists $\omega$ such that

$$\omega > \frac{k}{\overline{a}} \left\{ \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I \sigma}(B(0) + \frac{k\overline{a}}{\sigma}) + \sup_{B \in [0, \sup_{t\in[0,\epsilon]}\{B(0)e^{-\sigma t} + k\overline{a}\int_0^t e^{\sigma(u-t)}du\}]} \frac{\partial W_I}{\partial B}(\epsilon, B; \varphi, \omega) \right\}. \tag{34}$$

The proof of the proposition is in four steps.

**Step 1.** We prove the existence of a solution to the incumbent's problem with arbitrary bounds on the controls. Consider the following constrained problem (II):

$$\begin{aligned} \max_{p_I(t),\, a(t)} \Pi_I &= \max_{p_I(t),\, a(t)} \int_0^\epsilon \left( p_I(t) \left( \tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t) \right) - \frac{\omega}{2} a^2(t) \right) dt + W_I\big(\epsilon, B(\epsilon)\big), \\ \dot{B}(t) &= ka(t) - \sigma B(t), \ B(0) = B_0, \end{aligned} \tag{35}$$

with $0 \leq a(t) \leq \overline{a}$, and $0 \leq p_I(t) \leq \overline{p}$, where $\overline{a}$ and $\overline{p}$ are positive real numbers defined in the statement of the proposition.

The existence follows from Filippov's Theorem (see Filipov (1962)).

**Step 2.** Let us show that the solution is interior, i.e., satisfies $0 < p_I(t) < \overline{p}$ and $0 < a(t) < \overline{a}$ for all $t$.

The solution to problem (II) maximizes the following Hamiltonian at each date $t$:

$$\begin{aligned} H = p_I(t) \left( \tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t) \right) - \frac{\omega}{2} a^2(t) + \eta(t)\left(ka(t) - \sigma B(t)\right) \\ + \theta(t)p_I(t) + \zeta(t)(\overline{p} - p_I(t)) + \xi(t)a(t) + \psi(t)(m - a(t)), \end{aligned} \tag{36}$$

Therefore, the next first-order conditions must be satisfied:

$$\frac{\partial H}{\partial p_I} = 0 \iff \tilde{\alpha}_I B(t) - 2\tilde{\beta}_I p_I(t) + \theta(t) - \zeta(t) = 0, \tag{37}$$

$$\frac{\partial H}{\partial a} = 0 \iff -\omega a(t) + k\eta(t) + \xi(t) - \psi(t) = 0, \tag{38}$$

$$\dot{\eta}(t) = -\frac{\partial H}{\partial B} = -\tilde{\alpha}_I p_I(t) + \eta(t)\sigma, \qquad \eta(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B(\epsilon)), \tag{39}$$

$$\dot{B}(t) = \frac{\partial H}{\partial \eta} = ka(t) - \sigma B(t), \qquad B(0) = B_0, \tag{40}$$

$$p_I(t) \geq 0, \quad \theta(t) \geq 0, \quad \theta(t)p_I(t) = 0, \tag{41}$$

$$a(t) \geq 0, \quad \xi(t) \geq 0, \quad \xi(t)a(t) = 0, \tag{42}$$

$$p_I(t) \leq \overline{p}, \quad \zeta(t) \geq 0, \quad \zeta(t)(\overline{p} - p_I(t)) = 0, \tag{43}$$

$$a(t) \leq \overline{a}, \quad \psi(t) \geq 0 \quad \psi(t)(\overline{a} - a(t)) = 0 \tag{44}$$

Solving for $B(t)$, we have

$$B(t) = B_0 e^{-\sigma t} + k \int_0^t e^{\sigma(u-t)} a(u) du > B_0 e^{-\sigma t} > 0. \tag{45}$$

Note that $B(t)$ is always positive.

- Now let us show that $0 < p_I(t) < \overline{p}$, for all $t$.

Assume that $p_I(t) = 0$. Then, $\zeta(t) = 0$ and the first-order condition (37) writes $\tilde{\alpha}_I B(t) + \theta(t) = 0$. Since $\theta(t) \geq 0$ and $B(t)$ is positive, we get a contradiction.

