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Abstract: In this paper we present a novel formulation based on a certain median function to solve discretely constrained mixed complementarity problems (MCPs). Such problems seek to combine integer (discrete) solutions that are also equilibrium ones and have applications in engineering and economics. Several theoretical results show the correspondence between the easier-to-solve formulations presented and the original discretely constrained MCP. Lastly, the approach is tested on a variety of illustrative examples.

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1 Overview

In recent years, both the fields of equilibrium programming and integer programming have seen extensive modeling and algorithmic development. Most often equilibrium programming models are expressed as either mixed complementarity problems (MCPs) or variational inequalities [5, 6, 13, 14] but with the restriction that the variables are either implicitly or explicitly taken to be continuous-valued. Such an assumption is valid for example when considering volumetric flows in power or water networks where individual electrons or water molecules are not tracked. However, when considering individual items such as cars or people that might be counted, integer-valued variables or more generally discrete restrictions make more sense, hence the study of discretely constrained MCPs (DC-MCPs).

As shown in [11, 12], combining MCPs with integer restrictions results in a hard problem for which there may not always be a solution. In these works, a Pareto optimal approach [4] is advocated for finding nearly complementary and nearly integer solutions. When the integer variables in the problem are specialized to binary, many applications in engineering and economics are possible with these binary variables relating to the on/off status of machines, programs, or the like. For example, [17] considers an application of binary-constrained MCPs in power markets related to the concept of non-convex pricing and uplifts e.g., [2, 3, 15]. In this paper, we present a new approach to solve discretely constrained MCPs based on a certain median function [8, 10] and a resulting exact, easier-to-solve transformation. If the MCPs are linear, the resulting formulations are expressed as mixed-integer linear programs. We validate the approach with supporting theoretical and numerical results in Sections 2 and 3 and conclude in Section 4.

What is new about the work proposed in this paper is as follows:

1. As compared to the above cited works [11, 12, 17], the current approach is able to efficiently handle a more general class of discretely constrained MCPs, namely ones with general upper and lower bounds $u_i \geq l_i$, respectively for nonnegative variables. In the more traditional MCP approach, $u_i = +\infty, l_i = 0$ for nonnegative variables x_i and $u_j = +\infty, l_j = -\infty$ for free variables y_j. When potentially different upper and lower bounds than the traditional ones are allowed, combined with a mixture of continuous and discrete variables, a larger number of problems can be solved with the proposed methodology. Theorems 1–3 in Section 2.3.1 show why the proposed formulation works and Example #2d provides a numerical illustration of the related formulation (7). Then, this formulation is specialized to the more traditional case in Corollary 1 in Section 2.3.2.
2. As compared to the above cited works, the values to the integer variables don't need to be spelled out in advance. Rather, the solver is just told that integer values are desired. In principle, in [11, 12], in one of the seven variations tested, this could also be done by declaring integer variables to the solver. However, in practice, for the problems tried in these works, what worked well was to allow flexibility in relaxing both complementarity and integrality though for which the integer values did need to be spelled out ahead of time.
3. In the proposed work, given the new approach for minimizing deviations from complementarity via a certain median function H, a much more selective near (or relaxed) complementarity can be achieved. For example, in the case of a duopoly (Example #3), player p's complementarity problem can be described by finding the zero of a certain vector-valued, median function $H_p(x^p, y^p)$ (see equation (2)) for its own set of nonnegative variables x^p and free variables y^p . Rather than equivalently finding $\left(x^1, y^1, x^2, y^2\right)$ by solving a minimization problem whose objective of $\left\|\left(H_1(x^1, y^1)^T \quad H_2(x^2, y^2)^T\right)\right\|$ must equal zero, a weighted version of this objective function in the form $\sum_{p=1,2} \omega_p \left\|H_p(x^p, y^p)\right\|$ for positive weights ω_p can be applied. This latter approach permits a great deal of flexibility by allowing different players or more generally, parts of the MCP, to achieve different levels of relaxed complementarity with definite modeling advantages. Thus, the Pareto frontier [4] between achieving a complementary solution for one player (MCP part) can be determined. In a related way, alternatively, such targeted precision in relaxing complementarity can be achieved via constraints of the form $\|H_p(x^p, y^p)\| \leq \varepsilon_p$ for $\varepsilon_p > 0$ or a combination of such constraints for some of the players (parts of the MCP) and for the other players' (MCP parts), representation in the objective function as described above. This has implications for

example in power markets where nearly equilibrium prices or quantities might be acceptable for certain market segments if other considerations like minimizing uplift are taken in to account.

4. This paper as compared to the above works also makes clear that solving the DC-MCP amounts to a projection of a continuous MCP solution on to the set of integers (for appropriate variables) where the projection depends on the vector norm chosen for the function H (see Example #1).
5. Lastly, as shown in Example #4 for the spatial price equilibrium problem (SPE) and not shown in the above cited works, the DC-MCP formulation allows for if-then type restrictions on the variables for an MCP. As this example points out, a nuanced logic can then be encoded for the variables of the MCP which is a improvement towards a more realistic version of SPE as well as other equilibrium problems where if-then, logic-constrained equilibrium solutions are sought. In Example #4, logical, equity constraints on SPE flows are modeled to illustrate this if-then capability on top of finding MCP solutions.

2 Formulation and median function

2.1 Re-expressing the MCP as the zero of a particular median-related function H

Given the function $F : R^n \to R^n$ and vectors $l, u \in R^n \cup \{-\infty, +\infty\}$ with $l \leq u$, consider the mixed complementarity problem (MCP) [6] as follows. Find $x \in R^n$ so that

$$\begin{array}{ll} F_i(x) \geq 0 & x_i = l_i \\ F_i(x) = 0 & l_i < x_i < u_i \\ F_i(x) \leq 0 & x_i = u_i \end{array} \tag{1}$$

As described in [10] and [8], MCP(F) can be equivalently recast as finding the zero of the function $H : R^n \to R^n$ defined by

$$H_i(x) = x_i - mid(l_i, u_i, x_i - F_i(x)), \forall i \tag{2}$$

where $mid(a, b, c)$ represents the median of the three scalars a, b, c. The more traditional form of the MCP is to rewrite it in terms of a vector of nonnegative variables $x \in R^{n_x}$ with corresponding nonnegative F_i and complementarity and the rest of the variables $y \in R^{n_y}$ with associated equations. More specifically,

$$0 \leq F_i(x, y) \perp x_i \geq 0, i \in I_x = \{1, \dots, n_x\} \tag{3a}$$

$$0 = F_j(x, y), y_j \text{ free}, j \in I_y = \{1, \dots, n_y\} \tag{3b}$$

where $u \perp v$ means that the vectors u and v are complementarity (i.e., orthagonal) to each other. The specialization of (1) to (3) is acheived by choosing

$$\begin{array}{ll} l_i = 0, u_i = +\infty & i \in I_x \\ l_j = -\infty, u_j = +\infty & j \in I_y \end{array} \tag{4}$$

The discretely constrained,[1] mixed complementarity problem (DC-MCP) is to find vectors x, y that satisfy (1) and in addition have the discrete restrictions:

$$x_d \in Z_+, d \in D_x \subseteq I_x \tag{5a}$$

$$y_d \in Z, d \in D_y \subseteq I_y \tag{5b}$$

with the remaining x and y variables continuous-valued. This was a formulation also considered in [11, 12], and [17].

[1] In this paper, the term discretely-constrained is used interchangeably with integer-valued.

2.2 Formulation with different norms

Using the median function H given in (2), the DC-MCP can be succinctly represented as:

$$\begin{aligned} &\min_x \|H(x,y)\| && \text{(6a)}\\ s.t. \quad & x_i \in R_+, i \in I_x \backslash D_x && \text{(6b)}\\ & x_i \in Z_+, i \in D_x && \text{(6c)}\\ & y_j \in R, j \in I_y \backslash D_y && \text{(6d)}\\ & y_j \in Z, j \in D_y && \text{(6e)} \end{aligned}$$

The function $\|H(x,y)\|$ is in general non-smooth. For example, let $F : R^2 \to R^2$ be defined as $F_1(x,y) = x + y, F_2(x,y) = y$ where $l_1 = 0, u_1 = +\infty$, $l_2 = -\infty$, $u_2 = +\infty$. Then,

$$H(x,y) = \begin{pmatrix} H_1(x,y) \\ H_2(x,y) \end{pmatrix} = \begin{pmatrix} x - mid(0, +\infty, x - (x+y)) \\ y - mid(-\infty, +\infty, y - (y)) \end{pmatrix} = \begin{pmatrix} \begin{cases} x+y & y \le 0 \\ x & y > 0 \end{cases} \\ y \end{pmatrix}$$

So that

$$\begin{aligned} \|H(x,y)\|_1 &= \begin{cases} |x+y| + |y| & y \le 0 \\ |x| + |y| & y > 0 \end{cases} \\ &= \begin{cases} |x+y| - y & y \le 0 \\ |x| + y & y > 0 \end{cases} \\ &= \begin{cases} |y| - y = -2y & y \le 0, \text{for } x = 0 \\ y & y > 0, \ \text{for } x = 0 \end{cases} \end{aligned}$$

which for $x = 0$ fixed, is continuous in y but kinked, hence non-differentiable in the sense of Fréchet at the point $y = 0$ as shown in Figure 1. Note that the unique solution to the MCP for this function is $(x,y) = (0,0)$ for which $\|H(x,y)\|_1 = 0$. For any other choice of $(x,y)\,, \|H(x,y)\|_1 > 0$; for example, when $(x,y) = (1,0)\,, \|H(x,y)\|_1 = 1$.

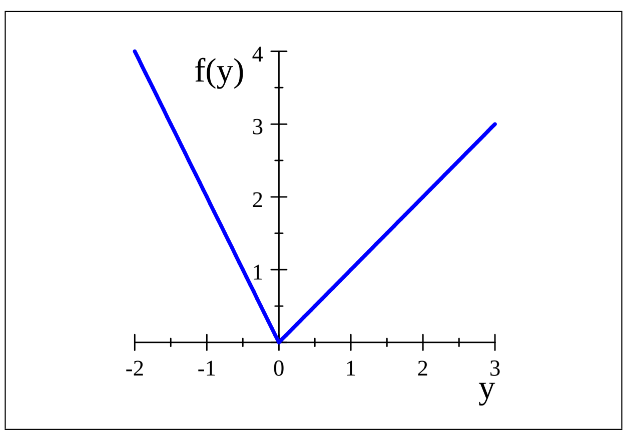

Figure 1: Example of the mid function being non-smooth

As shown in Section 2.3, we can exactly transform this median function through the use of binary variables and solve the resulting formulation with standard mixed-integer (nonlinear or linear) solvers.

Despite the fact that the objective function to (6) is bounded below by zero, this problem may not have a solution. Consider the case when $I_x = \{1\}\,, I_y = \varnothing, D_x = \{1\}\,, F_1\,(x_1) = x_1 - e^{-x_1}, u_1 = +\infty$, and $l_1 = -2$. In this case,

$$\begin{aligned} &x_1 = l_i = -2 \Rightarrow F_1\,(x_1) = [x_1 - e^{-x_1}]_{x_1=-2} \approx -9.389\,056\,099 \ngeq 0 \\ &-2 = l_i < x_i < u_i = +\infty, F_1\,(x_1) = [x_1 - e^{-x_1}] = 0 \Leftrightarrow x_1 \approx 0.567\,143\,290\,4 \\ &x_i = u_i = +\infty \text{ not possible} \end{aligned}$$

Thus, $x_1 \approx 0.567\,143\,290\,4$ is the only continuous solution to this MCP and in the range $[-2, +\infty)$, $|H_1(x_1)| = |x_1 - F_1(x_1)| = e^{-x_1}$ which is shown in Figure 2 below. Clearly, there is no positive integer large enough that achieves the lower bound of 0 for the objective function in this case. The problem here is the boundedness of the feasible region.

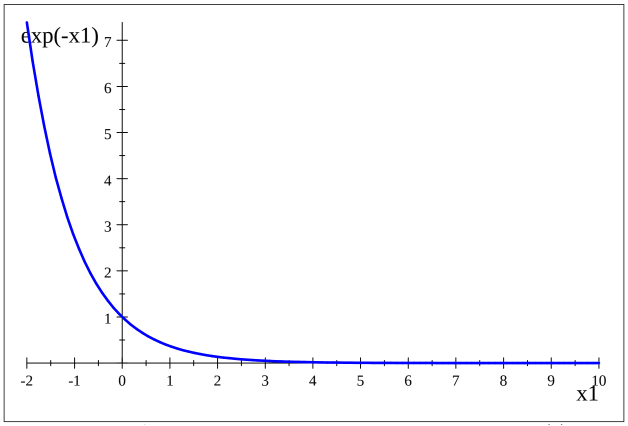

Figure 2: Counter-example to a solution always existing to (6)

2.3 Formulation with median-related binary variables

We first present the formulation and related theoretical results (Theorems 1–3) for the upper-bounded MCP in Section 2.3.1. This formulation is then specialized in Section 2.3.2 to the more traditional one and Corollary 1 provides appropriate results.

2.3.1 The general case for $l_i, u_i, i \in I_x$

The key idea is to use binary variables related to the various cases of the median function to transform the problem into an easier-to-solve form. First, using the L_1 vector norm, the term $\|H(x,y)\|_1 = \sum_i |H_i(x,y)|$ can be replaced as follows. Form a new variable $z_i = H_i(x,y)$ as the difference of two nonnegative variables z_i^+, z_i^-, i.e., $z_i \triangleq z_i^+ - z_i^-$. Then, rewrite $|H_i(x,y)| = z_i^+ + z_i^-$ for $i \in I_x$, noting that at most one of z_i^+, z_i^- can be nonzero, i.e., these variables are of the SOS1 type (specially ordered sets of type 1, [1]). Similarly, we let $w_j \triangleq w_j^+ - w_j^-, w_j^+, w_j^- \geq 0$, and rewrite $|H_j(x,y)| = w_j^+ + w_j^-$ for $j \in I_y$. Next, we take advantage of the fact that the median portion of the H function has two of its arguments, l_i and u_i ordered so that $l_i \leq u_i$ by definition. We write this function H in partitioned form conformal with the vectors x and y as:

$$\begin{aligned}
\begin{pmatrix} H_x(x,y) \\ H_y(x,y) \end{pmatrix} &\triangleq \begin{pmatrix} x - mid(l_x, u_x, x - F_x(x,y) \\ y - mid(l_y, u_y, y - F_y(x,y) \end{pmatrix} \\
&= \begin{pmatrix} x_i - mid(l_i, u_i, x_i - F_i(x,y), \forall i \in I_x \\ y_j - mid(-\infty, \infty, y_j - F_j(x,y), \forall j \in I_y \end{pmatrix}
\end{aligned}$$

where we have assumed that for each $i \in I_x$, x_i is constrained to be in $[l_i, u_i]$ but that for each $j \in I_y$, $y_j \in (-\infty, +\infty)$; here $u_j = +\infty, l_j = -\infty$. First, considering the general case for the upper and lower bounds l_x, u_x, we see that there are five cases for each component i relating to the x variables:

Case 1: $x_i - F_i(x,y) < l_i \leq u_i \Rightarrow z_i = H_i(x,y) = x_i - l_i$

Case 2: $l_i < x_i - F_i(x,y) < u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y)) = F_i(x,y)$

Case 3: $l_i = x_i - F_i(x,y) \leq u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y)) = F_i(x,y) = x_i - l_i$

Case 4: $l_i \leq u_i < x_i - F(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i$

Case 5: $l_i \leq u_i = x_i - F_i(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i = F_i(x,y)$

The following mixed-integer optimization problem relating to $\|H(x,y)\|_1$ then can be used to solve the general form of the DC-MCP given as (1), (5). Clearly a similar line of reasoning could be applied to the L_∞ norm for H or the L_2 norm but for the latter, a quadratic objective function would result. With the L_1 formulation shown below, for $F(x,y)$ linear (affine) then the resulting problem is a mixed-integer linear program (MILP) for which a number of standard solvers can be applied. For nonlinear $F(x,y)$, the resulting problem is a mixed-integer, nonlinear program (MINLP).

$$\min_{x,y,z^+,z^-,w^+,w^-,b,\widetilde{b}} f = \sum_{i\in I_x}\left(z_i^+ + z_i^-\right) + \sum_{j\in I_y}^{n}\left(w_j^+ + w_j^-\right) \tag{7a}$$

$$s.t. -Mb_i \le x_i - F_i(x,y) - l_i \le M(1-b_i), \forall i \in I_x \tag{7b}$$

$$-M\widetilde{b}_i \le x_i - F_i(x,y) - u_i \le M(1-\widetilde{b}_i), \forall i \in I_x \tag{7c}$$

$$-M\left(2 - b_i - \widetilde{b}_i\right) \le z_i^+ - z_i^- - x_i + l_i \le M\left(2 - b_i - \widetilde{b}_i\right) \tag{7d}$$

$$-M\left(1 + b_i - \widetilde{b}_i\right) \le z_i^+ - z_i^- - F_i\left(x,y\right) \le M\left(1 + b_i - \widetilde{b}_i\right) \tag{7e}$$

$$-M\left(b_i + \widetilde{b}_i\right) \le z_i^+ - z_i^- - x_i + u_i \le M\left(b_i + \widetilde{b}_i\right) \tag{7f}$$

$$w_j^+ - w_j^- = F_j(x,y), \forall j \in I_y \tag{7g}$$

$$x_i \in R_+, i \in I_x \backslash D_x \tag{7h}$$

$$x_i \in Z_+, i \in D_x \tag{7i}$$

$$y_j \in R, j \in I_y \backslash D_y \tag{7j}$$

$$y_j \in Z, j \in D_y \tag{7k}$$

$$b_i, \widetilde{b}_i \in \{0,1\}, \forall i \in I_x \tag{7l}$$

$$z_i^+, z_i^- \ge 0, \forall i \in I_x \tag{7m}$$

$$w_j^+, w_j^- \ge 0, \forall j \in I_y \tag{7n}$$

The constant M shown above should be chosen as small as possible to avoid numerical ill-conditioning and a degradation in solver performance but large enough to not constrain the decision variables more than what they would be without such disjunctive constraints [7]. Clearly M could also be chosen to vary by constraint but for ease of notation, only one value is chosen for all constraints. In some cases, when the function F is specified, better estimates than just trial and error can be applied [9]. See Theorem 4 in Section 3 for such a calculation in Example #3 for the capacitated duopoly problem.