Now let us show that $p_I(t) < \overline{p}$. Assume that $p_I(t) = \overline{p}$ for some $t$. Then, $\theta(t) = 0$, and from Equation (37), we have $p_I(t) = \overline{p} = \frac{\tilde{\alpha}_I B(t) - \zeta(t)}{2\tilde{\beta}_I}$. Since $\zeta(t) \geq 0$, it follows that $\overline{p} < \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I}$. From Equation (45) and the fact that $a(t) \leq \overline{a}$, we have

$$\overline{p} < \frac{\tilde{\alpha}_I}{2\tilde{\beta}_I}\left(B_0 e^{-\sigma t} + k\overline{a}\int_0^t e^{\sigma(u-t)}du\right) = \frac{\tilde{\alpha}_I}{2\tilde{\beta}_I}\left(B_0 e^{-\sigma t} + \frac{k\overline{a}}{\sigma}(1 - e^{-\sigma t})\right), \tag{46}$$

$$< \frac{\tilde{\alpha}_I}{2\tilde{\beta}_I}(B_0 + \frac{k\overline{a}}{\sigma}), \tag{47}$$

which contradicts (13). Therefore, we have $p_I(t) = \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I}$ for all $t$.

- $0 < a(t) < \overline{a}$ for all $t$.

Assume by way of contradiction that there is some $t$ at which $a(t) = 0$. Then, $\psi(t) = 0$ and (38) can be written as $k\eta(t) + \xi(t) = 0$. Since $\xi(t) \geq 0$, therefore we have $\eta(t) = \frac{\xi(t)}{k} \leq 0$.

Let us show that $t \neq \epsilon$. Indeed, $\eta(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B(\epsilon))$. Under our assumption $\frac{\partial W_I}{\partial B}$ is positive. Since $B_0 > 0$, we have seen that $B(t)$ is always positive.

Now solving for $\eta(t)$ from (39) (and using $p_I(t) = \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I}$) we get

$$\eta(t) = e^{\sigma t}\left(\int_t^\epsilon e^{-\sigma u}\frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}B(u)du + e^{-\sigma\epsilon}\eta(\epsilon)\right). \tag{48}$$

It follows that $\eta(t) > 0$, which is a contradiction.

Now assume again by way of contradiction that there is a date $t$ at which $a(t) = \overline{a}$. Then, $\xi(t) = 0$ and we must have $-\omega\overline{a} + k\eta(t) - \psi(t) = 0$. Therefore, $k\eta(t) = \omega\overline{a} + \psi(t)$, or $\eta(t) \geq \frac{\omega\overline{a}}{k}$. Now from (45), we have

$$\begin{aligned}
B(t) &= B_0 e^{-\sigma t} + k\int_0^t e^{\sigma(u-t)}a(u)du,\\
&\leq B_0 e^{-\sigma t} + km\int_0^t e^{\sigma(u-t)}du\\
&= B_0 e^{-\sigma t} + \frac{k\overline{a}}{\sigma}(1 - e^{-\sigma t})\\
&\leq B_0 + \frac{k\overline{a}}{\sigma}.
\end{aligned}$$

Using (48) we get

$$\begin{aligned}
\frac{\omega\overline{a}}{k} &\leq \eta(t) \leq e^{\sigma t}\left(\int_t^\epsilon e^{-\sigma u}\frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}(B_0 + \frac{k\overline{a}}{\sigma})du + e^{-\sigma\epsilon}\eta(\epsilon)\right)\\
&= \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I\sigma}(B_0 + \frac{k\overline{a}}{\sigma})(1 - e^{\sigma(t-\epsilon)}) + e^{\sigma(t-\epsilon)}\eta(\epsilon)\\
&< \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I\sigma}(B_0 + \frac{k\overline{a}}{\sigma}) + \eta(\epsilon) = \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I\sigma}(B_0 + \frac{k\overline{a}}{\sigma}) + \frac{\partial W_I}{\partial B}(B(\epsilon))\\
&\leq \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I\sigma}(B_0 + \frac{k\overline{a}}{\sigma}) + \sup_{B\in[0,\,\sup_{t\in[0,\epsilon]}\{B_0e^{-\sigma t} + k\overline{a}\int_0^t e^{\sigma(u-t)}du\}]}\frac{\partial W_I}{\partial B}(\epsilon, B).
\end{aligned}$$

But this last inequation contradicts the assumption (34).