The following results show the correspondence between the above problem (7) and solving the DC-MCP expressed as (6).

Theorem 1 *For each $i \in I_x$, assume that $l_i < u_i$. Consider any feasible solution $\left(x,y,z^+,z^-,w^+,w^-,b,\widetilde{b}\right)$ to (7). Then, for $z_i \triangleq z_i^+ - z_i^-, w_j \triangleq w_j^+ - w_j^-$,*

$$\begin{aligned} z_i &= H_i\left(x,y\right), \forall i \in I_x \\ w_j &= H_j\left(x,y\right), \forall j \in I_y \end{aligned}$$

Proof. First consider the constraints (7b) and (7c) and noting the premise that $l_i < u_i$ we see the following.

Case 1: $x_i - F_i(x,y) - l_i < 0$. Then, (7b) is feasible only if $b_i = 1$. Likewise since $x_i - F_i(x,y) - u_i < x_i - F_i(x,y) - l_i < 0$, this means that (7c) is feasible only when $\widetilde{b}_i = 1$.

Case 2: $l_i < x_i - F_i(x,y) < u_i$. Then, (7b) is feasible only if $b_i = 0$. Since $x_i - F_i(x,y) - u_i < 0$, this means that (7c) is feasible only when $\widetilde{b}_i = 1$.

Case 3: $l_i = x_i - F_i(x,y) < u_i$. Then, (7b) is feasible if $b_i = 0$ or 1. Since $x_i - F_i(x,y) - u_i < 0$, this means that (7c) is feasible only when $\widetilde{b}_i = 1$. We designate case 3a: $b_i = 0, \widetilde{b}_i = 1$, case 3b: $b_i = 1, \widetilde{b}_i = 1$.

Case 4: $l_i < u_i < x_i - F_i(x,y)$. Then, (7b) is feasible only if $b_i = 0$. Since $x_i - F_i(x,y) - u_i > 0$, this means that (7c) is feasible only when $\widetilde{b}_i = 0$.

Case 5: $l_i < u_i = x_i - F_i(x,y)$. Then, (7b) is feasible only if $b_i = 0$. Since $x_i - F_i(x,y) - u_i = 0$, this means that (7c) is feasible when $\widetilde{b}_i = 0$ or 1. We designate case 5a: $b_i = 0, \widetilde{b}_i = 0$, case 5b: $b_i = 0, \widetilde{b}_i = 1$.

Next consider the constraints (7d), (7e), and (7f). For cases 1 and 3b, both the left-hand and right-hand sides of (7d) are zero which forces $z_i = z_i^+ - z_i^- = x_i - l_i$ so that z_i corresponds correctly to $H_i(x,y)$. For the other cases, the left-hand side is either $-M$ or $-2M$ which is sufficienctly negative and small to not constrain the problem assuming M was selected large enough. The right-hand side is sufficiently large using the same reasoning for the other cases. Now consider the constraint (7e). The left-hand and right-hand sides are zero only when $b_i = 0$, $\widetilde{b}_i = 1$ corresponding to case 2, case 3a and case 5b. For cases 1, 3b , case 4, and case 5a, due to the choices of the binary variables b_i, $\widetilde{b}_i$, the left-hand and right-hand sides are sufficiently small and sufficiently large, respectively, to make this constraint non-binding. For those values of the binding variable, $z_i = z_i^+ - z_i^- = F_i(x,y)$. Lastly, for the constraint (7f), it is only when both $b_i = 0$, $\widetilde{b}_i = 0$ (cases 4 and 5a) that the constraint is binding. For those values of the binding variable, $z_i = z_i^+ - z_i^- = x_i - u_i$. Constraint (7g) shows that $w_j = w_j^+ - w_j^- = F_j(x,y)$, always corresponding to $u_j = +\infty$, $l_j = -\infty$ so that the correspondence between z_j and $H_j(x,y)$ for $j \in I_y$ follows.

Thus, we see that the pattern for $\left(b_i, \widetilde{b}_i\right)$ for each of the five cases leads to the following:

case	b_i	$\widetilde{b}_i$	z_i	$H_i(x,y)$
1	1	1	$x_i - l_i$	$x_i - l_i$
2	0	1	$F_i(x,y)$	$F_i(x,y)$
$3a$	0	1	$F_i(x,y)$	$F_i(x,y) = x_i - l_i$
$3b$	1	1	$x_i - l_i$	$F_i(x,y) = x_i - l_i$
4	0	0	$x_i - u_i$	$x_i - u_i$
$5a$	0	0	$x_i - u_i$	$F_i(x,y) = x_i - u_i$
$5b$	0	1	$F_i(x,y)$	$F_i(x,y) = x_i - u_i$

(Note that $b_i = 1, \widetilde{b}_i = 0$ is not possible since it would mean that $u_i \le x_i - F_i(x,y) \le l_i$ which violates the premise that $l_i < u_i$ for each $i \in I_x$.)

Thus, the constraints (7b)–(7g) enforce that $z_i = H_i(x,y)$ for all $i \in I_x$ and $w_j = H_j(x,y)$ for all $j \in I_y$. □

As shown in the next result, the SOS1 property is maintained for $\left(z_i^+, z_i^-\right)$ as well as for $\left(w_j^+, w_j^-\right)$ thus making the correspondence between $\|H(x,y)\|_1$ and the objective function (7a).

Theorem 2 *For each $i \in I_x$, assume that $l_i < u_i$. Consider any optimal solution $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ to (7). Then at most one of $\left(z_i^+, z_i^-\right)$ is nonzero and at most one of $\left(w_i^+, w_i^-\right)$ is nonzero.*

Proof. Consider an optimal solution of (7) where for sake of contradiction, there is at least one i with both $\left(z_i^+\right)^*, \left(z_i^-\right)^* > 0$. Clearly we can form new variables $\left(z_i^+\right)^{*,new} = \left(z_i^+\right)^* - \Delta_i, \left(z_i^-\right)^{*,new} - \Delta_i$ where $\Delta_i \triangleq \min\left(z_i^{+*}, z_i^{-*}\right) > 0$ so that $\left(z_i^+\right)^{*,new}, \left(z_i^-\right)^{*,new}$ satisfy $\left(z_i^+\right)^* - \left(z_i^-\right)^* = \left(z_i^+\right)^{*,new} - \left(z_i^-\right)^{*,new}$ and all the other constraints are satisfied but achieve a lower objective function value by $2\Delta_i$. This is a contradiction. The same logic applies for $\left(w_j^+, w_j^-\right)$. □

The next theorem shows the correspondence between the solution sets of the DC-MCP (1), (5) and the formulations (6) and (7).

Theorem 3 *Consider any optimal solution* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *to (7) with corresponding optimal objective function value* f^*. *Then,*

a. (x^*, y^*) *solves the DC-MCP (6) if and only if* , $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *solves (7).*

b. $f^* = 0 \Leftrightarrow (x^*, y^*)$ *solve the DC-MCP (1), (5).*

Proof.

a. From Theorem 1, if (x^*, y^*) solves the DC-MCP (6) then they are feasible to that problem so that $z_i = H_i(x, y)$ for all $i \in I_x, w_j = H_j(x, y)$ for all $j \in I_y$. From Theorem 2 we see that $|z_i| = z_i^+ + z_i^-, |w_j| = w_j^+ + w_j^-$, since at most one of $\left(z_i^+, z_i^-\right)$ is nonzero and at most one of $\left(w_i^+, w_i^-\right)$ is nonzero. Thus, (7) can be rewritten equivalently as (6).

b. First, if $f^* > 0$ then there must exist an index $i \in I_x$ with $H_i(x^*, y^*) \neq 0$ or a $j \in I_y$ with $H_j(x, y) \neq 0$. This means that $\|H(x^*, y^*)\|_1 \neq 0$ or that (x^*, y^*) is not a continuous solution to (1). Hence, it's not a solution to the DC-MCP (1), (5). On the other hand, if $f^* = 0$ then we need to show that (x^*, y^*) is a solution to (1). But since the optimal objective function value to (7) is zero and the solution corresponding to this necessarily satisfies all the constraints in (7), we see that $z_i^* = H_i(x^*, y^*) = 0, \forall i \in I_x$ and $w_j^* = H_j(x^*, y^*) = 0, \forall j \in I_y$. which along with the integer restrictions satisfied means that (x^*, y^*) solves the DC-MCP, (5).

□

Remarks: See Example #2d for an illustrative demonstration of (7) and its correspondence to a DC-MCP solution.