**Step 3.** We can check that (14)–(16) is the interior solution of the first-order conditions (when $\theta(t) = \zeta(t) = \xi(t) = \psi(t) = 0$) provided that $\sigma^2 \geq \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}\frac{k^2}{\omega}$. Indeed, the first-order optimality conditions become

$$
\begin{aligned}
&\tilde{\alpha}_I B(t) - 2\tilde{\beta}_I p_I(t) = 0 \Leftrightarrow p_I(t) = \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I} \\
&-\omega a(t) + k\eta(t) = 0 \Leftrightarrow a(t) = \frac{\eta(t)\, k}{\omega}, \\
&\dot{\eta}(t) = -\frac{\partial H}{\partial B} = -\tilde{\alpha}_I p_I(t) + \eta(t)\,\sigma, \qquad \eta(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B(\epsilon)), \\
&\dot{B}(t) = \frac{\partial H}{\partial \eta} = ka(t) - \sigma B(t), \qquad B(0) = B_0.
\end{aligned}
$$

Substituting for the controls in the state and adjoint equations yields the following linear differential system:

$$
\begin{aligned}
\dot{\eta} &= \sigma\eta - \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I} B, \qquad \eta(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B(\epsilon)), \\
\dot{B} &= \frac{k^2}{\omega}\eta - \sigma B, \qquad B(0) = B_0.
\end{aligned}
$$

This system can be written $\dot{Y} = AY$, where $Y = \begin{pmatrix} \eta \\ B \end{pmatrix}$, $\dot{Y} = \begin{pmatrix} \dot{\eta} \\ \dot{B} \end{pmatrix}$ and $A = \begin{pmatrix} \sigma & -\frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I} \\ \frac{k^2}{\omega} & -\sigma \end{pmatrix}$.

Since $\sigma^2 \geq \dfrac{\tilde{\alpha}_I^2 k^2}{2\tilde{\beta}_I \omega}$, the two real eigenvalues are $v$ and $-v$, where $v = \sqrt{\sigma^2 - \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}\frac{k^2}{\omega}}$ and the corresponding eigenvectors are

$$
V_1 = \begin{pmatrix} 1 \\ \dfrac{2\tilde{\beta}_I(\sigma - v)}{\tilde{\alpha}_I^2} \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 \\ \dfrac{2\tilde{\beta}_I(\sigma + v)}{\tilde{\alpha}_I^2} \end{pmatrix}.
$$

Therefore, the general solution of the linear-differential system is given by

$$
Y(t) = r_1 e^{vt} V_1 + r_2 e^{-vt} V_2,
$$

that is,

$$
\begin{aligned}
\eta(t) &= r_1 e^{vt} + r_2 e^{-vt}, \\
B(t) &= r_1 \frac{2\tilde{\beta}_I(\sigma - v)}{\tilde{\alpha}_I^2} e^{vt} + r_2 \frac{2\tilde{\beta}_I(\sigma + v)}{\tilde{\alpha}_I^2} e^{-vt}.
\end{aligned}
$$

Further, $r_1$ and $r_2$ are uniquely determined (using Cramer's rule for example) so that the solution satisfies the boundary conditions $\eta(\epsilon) = \frac{\partial W_I}{\partial B}(\epsilon, B(\epsilon))$ and $B(0) = B_0$. We obtain

$$
r_1 = \frac{2\tilde{\beta}_I(\sigma + v)\eta(\epsilon)\, e^{v\epsilon} - \tilde{\alpha}_I^2 B_0}{2\beta_I((\sigma + v)\, e^{2v\epsilon} - (\sigma - v))}, \tag{49}
$$

$$
r_2 = \frac{e^{2v\epsilon}\tilde{\alpha}_I^2 B_0 - 2\tilde{\beta}_I(\sigma - v)\,\eta(\epsilon)\, e^{v\epsilon}}{2\tilde{\beta}_I((\sigma + v)\, e^{2v\epsilon} - (\sigma - v))}. \tag{50}
$$

Consequently, we get

$$
\begin{aligned}
B(t) &= r_1 \frac{2\tilde{\beta}_I(\sigma - v)}{\tilde{\alpha}_I^2} e^{vt} + r_2 \frac{2\tilde{\beta}_I(\sigma + v)}{\tilde{\alpha}_I^2} e^{-vt}, \\
p_I(t) &= \frac{\tilde{\alpha}_I B(t)}{2\tilde{\beta}_I} = r_1 \frac{(\sigma - v)}{\tilde{\alpha}_I} e^{vt} + r_2 \frac{(\sigma + v)}{\tilde{\alpha}_I} e^{-vt}, \\
a(t) &= \frac{(r_1 e^{vt} + r_2 e^{-vt})k}{\omega}.
\end{aligned}
$$

Substituting for $r_1$ and $r_2$ into $B(t)$, $p_I(t)$ and $a(t)$, we obtain (14)–(16).

**Step 4.** To guarantee that the solution of the first-order conditions exists and is unique, the following equation must have a unique solution in $B(\epsilon)$:

$$B^D(\epsilon) = \frac{2\tilde{\beta}\frac{\partial W_I}{\partial B}(B^D(\epsilon))\left(\sigma^2 - v^2\right)\left(e^{2v\epsilon} - 1\right) + B_0\left((\sigma+v)e^{v\epsilon} - (\sigma - v)e^{v\epsilon}\right)\tilde{\alpha}_I^2}{\tilde{\alpha}_I{}^2\left((\sigma+v)e^{2v\epsilon} - (\sigma - v)\right)}, \tag{51}$$

$$= 2\frac{\partial W_I}{\partial B}(B^D(\epsilon))\frac{\tilde{\beta}_I}{\tilde{\alpha}_I^2}\frac{(e^{2v\epsilon}-1)(\sigma - v)}{e^{2v\epsilon} - \frac{\sigma - v}{\sigma + v}} + \frac{2B_0 v e^{v\epsilon}}{(\sigma+v)\left(e^{2v\epsilon} - \frac{\sigma-v}{\sigma+v}\right)}, \tag{52}$$

where $\frac{\partial W_I}{\partial B}(B) = 2x(t)B + y(t)$. This equation has a positive solution $B^D(\epsilon)$ if, and only if,

$$4x(\epsilon)\frac{\tilde{\beta}_I}{(\tilde{\alpha}_I)^2}\frac{(e^{2v\epsilon}-1)(\sigma - v)}{e^{2v\epsilon} - \frac{\sigma-v}{\sigma+v}} < 1. \tag{53}$$

Let us now show that for $\omega \geq \frac{\tilde{\alpha}_I^2}{2\tilde{\beta}_I}\frac{k^2}{\sigma^2}$ and high enough, *both* inequations (34) and (53) hold. Recall that

$$x(t) = \frac{\beta_I \triangle_I^2\left(e^{2\sqrt{\Psi}(T-t)} - 1\right)}{\Omega + \sqrt{\Psi} - e^{2\sqrt{\Psi}(T-t)}\left(\Omega - \sqrt{\Psi}\right)}.$$

We can see that $\lim_{\omega \to +\infty} x(\epsilon)$ exists in $R$.

In addition, $\lim_{\omega \to +\infty} \sigma - v = 0$. Therefore, condition (53) holds when $\omega$ is high enough since $x(\epsilon, \omega)$ is bounded. Hence, there is a unique value $B^D(\epsilon)$ that satisfies Equation (52). Finally, when $\omega$ is high enough, condition (34) is also satisfied. We let $\underline{\omega}(\varphi)$ be a threshold such that when $\omega$ is above this threshold, the above inequations hold.

**Conclusion of the proof:** There is a threshold $\underline{\omega}(\varphi)$ such that, for any $\omega > \underline{\omega}(\varphi)$, there exists an interior solution to the problem, and the first-order conditions provide a unique solution. Therefore, we can conclude that for any $\omega > \underline{\omega}(\varphi)$, and that this unique solution is indeed the solution of the first-stage problem. □

# Proof of Lemma 3

**Proof.** The value function of the incumbent in the monopoly case is determined by solving the following Hamilton-Jacobi-Bellman equation:

$$-\frac{\partial V_I}{\partial t}(t,B) = \max_{a,p_I}\left\{p_I(\tilde{\alpha}_I B - \tilde{\beta}_I p_I) - \frac{\omega}{2}a(t)^2 + \frac{\partial V_I}{\partial B}(t,B)\,(ka - \sigma B)\right\},$$

with $V_I(T,B) = sB$ for all $B$. We shall look for a solution of the following kind:

$$V_I(t,B) = x_M(t)B^2 + y_M(t)B + z_M(t).$$

Solving the optimality conditions for the right-hand side of the HJB equation we get

$$p_I = \frac{\tilde{\alpha}_I}{2\tilde{\beta}_I}B, \tag{54}$$

$$a(t) = \frac{k}{\omega}\frac{\partial V_I}{\partial B}(t,B). \tag{55}$$

The HJB equation can now be written as follows:

$$-(\dot{x}_M(t)B^2 + \dot{y}_M(t)B + \dot{z}_M) = \frac{\tilde{\alpha}_I^2}{4\tilde{\beta}_I}B^2 + \frac{k^2}{2\omega}\left(\frac{\partial V_I}{\partial B}\right)^2 - \frac{\partial V_I}{\partial B}\sigma B.$$