2.3.2 The traditional case for $l_i = 0, u_i = +\infty, i \in I_x$

When we specify $l_i = 0,\ u_i = +\infty\ \forall i \in I_x$ corresponding to the traditional MCP and make a boundedness assumption, the resulting formulation is more efficient (less binary variables) as discussed next. First, we want to exclude cases 4 and 5:

$$\begin{aligned}
&\text{Case 4}: \quad l_i \le u_i < \ x_i - F_i(x, y) \Rightarrow z_i = H_i(x, y) = x_i - u_i \\
&\text{Case 5}: \quad l_i \le u_i = \ x_i - F_i(x, y) \Rightarrow z_i = H_i(x, y) = x_i - u_i = F_i(x, y)
\end{aligned}$$

There are several ways to exclude cases 4 and 5. For example, suppose that the following assumption is in force.

Assumption 1 There exists a finite $u_i^{\max} \in R_+$ such that $x_i - F_i(x, y) \le u_i^{\max}$ for all $x \in R_+^{n_x}$, $y \in R^{n_y}$.

Then, if u_i is selected greater than $u_i^{\max}$, we have

$$x_i - F_i(x, y) \le u_i^{\max} < u_i$$

so that cases 4 and 5 are not possible. Assumption 1 is mild but rules out functions like $F_i(x, y) = -\frac{1}{x_i}$ where $x_i - F_i(x, y) \to +\infty$ as $x_i \to 0$, which for any finite choice of u_i would not necessarily rule out cases 4 and 5. Another way to exclude cases 4 and 5 is to set $u_i = +\infty$ so that case 4 and case 5's conditions combined

$$l_i \le u_i = +\infty \le x_i - F_i(x, y)$$

are never true for finite x, y.[2] Thus, for specificity but without loss of generality, from here on we take $l_x = 0, u_x = +\infty$ so that the resulting three cases are:

[2] Here the extreme case when $x_i = 0$, of comparing $+\infty$ on both sides of the strict inequality $l_i \le u_i = +\infty < 0 - \left(-\frac{1}{0}\right) = +\infty$, when for example, $F_i(x, y) = -\frac{1}{x_i}$, is not considered.

Case 1: $x_i - F_i(x, y) < 0 = l_i \le u_i = \infty \Rightarrow z_i = H_i(x, y) = x_i$

Case 2: $0 = l_i < x_i - F_i(x, y) \le u_i = \infty \Rightarrow z_i = H_i(x, y) = F_i(x, y)$

Case 3: $0 = l_i = x_i - F_i(x, y) \le u_i = \infty \Rightarrow z_i = H_i(x, y) = F_i(x, y) = x_i$

To handle these mutually exclusive cases for the median function, we consider just the binary variables b_i. We have the following l_1 version of the problem where $l_i = 0, u_i = +\infty, \forall i \in I_x$.

$$\min_{x,y,z^+,z^-,w^+,w^-,b} f = \sum_{i \in I_x} \left(z_i^+ + z_i^-\right) + \sum_{j \in I_y}^{n} \left(w_j^+ + w_j^-\right) \tag{8a}$$
$$s.t. - Mb_i \le x_i - F_i(x, y) \le M(1 - b_i), \quad \forall i \in I_x \tag{8b}$$
$$-M\left(1 - b_i\right) \le z_i^+ - z_i^- - x_i \le M\left(1 - b_i\right), \quad \forall i \in I_x \tag{8c}$$
$$-M\left(b_i\right) \le z_i^+ - z_i^- - F_i(x, y) \le M\left(b_i\right), \quad \forall i \in I_x \tag{8d}$$
$$w_j^+ - w_j^- = F_j(x, y), \quad \forall j \in I_y \tag{8e}$$
$$x_i \in R_+, i \in I_x \backslash D_x \tag{8f}$$
$$x_i \in Z_+, i \in D_x \tag{8g}$$
$$y_j \in R, j \in I_y \backslash D_y \tag{8h}$$
$$y_j \in Z, j \in D_y \tag{8i}$$
$$b_i \in \{0, 1\}, \quad \forall i \in I_x \tag{8j}$$
$$z_i^+, z_i^- \ge 0, \quad \forall i \in I_x \tag{8k}$$
$$w_j^+, w_j^- \ge 0, \quad \forall j \in I_y \tag{8l}$$

It is clear that the formulation (8) is a special case of (7) with $\widetilde{b_i}$ fixed to 1 for all $i \in I_x$ since $x_i - F_i\left(x, y\right) < u_i = +\infty$ always. Consequently, (7c) is eliminated, (7d) is specialized to (8c), (7e) to (8d) and (7f) is left off as it is never active. The previous theorems all hold for (8) as described below in Corollary 1.

Corollary 1 *Consider any optimal solution* $(x^*, y^*, z^{+*}, z^{-*}, w^{+*}, w^{-^*}, b^*)$*to (8) with corresponding objective function value* f^**. Then,*

a. (x^*, y^*) *solves the DC-MCP (6) if and only if,* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*\right)$ *solves (8).*

b. $f^* = 0 \Leftrightarrow x^*, y^*$ *solves the DC-MCP (1), (5).*

Proof. Following the analysis of the proof of Theorem 1, we see that the only possible cases are

case	b_i	z_i	$H_i\left(x, y\right)$
1	1	x_i	x_i
2	0	$F_i\left(x, y\right)$	$F_i\left(x, y\right)$
$3a$	0	$F_i(x, y)$	$F_i\left(x, y\right) = x_i$
$3b$	1	x_i	$F_i\left(x, y\right) = x_i$

so that $z_i = H_i\left(x, y\right)\ \forall i \in I_x$ and similarly we have $w_j = H_j\left(x, y\right)\ \forall j \in I_y$. The logic of Theorem 2 also holds here so that the objective function (8a) corresponds to $\|H(x, y)\|_1$ and thus part a. follows. As for part b., the analysis is the same as in Theorem 3. $\square$

3 Numerical results and analysis

In the section, numerical results using both the formulations (7) and (8) are presented and discussed; all test problems are linear MCPs. As most problems in the literature use the traditional MCP formulation $(u_i = +\infty, l_j = -\infty)$, (8) is the default problem solved unless specified otherwise. In particular, the following illustrative examples are given.

Examples #1abcd: Small illustrative problems with a short discussion of projection on to the set of integer values.

Examples #2abcd: Larger illustrative problems showing that rounding doesn't work (#2abc). Modification of Example #2c to include more general lower and upper bounds $l_i \leq u_i$ (general case relating to (7)).

Examples #3abc: Energy production duopoly problem showing respectively, only continuous variables, integer variables, modification with different weights $\omega_p > 0$, for $\|H_p\|$ for each player p. Proof of a valid value for the disjunctive constant M for this problem in Theorem 4 and Theorem 5 characterizes all the integer solutions of this problem to gain insight.

Examples #4abc: Energy spatial price equilibrium problem with all continuous variables (Example #4a), all discrete variables (Example #4b), all discrete variables and logical, if-then, equity-enforcing constraints on the flow (Example #4c).

3.1 Small illustrative examples

Example #1a

To demonstrate some aspects of the above models, consider the following small but illustrative example. Assume only two nonnegative variables x_1, x_2 that are both constrained to be integers and the following linear complementarity problem (LCP) form of DC-MCP with

$$F(x) = q + Mx = \begin{pmatrix} -2 \\ 10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{9}$$

As M in the above example is a P-matrix [5], there is a unique solution for every right-hand side q. The unique solution to this LCP is $(x_1, x_2) = (1, 0)$ which does satisfy the integer restrictions. The formulation (8) specialized to this problem is the following taking $M = 1000$:

$$\min_{x,z^+,z^-,b} f = \sum_{i=1,2} \left(z_i^+ + z_i^-\right) \tag{10a}$$

$$s.t. -1000b_i \leq x_i - F_i(x,y) \leq 1000(1-b_i), i = 1,2 \tag{10b}$$

$$-1000\,(1-b_i) \leq z_i^+ - z_i^- - x_i \leq 1000\,(1-b_i)\,, i = 1,2 \tag{10c}$$

$$-1000\,(b_i) \leq z_i^+ - z_i^- - F_i(x,y) \leq 1000\,(b_i)\,, \quad \forall i \in I_x \tag{10d}$$

$$x_i \in Z_+, i = 1,2 \tag{10e}$$

$$b_i \in \{0,1\}\,, \quad \forall i = 1,2 \tag{10f}$$

$$z_i^+, z_i^- \geq 0, \quad \forall i = 1,2 \tag{10g}$$

The solution returned by CPLEX using GAMS is $(x_1^*, x_2^*) = (1,0), (b_1^*, b_2^*) = (0,1)$ corresponding to (case 2, case 1) as expected given that $x_1^* - F_1\left(x_1^*, x_2^*\right) = 1 - 0 = 1$ and $x_2^* - F_2\left(x_1^*, x_2^*\right) = 0 - 10 = -10$. As an integer solution to the LCP achieved, as expected the objective function $f = 0$ as well as $z_i = z_i^+ = z_i^- = 0, i = 1,2$.