Using $V_I(t,B) = x_M(t)B^2 + y_M(t)B + z_M(t)$, we get

$$-\left(\dot{x}_M(t)B^2+\dot{y}_M(t)B+\dot{z}_M\right)=\frac{\tilde{\alpha}_I^2}{4\tilde{\beta}_I}B^2+\frac{k^2}{2\omega}\Big(4x_M^2(t)B^2(t)+4x_M(t)y_M(t)B(t)+y_M^2(t)\Big) - \sigma B\Big(2x_M(t)B+y_M(t)\Big). \quad (56)$$

By identification we get

$$-\dot{x}_M(t)=2\frac{k^2}{\omega}x_M^2(t)-2\sigma x_M(t)+\frac{\tilde{\alpha}_I^2}{4\tilde{\beta}_I}, \quad (57)$$

$$-\dot{y}_M(t)=2\frac{k^2}{\omega}x_M(t)y_M(t)-\sigma y_M(t), \quad (58)$$

$$-\dot{z}_M(t)=\frac{k^2}{2\omega}y_M^2(t). \quad (59)$$

Following the same procedure as in the proof of Proposition 1, we get

$$x_M(t)=\frac{\tilde{\alpha}_I^2}{4\tilde{\beta}_I}\left(\frac{(e^{2v(T-t)}-1)}{v-\sigma+(v+\sigma)e^{2v(T-t)}}\right), \quad (60)$$

$$y_M(t)=se^{\int_t^T\left(2x(i)\frac{k^2}{\omega}-\sigma\right)di}, \quad (61)$$

$$z_M(t)=\int_t^T\frac{k^2}{2\omega}y_M^2(i)di. \quad (62)$$

It suffices to substitute for these coefficients in the optimality conditions to get the results in the proposition. □

# Proof of Proposition 4

**Proof.** When $\varphi\leq 0$, for $t\in[\epsilon,T)$ the value of $B(t)$ is given by

$$B^D(t)=e^{\sigma(\epsilon-t)}B(\epsilon)+\int_\epsilon^t e^{\sigma(i-t)}(ka(i)+\varphi q_E(i))di.$$

Now, suppose that entry does not occur and assume, nevertheless, that the incumbent implements during the period $[\epsilon,T)$ the same advertising that it would have implemented with entry. Let $\tilde{B}\,(t)$ be the reputation's value resulting from this policy. As $\varphi\leq 0$, we clearly have at each date $t$

$$B^D(t)<\tilde{B}(t).$$

Recall that the incumbent's optimization problem is given by

$$\Pi_I^D=\max_{p_I(t),a(t)}\left\{\int_0^\epsilon\Big(p_I\,(t)\Big(\tilde{\alpha}_IB\,(t)-\tilde{\beta}_Ip_I\,(t)\Big)-\frac{\omega}{2}a^2\,(t)\Big)\,dt+W_I(\epsilon,B(\epsilon))\right\},$$

and that the optimized profit is

$$\Pi_I^D=\int_0^\epsilon\Big(p_I^D\,(t)\Big(\tilde{\alpha}_IB\,(t)-\tilde{\beta}_Ip_I^D\,(t)\Big)-\frac{\omega}{2}a^2\,(t)\Big)\,dt+W_I(\epsilon,B(\epsilon)).$$

We clearly have

$$\begin{aligned}\Pi_I^D &= \int_0^\epsilon\Big(p_I^D\,(t)\Big(\tilde{\alpha}_IB\,(t)-\tilde{\beta}_Ip_I^D\,(t)\Big)-\frac{\omega}{2}a^2\,(t)\Big)\,dt+W_I(\epsilon,B(\epsilon))\\ &< \int_0^\epsilon\Big(p_I^D\,(t)\Big(\tilde{\alpha}_IB\,(t)-\tilde{\beta}_Ip_I^D\,(t)\Big)-\frac{\omega}{2}a^2\,(t)\Big)\,dt+V_I(\epsilon,\tilde{B}(\epsilon)),\\ &< \int_0^\epsilon\Big(p_I^D\,(t)\Big(\tilde{\alpha}_IB\,(t)-\tilde{\beta}_Ip_I^D\,(t)\Big)-\frac{\omega}{2}a^2\,(t)\Big)\,dt+V_I(\epsilon,B^M(\epsilon)),\\ &= \Pi_I^M.\end{aligned}$$