Example #1b

Now consider a variation of the above DC-LCP with only the first component of q changing:

$$F(x) = q + Mx = \begin{pmatrix} -1 \\ 10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11}$$

Again, since M is a P-matrix, there is exactly one solution to the continuous relaxation which in this case is $(x_1, x_2) = (0.5, 0)$. Since this continuous relaxation of DC-MCP does not have an integer solution, the integer-constrained solution must have an objective function value greater than zero as shown earlier. In particular, the solution to (8) is $(x_1^*, x_2^*) = (0,0)$ with the associated optimal binary variables $(b_1^*, b_2^*) = (0,1)$ corresponding to (case 2, case 1) as expected given that $x_1^* - F_1\left(x_1^*, x_2^*\right) = 0 - (-1) = 1$ and $x_2^* - F_2\left(x_1^*, x_2^*\right) =$

$0 - 10 = -10$. What is interesting to note is why the solver chose an optimal objective function value of $f = 1$ with $z_1^* \triangleq z_1^{+*} - z_1^{-*} = 0 - 1 = -1, z_2^* \triangleq z_2^{+*} - z_2^{-*} = 0 - 0 = 0$ so that $f = |z_1| + |z_2| = 1 + 0 = 1$. Consider the first set of constraints in (10) and the possible values for b_1 and b_2 given that $x_1, x_2 \geq 0$.

First, given the form of the LCP, it is clear that the analysis decouples by i. If $b_1 = 0$, then with $z_i \triangleq z_i^+ - z_i^-$, the first three constraints in (10) (when $i = 1$) are:

$$0 \leq 1 - x_1 \leq 1000 \tag{12a}$$
$$-1000 \leq z_1 - x_1 \leq 1000 \tag{12b}$$
$$0 \leq z_1 - 2x_1 + 1 \leq 0 \tag{12c}$$

which means that $|z_1| = |2x_1 - 1|$ with $0 \leq x_1 \leq 1$. Clearly, at $x_1 = 0, |z_1| = 1$ is lowest corresponding to $z_1^* = -1 = z_1^{+*} - z_1^{-*} = 0 - 1$. A similar analysis but for $b_1 = 1$ gives $z_1 = x_1$ with $x_1 \geq 1$ so $|z_1| \geq 1$.

Considering b_2, we see that for $b_2 = 0$, $z_2 = 10x_2 + 10$ with $-\frac{10}{9} \geq x_2$ which is infeasible since $x_2 \geq 0$. For $b_2 = 1$, $z_2 = x_2$ with $x_2 \geq \max\left\{-\frac{10}{9}, 0\right\} = 0$. Thus, $|z_2| = 0$.

The upshot of this analysis is that the formulation has found the projection of the continuous-valued LCP solution set on to the set of integers for x_1, x_2 which is to take both values equal to zero. Given the nonnegativity restrictions on the variables, there are four neighboring integer points to consider: $(x_1, x_2) = \{(0,0),(0,1),(1,0),(1,1)\}$. In the L_1 sense, $\{(0,0),(1,0)\}$ are closest projections to $(0.5, 0)$. It is interesting to note that $(x_1, x_2) = (1, 0)$ is an alternate solution achieving the same objective function value of 1 but just with different values $z_1^* = 1 = z_1^{+*} - z_1^{-*} = 1 - 0$ and $z_2^* = 0 = z_2^{+*} = z_2^{-*}$.

Example #1c

Now consider a variation of the above DC-LCP from Example #1a but adding one free variable y_1:

$$F(x,y) = \begin{pmatrix} -2 \\ 10 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix} \tag{13}$$

This problem like the previous two, by design decouples by each variable. Thus, assuming that all the variables are integer-valued, the solution should be the same for x_1, x_2 from before and the only new part is to solve the third row as an equation relating to the free variable y_1. Clearly, the solution has $y_1 = 5$.

Example #1d

Now consider a variation of the above DC-LCP from Example #1b but adding one free variable y_1:

$$F(x,y) = \begin{pmatrix} -1 \\ 10 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix} \tag{14}$$

This problem like the previous three, by design decouples by each variable. Thus solution reported by GAMS is $(x_1, x_2, y_1) = (1, 0, 5)$ with (x_1, x_2) matching the alternate solution described in example #1b along with $y_1 = 5$.

3.2 Larger generated examples

To test that formulation 1 is working, several larger, generated examples were generated as follows. First, a given solution $(\overline{x}, \overline{y})$ was selected ahead of time. Then, using this solution, a generated square matrix A was generated. Conformal with the dimensions of x and y, the matrix A has submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Example #2a

The first example of this second group was to take A_{11} and A_{22} both P-matrices with $A_{12} = 0 = A_{21}$ so that the resulting matrix A is a P-matrix, hence unique solutions for all right-hand sides q. In this first test, $A_{11} = diag(1, 2, \ldots, 10)$, $A_{11} = diag(11, 12, \ldots, 20)$. Here $I_x = D_x = \{1, \ldots, 10\}, I_y = D_y = \{1, \ldots, 10\}$. The given integer solution to test is:

$$\begin{array}{rcl} \overline{x}^T & = & \begin{pmatrix} 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \overline{y}^T & = & \begin{pmatrix} -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{pmatrix} \end{array}$$

With the vector $q^T = \begin{pmatrix} q_x^T & q_y^T \end{pmatrix}$ and

$$\begin{array}{rcl} q_x^T & = & \begin{pmatrix} -5 & -8 & -9 & -8 & -5 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\ q_y^T & = & \begin{pmatrix} 55 & 48 & 39 & 28 & 15 & 0 & -17 & -36 & -57 & -80 \end{pmatrix} \end{array}$$

it can easily be verified that the given $(\overline{x}, \overline{y})$ is an integer-valued solution to the corresponding DC-MLCP so that the MIP has a zero objective function value. For the resulting tests $M = 1,000$ and $M = 2000$ were used (the latter one as a check). The resulting binary variable values are: $b_i = 0, i = 1, \ldots, 5$, corresponding to case 2, $b_i = 1, i = 6, \ldots, 10$ with $b_6, \ldots, b_9$ relating to case 3 (could also be $b_i = 0$) and b_{10} corresponding to case 1.

Example #2b

This is the same as Example #2a except that the right-hand side vector q is changed for $q_{x1} = -5.1$ and $q_{y3} = 48.1$ to induce the non-integer, continuous-valued MLCP solution

$$\begin{array}{rcl} \overline{x}^T & = & \begin{pmatrix} 5.1 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \overline{y}^T & = & \begin{pmatrix} -5 & -4 & -3.7 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{pmatrix} \end{array}$$

with $I_x = D_x = \{1, \ldots, 10\}$. The DC-MLCP solution is the following (bolded items are different from the continuous solution)

$$\begin{array}{rcl} \overline{x}^T & = & \begin{pmatrix} \mathbf{6} & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \overline{y}^T & = & \begin{pmatrix} -5 & -4 & -\mathbf{4} & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{pmatrix} \end{array}$$

with an optimal objective function value of 4.8 representing $z_1^+ = 0.9, z_1^- = 0, |z_1| = 0.9$, $w_3^+ = 0.9, w_3^- = 3.9, |w_3| = 3.9$ and all other variable values equal to zero. It is interesting to note that in this case, the DC-MLCP solution is obtained from rounding up the fractional x_1 value and rounding up (in absolute value) the y_3 value given that the system under consideration easily decouples by variable.

Example #2c

This is the same as Example #2a except that the right-hand side vector q is changed to have more fractional values than Example #2b. Here

$$\begin{array}{rcl} q_x^T & = & \begin{pmatrix} -5.1 & -8.4 & -11.4 & -8.4 & -8 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\ q_y^T & = & \begin{pmatrix} 61.6 & 48 & 48.1 & 32.2 & 16.5 & 0 & -25.5 & -52.2 & -57 & -80 \end{pmatrix} \end{array}$$

with a non-integer, continuous-valued MLCP solution

$$\begin{array}{rcl} \overline{x}^T & = & \begin{pmatrix} 5.1 & 4.2 & 3.8 & 2.1 & 1.6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \overline{y}^T & = & \begin{pmatrix} -5.6 & -4.0 & -3.7 & -2.3 & -1.1 & 0 & 1.5 & 2.9 & 3.0 & 4.0 \end{pmatrix}. \end{array}$$

The DC-MLCP solution is

$$\begin{array}{rcl} \overline{x}^T & = & \left(\begin{array}{cccccccccc} \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ \overline{y}^T & = & \left(\begin{array}{cccccccccc} \mathbf{-6} & \mathbf{-4} & \mathbf{-4} & \mathbf{-2} & \mathbf{-1} & 0 & \mathbf{1} & \mathbf{3} & 3 & 4 \end{array} \right) \end{array}$$

(with bolded items corresponding to different values than the continuous solution) with nonzero values $z_1^+ = 0.9, z_2^+ = 1.6, z_3^+ = 0.6, z_4^+ = 3.0, z_5^+ = 2.0$, and $w_1^- = 4.4, w_3^- =, 3.9, w_4^+ = 4.2, w_5^+ = 1.5, w_7^+ = 8.5, w_8^+ = 1.8$, so that $\sum_i |z_i| = 8.1, \sum_j |w_j| = 24.3$ with an optimal objective function value of 32.4. It is interesting to note that in this case, the DC-MLCP solution is not obtained from just rounding up (in absolute value) the fractional continuous values. For example, y_4 is -2.3 in the continuous solution but -2 in the DC-MLCP one and similarly for y_5, y_7.

Example #2d, more general bounds $l_i \leq u_i$

In this problem we demonstrate how general bounds $l_i \leq u_i$ can be used in the DC-MCP formulation for the x variables and $l_j = -\infty, u_j = +\infty$ for the y variables (unchanged).

First, to verify that the adjusted GAMS code from Example #2c is working for this more generalized problem, the case where $l_i = 0, u_i = +\infty$ was run. With the same (x, y) continuous solution as reported in Example #2c the associated F_i terms were:

$$\begin{array}{rcl} F_x^T & = & \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right) \\ F_y^T & = & \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \end{array}$$

consistent with the definition of the more general, bounded MCP. (Here $l_i = 0$, $u_i = 1000$, $M = 10000$).

Next, given the solution to Example #2c, $l_i = 1, u_i = 5\ \forall i \in I_x$ was selected to induce differences with the traditionally bounded MCP.

A continuous solution to this more general-bounded MCP is the following ($M = 10000$):

$$\begin{array}{rcl} \overline{x}^T & = & \left(\begin{array}{cccccccccc} 5 & 4.2 & 3.8 & 2.1 & 1.6 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \right) \\ \overline{y}^T & = & \left(\begin{array}{cccccccccc} -5.6 & -4 & -3.7 & -2.3 & -1.1 & 0 & 1.5 & 2.9 & 3 & 4 \end{array} \right) \end{array}$$

and the corresponding F_i values are:

$$\begin{array}{rcl} F_x^T & = & \left(\begin{array}{cccccccccc} \mathbf{-0.1} & 0 & 0 & 0 & 0 & 7 & 8 & 9 & 10 & 11 \end{array} \right) \\ F_y^T & = & \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \end{array}$$

Notice that different from the traditional approach, since $x_1 = u_1 = 5$, the corresponding F_1 is nonpositive (shown in bold). Since $1 = l_i < x_1 < u_1 = 5, i = 2, 3, 4, 5$ the corrsponding F_i values equal zero. Lastly, for $i = 6, 7, 8, 9, 10$ since $x_i = l_i = 1$, the corresponding F_i values are nonnegative.

For the integer-restricted version of this problem the following solution was generated ($M = 10000$):

$$\begin{array}{rcl} \overline{x}^T & = & \left(\begin{array}{cccccccccc} \mathbf{5} & \mathbf{4} & 4 & \mathbf{2} & 2 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \right) \\ \overline{y}^T & = & \left(\begin{array}{cccccccccc} -6 & -4 & -4 & -2 & -1 & 0 & 1 & 3 & 3 & 4 \end{array} \right). \end{array}$$

which differs from Example #2c's integer solution as shown by the bolded x terms (the y terms are the same). The objective function is 26.7 with corresponding F values as follows:

$$\begin{array}{rcl} F_x^T & = & \left(\begin{array}{cccccccccc} -0.1 & -0.4 & 0.6 & -0.4 & 2 & 7 & 8 & 9 & 10 & 11 \end{array} \right) \\ F_y^T & = & \left(\begin{array}{cccccccccc} -4.4 & 0 & -3.9 & 4.2 & 1.5 & 0 & -8.5 & 1.8 & 0 & 0 \end{array} \right). \end{array}$$

showing for example, that for $i = 1$, F_i and x_i satisfy the defining condition: $x_i = u_i \Rightarrow F_i(x, y) \leq 0$. For $i = 6, 7, 8, 9, 10$ the defining condition $x_i = l_i \Rightarrow F_i(x, y) \geq 0$ is also met. However, the remaining condition, that $l_i < x_i < u_i \Rightarrow F_i(x, y) = 0$ is not met for the remaining x_i and corresponding F_i terms, $i = 2, 3, 4, 5$ as well as most of the y_j and F_j terms due to the imposition of the integer restrictions on both the x and the y variables. When the y variables are relaxed to be only continuous-valued, the resulting optimal objective function is 2.4 much lower and consistent with less restrictions on the y variables and the solution is:

$$\begin{array}{rcl} \overline{x}^T & = & \left(5 \quad \mathbf{4} \quad 4 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \right) \\ \overline{y}^T & = & \left(\mathbf{-5.6} \quad \mathbf{-4} \quad \mathbf{-3.7} \quad \mathbf{-2.3} \quad \mathbf{-1.1} \quad 0 \quad \mathbf{1.5} \quad \mathbf{2.9} \quad 3 \quad 4 \right) \end{array}$$

with the x variables unchanging but most of the y variables different from the previous answer just stated above. The corresponding F values are:

$$\begin{array}{rcl} F_x^T & = & \left(-0.1 \quad -0.4 \quad 0.6 \quad -0.4 \quad 2 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \right) \\ F_y^T & = & \left(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right). \end{array}$$

showing that for all the y terms the general MCP conditions are met. The same is almost true for the x terms with the exception of $i = 2, 3, 4, 5$ due to the integer restrictions on x. This example shows some nuanced answers when the general bounded MCP problem is solved with integer restrictions.

3.3 Energy production, capacitated duopoly example with selected complementarity relaxation weighting

For this problem, we have two energy producers $p = 1, 2$ with the pth optimization problem as follows:

$$\max_{q_p} p\left(\sum_p q_p\right) q_p - c_p\left(q_p\right) \tag{15a}$$

$$s.t.\ 0 \leq q_p \leq q_p^{\max} \qquad (\lambda_p) \tag{15b}$$

where $q_p^{\max} > 0$ represents the maximum energy production for producer p (e.g., 10 MW) and λ_p is the associated capacity price (dual variable). As long as the objective function $f(q_p) = p\left(\sum_p q_p\right) q_p - c\left(q_p\right)$ is concave in q_p, since the constraints are linear, the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient for optimality [13]. Taking the inverse energy market demand function $p\left(\sum_p q_p\right)$ to be linear, i.e., $p\left(\sum_p q_p\right) = \alpha - \beta\left(\sum_p q_p\right)$ for constants $\alpha, \beta > 0$ and the energy production cost function $c_p\left(q_p\right) = \gamma_p q_p$ for $\gamma_p > 0$ means that the objective function is concave. The associated necessary and sufficient KKT conditions are to find $q_1, q_2, \lambda_1, \lambda_2$:

$$0 \leq \begin{pmatrix} \gamma_1 - \alpha \\ \gamma_2 - \alpha \\ q_1^{\max} \\ q_2^{\max} \end{pmatrix} + \begin{pmatrix} 2\beta & \beta & 1 & 0 \\ \beta & 2\beta & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \perp \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \geq 0 \tag{16}$$

which is an LCP in four x variables (nonnegatively constrained). The related H function for this problem is:

$$\left\|H(q_1, q_2, \lambda_1, \lambda_2)\right\|_1 \triangleq \sum_{i=1,4} \left|H_i(q_1, q_2, \lambda_1, \lambda_2)\right| = 0 \Leftrightarrow \left|H_i(q_1, q_2, \lambda_1, \lambda_2)\right| = 0 \text{ for all } i$$

Thus, an equivalent objective function that could be used would be

$$\sum_{i=1,4} \omega_i \left|H_i(q_1, q_2, \lambda_1, \lambda_2)\right| \tag{17}$$

with the positive weights ω_i chosen to give preference to the four elements of H_i. Alternatively, to allow specialized levels of complementarity relaxation, constraints of the form

$$\left|H_i(q_1, q_2, \lambda_1, \lambda_2)\right| \leq \varepsilon_i \tag{18}$$

could also be added for complementarity tolerances ε_i or in combination with the weighted objective shown just above. This could be accomplished via the constraints $|z_i| \leq \varepsilon_i, |w_j| \leq \varepsilon_j$ given the earlier correspondence that was shown with the H function. These ideas are explored in the variations of Example #3 below. First, we prove two results for this discretely constrained duopoly problem which give insight to this and other problems.

As mentioned above the constant M in the DC-MCP formulation (8) can be determined if the problem specifics are considered. The next result determines valid values for M for the duopoly problem above.