□

## Proof of Proposition 5

**Proof. Part 1.** From Proposition 1 we have that $W_I$ is well defined and that

$$W_I(\epsilon, B; \varphi) = x(\epsilon; \varphi)B^2 + y(\epsilon; \varphi)B + z(\epsilon; \varphi).$$

Since $x(t; \varphi)$ and $z(t; \varphi)$ are positive, $W_I(\epsilon, B; \varphi) > y(\epsilon; \varphi)B$. So to prove that

$$V_I(\epsilon, B^M(\epsilon)) < W_I(\epsilon, B^M(\epsilon))$$

holds it is sufficient to prove that there exists $\varphi > 0$ such that

$$y(\epsilon; \varphi)B^M(\epsilon) > V_I(\epsilon, B^M(\epsilon)).$$

Recall that

$$y(t) = se^{\int_t^T \left( \frac{\Gamma}{\beta_I \triangle_I^2} x(i) + \Omega \right) di},$$

so that

$$y(\epsilon) \geq se^{\int_\epsilon^T \Omega di} = se^{\Omega(T-\epsilon)}$$

Let $\check{\varphi} > 0$ be such that

$$se^{\left(\check{\varphi}\beta_E\triangle_E + \triangle_I \frac{\check{\varphi}\gamma^3}{4\beta_E\beta_I - \gamma^2} - \sigma\right)(T-\epsilon)} > \frac{V_I(\epsilon, B^M(\epsilon))}{B^M(\epsilon)}.$$

Then, we have proved that, for all $\varphi > \check{\varphi}$,

$$W_I(\epsilon, B^M(\epsilon); \varphi) > V_I(\epsilon, B^M(\epsilon)).$$

**Part 2.** Using Proposition 5 we have

$$\Pi_I^M = \int_0^T \left( p_I^M(t) \left( \tilde{\alpha}_I B^M(t) - \tilde{\beta}_I p_I^M(t) \right) - \frac{\omega}{2} \left( a^M \right)^2 (t) \right) dt + sB^M(T), \tag{63}$$

$$= \int_0^\epsilon \left( p_I^M(t) \left( \tilde{\alpha}_I B^M(t) - \tilde{\beta}_I p_I^M(t) \right) - \frac{\omega}{2} \left( a^M \right)^2 (t) \right) dt + V_I(\epsilon, B^M(\epsilon)) \tag{64}$$

$$< \int_0^\epsilon \left( p_I^M(t) \left( \tilde{\alpha}_I B^M(t) - \tilde{\beta}_I p_I^M(t) \right) - \frac{\omega}{2} \left( a^M \right)^2 (t) \right) dt + W_I(\epsilon, B^M(\epsilon)) \tag{65}$$

$$< \max_{p_I(t), a(t)} \left\{ \int_0^\epsilon \left( p_I(t) \left( \tilde{\alpha}_I B(t) - \tilde{\beta}_I p_I(t) \right) - \frac{\omega}{2} (a)^2 (t) \right) dt + W_I(\epsilon, B(\epsilon)) \right\} \tag{66}$$

$$= \Pi_I^D \tag{67}$$

□

## Proof of Proposition 7

**Proof.** Since $\frac{\partial W_I}{\partial B}(\epsilon, B) = 2x(\epsilon; \varphi)B + y(\epsilon; \varphi)$ and $x$ and $B$ are positive, to establish the result, it suffices to prove that there exists $\varphi > 0$ such that

$$y(\epsilon; \varphi) > \frac{\partial V_I}{\partial B}(\epsilon, B^M(\epsilon)).$$

Following the same steps as as in the proof of Proposition 5, the above condition can be rewritten as

$$se^{\left(\hat{\varphi}\beta_E\triangle_E + \triangle_I \frac{\hat{\varphi}\gamma^3}{4\beta_E\beta_I - \gamma^2} - \sigma\right)(T-\epsilon)} > \frac{\partial V_I(\epsilon, B^M(\epsilon))}{B^M(\epsilon)}.$$

□

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