Theorem 4 *If $\beta \geq \frac{1}{2}, \alpha > 0, q_i^{\max} > 0,\ \gamma_i > 0, i = 1, 2$, then the following is a valid value for the constant M in (8) applied to the capacitated duopoly problem (16).*

$$M = \alpha + \max_{i=1,2} \{q_i^{\max}\}$$

Proof. To determine the constant M in (8), it is necessary to bound the expressions

$$\begin{aligned} x_i - F_i(x), i &\in I_x \\ z_i - x_i, i &\in I_x \\ z_i - F_i(x), i &\in I_x \end{aligned}$$

noting that only nonnegative variables denoted by x_i are present in this problem. By construction, both the energy production variables q_1, q_2 have upper bounds, respectively given as $q_1^{\max}, q_2^{\max}$. Thus, it suffices to find upper bounds for the nonnegative multipliers λ_1, λ_2. If $\lambda_i > 0$ then the corresponding inequality must hold as an equality. Given the assumption about positive values for $q_i^{\max}$, this means that

$$\lambda_i > 0 \Rightarrow q_i = q_i^{\max} > 0, i = 1, 2$$

which in turn means that the first two inequalities in (16) must hold as equalities or that

$$\lambda_1 = \alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max} \tag{19a}$$
$$\lambda_2 = \alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max} \tag{19b}$$

as long as $\alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max} > 0$ and $\alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max} > 0 \Leftrightarrow \alpha > \max\left\{\gamma_1 + 2\beta q_1^{\max} + \beta q_2^{\max}, \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max}\right\}$. In that case

$$\lambda_i \leq \max\{\alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max}, \alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max}\} \tag{20a}$$
$$= \alpha + \max\{-\gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max}, -\gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max}\} \tag{20b}$$
$$< \alpha \tag{20c}$$

since $\beta > 0, \gamma_i > 0, q_i^{\max} > 0, i = 1, 2$. On the other hand, if either $\alpha \leq \{\gamma_1 + 2\beta q_1^{\max} + \beta q_2^{\max}\}$ or $\alpha \leq \{\gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max}\}$ then the case $\lambda_1 > 0, \lambda_2 > 0$ is not possible.

If $\lambda_1 = 0, \lambda_2 > 0 \Rightarrow q_2 = q_2^{\max} > 0$, then

$$\lambda_2 = \alpha - \gamma_2 - \beta q_1 - 2\beta q_2^{\max} < \alpha \tag{21}$$

If $\lambda_2 = 0,\ \lambda_1 > 0 \Rightarrow q_1 = q_1^{\max} > 0$, then

$$\lambda_1 = \alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2 < \alpha \tag{22}$$

If $\lambda_1 = 0, \lambda_2 = 0$ then any positive upper bound will do. Thus, $\lambda_i < \alpha$ for each of these cases. Consequently, a bound on all the $x_i, i \in I_x$ is $\max\{\alpha, q_1^{\max}, q_1^{\max}\}$. The terms $x_i - F_i(x)$ are of the form:

$$\left\{ \begin{array}{ll} -\gamma_i + \alpha + (1 - 2\beta) q_i - \beta q_j - \lambda_i & i \neq j, i = 1, 2 \\ \lambda_i - q_i^{\max} + q_i & i = 1, 2 \end{array} \right.$$

which in light of the assumption that $\beta \geq \frac{1}{2}$ means that

$$\begin{cases} < \alpha & i \neq j, i = 1,2 \\ < \alpha + q_i^{\max} & i = 1,2 \end{cases}$$
$$< \quad \alpha + \max_{i=1,2} \{q_i^{\max}\}$$

As $x_i \geq 0$ and

$$z_i = mid\,(0, +\infty, x_i - F_i\,(x)) = \max\,(0, x_i - F_i\,(x)) \leq \alpha + \max_{i=1,2} \{q_i^{\max}\}$$

Since $x_i \geq 0$ we have

$$z_i - x_i \leq \alpha + \max_{i=1,2} \{q_i^{\max}\}$$

Lastly, as $F_i\,(x) \geq 0$,

$$z_i - F_i\,(x) \leq \alpha + \max_{i=1,2} \{q_i^{\max}\}$$

Thus, a valid value for M in (8) is $\alpha + \max_{i=1,2} \{q_i^{\max}\}$. □

For this particular problem, as shown in the theorem below, we can completely characterize the set of continuous as well as integer solutions to the capacitated duopoly MCP. Such an analysis is valuable not only for this problem but for gaining insight into other similar ones.

Theorem 5 *If $\beta > 0, q_i^{\max} > 0, i = 1,2$, then the following is a characterization of the solution set to (16) with $(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*)^T =$*

$$\left\{ \begin{array}{l} \textit{Case 1:} \begin{pmatrix} \frac{-2\gamma_1+\gamma_2+\alpha}{3\beta} \\ \frac{\gamma_1-2\gamma_2+\alpha}{3\beta} \\ 0 \\ 0 \end{pmatrix} \\ \textit{if } \alpha \in (2\gamma_1 - \gamma_2, 2\gamma_1 - \gamma_2 + 3\beta q_1^{\max}) \cap (-\gamma_1 + 2\gamma_2, -\gamma_1 + 2\gamma_2 + 3\beta q_2^{\max}) \end{array} \right\} \tag{23a}$$

$$\left\{ \begin{array}{l} \textit{Case 2:} \begin{pmatrix} \frac{-\gamma_1+\alpha-\beta q_2^{\max}}{2\beta} \\ q_2^{\max} \\ 0 \\ \frac{\gamma_1-2\gamma_2+\alpha-3\beta q_2^{\max}}{2} \end{pmatrix} \\ \textit{if } \alpha \in (\gamma_1 + \beta q_2^{\max}, \gamma_1 + \beta q_2^{\max} + 2\beta q_1^{\max}) \cap [-\gamma_1 + 2\gamma_2 + 3\beta q_2^{\max}, \infty) \end{array} \right\} \tag{23b}$$

$$\left\{ \begin{array}{l} \textit{Case 3:} \begin{pmatrix} q_1^{\max} \\ \frac{-\gamma_2+\alpha-\beta q_1^{\max}}{2\beta} \\ \frac{-2\gamma_1+\gamma_2+\alpha-3\beta q_1^{\max}}{2} \\ 0 \end{pmatrix} \\ \textit{if } \alpha \in (\gamma_2 + \beta q_1^{\max}, \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max}) \cap [\, 2\gamma_1 - \gamma_2 + 3\beta q_1^{\max}, \infty) \end{array} \right\} \tag{23c}$$

$$\left\{ \begin{array}{l} \textit{Case 4:} \begin{pmatrix} q_1^{\max} \\ q_2^{\max} \\ \alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max} \\ \alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max} \end{pmatrix} \\ \textit{if } \alpha \geq \max\{\gamma_1 + 2\beta q_1^{\max} + \beta q_2^{\max}, \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max}\} \end{array} \right\} \tag{23d}$$

$$\left\{ \begin{array}{l} \textit{Case 5:} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \textit{if } \alpha = \gamma_1 = \gamma_2 \end{array} \right\} \tag{23e}$$

$$\left\{ \begin{array}{l} \textit{Case 6:} \begin{pmatrix} 0 \\ \frac{\alpha-\gamma_2}{2\beta} \\ 0 \\ 0 \end{pmatrix} \\ \textit{if } \ \alpha \in (\gamma_2, \gamma_2 + 2\beta q_2^{\max}), \alpha \leq 2\gamma_1 - \gamma_2 \end{array} \right\} \tag{23f}$$

$$\left\{\begin{array}{l} \textit{Case 7:} \begin{pmatrix} 0 \\ q_2^{\max} \\ 0 \\ \alpha-\gamma_2-2\beta q_2^{\max} \end{pmatrix} \\ \textit{if } \alpha \geq \gamma_2+2\beta q_2^{\max}, q_2^{\max} \geq \frac{\alpha-\gamma_1}{\beta} \end{array}\right\} \tag{23g}$$

$$\left\{\begin{array}{l} \textit{Case 8:} \begin{pmatrix} \frac{\alpha-\gamma_1}{2\beta} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \textit{if } 2\gamma_2-\gamma_1 \geq \alpha, \alpha \in (\gamma_1, \gamma_1+2\beta q_1^{\max}) \end{array}\right\} \tag{23h}$$

$$\left\{\begin{array}{l} \textit{Case 9:} \begin{pmatrix} q_1^{\max} \\ 0 \\ \alpha-\gamma_1-2\beta q_1^{\max} \\ 0 \end{pmatrix} \\ \textit{if } \alpha \in [\gamma_1+2\beta q_1^{\max}, \gamma_2+\beta q_1^{\max}] \end{array}\right\} \tag{23i}$$

Furthermore, these solutions are integer-valued as long as:

$$\begin{cases} \textit{Case 1:} \left\{\frac{-2\gamma_1+\gamma_2+\alpha}{3\beta}, \frac{\gamma_1-2\gamma_2+\alpha}{3\beta}\right\} \in Z_+ \\ \textit{Case 2:} \left\{\frac{-\gamma_1+\alpha-\beta q_2^{\max}}{2\beta}, q_2^{\max}, \frac{\gamma_1-2\gamma_2+\alpha-3\beta q_2^{\max}}{2}\right\} \in Z_+ \\ \textit{Case 3:} \left\{q_1^{\max}, \frac{-\gamma_2+\alpha-\beta q_1^{\max}}{2\beta}, \frac{-2\gamma_1+\gamma_2+\alpha-3\beta q_1^{\max}}{2}\right\} \in Z_+ \\ \textit{Case 4:} \{q_1^{\max}, q_2^{\max}, \alpha-\gamma_1-2\beta q_1^{\max}-\beta q_2^{\max}, \alpha-\gamma_2-\beta q_1^{\max}-2\beta q_2^{\max}\} \in Z_+ \\ \textit{Case 5: always integer-valued} \\ \textit{Case 6:} \frac{\alpha-\gamma_2}{2\beta} \in Z_+ \\ \textit{Case 7:} \{q_2^{\max}, \alpha-\gamma_2-2\beta q_2^{\max}\} \in Z_+ \\ \textit{Case 8:} \frac{\alpha-\gamma_1}{2\beta} \in Z_+ \\ \textit{Case 9:} \{q_1^{\max}, \alpha-\gamma_1-2\beta q_1^{\max}\} \in Z_+ \end{cases}$$

Proof. The capacitated duopoly MCP in (16) has four variables: $q_1, q_2, \lambda_1, \lambda_2$. There are nine cases to consider based on the three states for each of the production variables $q_i, i = 1, 2$ since the Lagrange multipliers λ_1, λ_2 are easily determined from the production values. The nine cases are based on considering the following mutually exclusive cases for each i: **a)** $0 = q_i < q_i^{\max}$, **b)** $0 < q_i < q_i^{\max}$, **c)** $0 < q_i = q_i^{\max}$ where it is assumed that $q_i^{\max} > 0$.

Case 1: $0 < q_1 < q_1^{\max}, 0 < q_2 < q_2^{\max}$ In this case $\lambda_1 = \lambda_2 = 0$ and (16) reduces to solving the system

$$\begin{pmatrix} 2\beta & \beta \\ \beta & 2\beta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \alpha-\gamma_1 \\ \alpha-\gamma_2 \end{pmatrix} \tag{24}$$

which in light of the assumption that $\beta > 0$ gives the unique solution

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = \left(\frac{-2\gamma_1+\gamma_2+\alpha}{3\beta}, \frac{\gamma_1-2\gamma_2+\alpha}{3\beta}, 0, 0\right)$$

as long as $0 < \frac{-2\gamma_1+\gamma_2+\alpha}{3\beta} < q_1^{\max} \Leftrightarrow \alpha \in (2\gamma_1-\gamma_2, 2\gamma_1-\gamma_2+3\beta q_1^{\max})$ and $0 < \frac{\gamma_1-2\gamma_2+\alpha}{3\beta} < q_2^{\max} \Leftrightarrow \alpha \in (-\gamma_1+2\gamma_2, -\gamma_1+2\gamma_2+3\beta q_2^{\max})$

which is integer as long as both $\frac{-2\gamma_1+\gamma_2+\alpha}{3\beta}, \frac{\gamma_1-2\gamma_2+\alpha}{3\beta}$ are integers.

Case 2: $0 < q_1 < q_1^{\max}, 0 < q_2 = q_2^{\max}$ The first interval for q_1 insures that $\lambda_1 = 0$. Solving the first two equations for q_1 and λ_2 gives

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = \left(\frac{-\gamma_1 + \alpha - \beta q_2^{\max}}{2\beta}, q_2^{\max}, 0, \frac{\gamma_1 - 2\gamma_2 + \alpha - 3\beta q_2^{\max}}{2}\right)$$

as long as $0 < \frac{-\gamma_1+\alpha-\beta q_2^{\max}}{2\beta} < q_1^{\max} \Leftrightarrow \alpha \in (\gamma_1 + \beta q_2^{\max}, \gamma_1 + \beta q_2^{\max} + 2\beta q_1^{\max})$ and $0 \le \frac{\gamma_1-2\gamma_2+\alpha-3\beta q_2^{\max}}{2}$ $\Leftrightarrow \alpha \in [-\gamma_1 + 2\gamma_2 + 3\beta q_2^{\max}, \infty)$.

Case 3: $0 < q_1 = q_1^{\max}, 0 < q_2 < q_2^{\max}$ Necessarily $\lambda_2 = 0$. A similar analysis to case 2 reveals the unique solution of

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = \left(q_1^{\max}, \frac{-\gamma_2 + \alpha - \beta q_1^{\max}}{2\beta}, \frac{-2\gamma_1 + \gamma_2 + \alpha - 3\beta q_1^{\max}}{2}, 0\right)$$

as long as $0 < \frac{-\gamma_2+\alpha-\beta q_1^{\max}}{2\beta} < q_2^{\max} \Leftrightarrow \alpha \in (\gamma_2 + \beta q_1^{\max}, \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max})$ and $0 \le \frac{-2\gamma_1+\gamma_2+\alpha-3\beta q_1^{\max}}{2}$ $\Leftrightarrow \alpha \in [2\gamma_1 - \gamma_2 + 3\beta q_1^{\max}, \infty)$.

Case 4: $0 < q_1 = q_1^{\max}, 0 < q_2 = q_2^{\max}$ So the first two constraints show that

$$\begin{aligned} 2\beta q_1^{\max} + \beta q_2^{\max} + \lambda_1 &= \alpha - \gamma_1 \\ \beta q_1^{\max} + 2\beta q_2^{\max} + \lambda_2 &= \alpha - \gamma_2 \end{aligned}$$

or that $\lambda_1 = \alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max}, \lambda_2 = \alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max}$. But both $\lambda_i, i = 1, 2$ must be nonnegative so this means that

$$\begin{aligned} \alpha &\ge \gamma_1 + 2\beta q_1^{\max} + \beta q_2^{\max} \\ \alpha &\ge \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max} \end{aligned}$$

Thus, as long $\alpha \ge \max\{\gamma_1 + 2\beta q_1^{\max} + \beta q_2^{\max}, \gamma_2 + \beta q_1^{\max} + 2\beta q_2^{\max}\}$, then

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = (q_1^{\max}, q_2^{\max}, \alpha - \gamma_1 - 2\beta q_1^{\max} - \beta q_2^{\max}, \alpha - \gamma_2 - \beta q_1^{\max} - 2\beta q_2^{\max})$$

Case 5: $0 = q_1 < q_1^{\max}, 0 = q_2 < q_2^{\max}$ Necessarily we must have $\lambda_1 = \lambda_2 = 0$. Combined with zero values for the production variables and the fact that the first two inequalities must hold as equations, this can only happen if: $\alpha = \gamma_1 = \gamma_2$. If so, then the solution is $(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = (0, 0, 0, 0)$.

Case 6: $0 = q_1 < q_1^{\max}, 0 < q_2 < q_2^{\max}$ Necessarily we must have $\lambda_1 = \lambda_2 = 0$. The unique solution is

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = \left(0, \frac{\alpha - \gamma_2}{2\beta}, 0, 0\right)$$

as long as $0 < \frac{\alpha-\gamma_2}{2\beta} < q_2^{\max} \Leftrightarrow \alpha \in (\gamma_2, \gamma_2 + 2\beta_2^{\max})$ and $2\gamma_1 - \gamma_2 \ge \alpha$.

Case 7: $0 = q_1 < q_1^{\max}, 0 < q_2 = q_2^{\max}$ Necessarily we must have $\lambda_1 = 0$. The first two constraints imply that

$$\begin{aligned} \beta q_2^{\max} &\ge \alpha - \gamma_1 \Leftrightarrow q_2^{\max} \ge \frac{\alpha - \gamma_1}{\beta} \\ 2\beta q_2^{\max} + \lambda_2 &= \alpha - \gamma_2 \Leftrightarrow \lambda_2 = \alpha - \gamma_2 - 2\beta q_2^{\max} \end{aligned}$$

Thus,

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = (0, q_2^{\max}, 0, \alpha - \gamma_2 - 2\beta q_2^{\max})$$

as long as $0 \le \alpha - \gamma_2 - 2\beta q_2^{\max} \Leftrightarrow \alpha \ge \gamma_2 + 2\beta q_2^{\max}$ and $q_2^{\max} \ge \frac{\alpha-\gamma_1}{\beta}$.

Case 8: $0 < q_1 < q_1^{\max}, 0 = q_2 < q_2^{\max}$ Necessarily we must have $\lambda_1 = \lambda_2 = 0$. Consequently, $q_1 = \frac{\alpha-\gamma_1}{2\beta}$ as long as $2\gamma_2 - \gamma_1 \ge \alpha$, $\alpha \in (\gamma_1, \gamma_1 + 2\beta q_1^{\max})$. Thus,

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = \left(\frac{\alpha - \gamma_1}{2\beta}, 0, 0, 0\right)$$

Case 9: $0 < q_1 = q_1^{\max}, 0 = q_2 < q_2^{\max}$ Necessarily we must have $\lambda_2 = 0$. Solving the first equaion shows that $\lambda_1 = \alpha - \gamma_1 - 2\beta q_1^{\max}$ as long as $\alpha - \gamma_1 - 2\beta q_1^{\max} \geq 0 \Leftrightarrow \alpha \geq \gamma_1 + 2\beta q_1^{\max}$. The second inequality shows that $q_1^{\max} \geq \frac{\alpha-\gamma_2}{\beta}$ and thus if these conditions are met we have

$$(q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = (q_1^{\max}, 0, \alpha - \gamma_1 - 2\beta q_1^{\max}, 0)$$

□

To numerically demonstrate the above nine cases along with the resulting continuous-MCP solutions, consider the table below.

Table 1: Results for the capacitated duopoly problem

case	α	β	γ_1	γ_2	$q_1^{\max}$	$q_2^{\max}$	q_1^*	q_2^*	λ_1^*	λ_2^*
1	10	2	4	4	7	7	1	1	0	0
1	10	2.5	4	4	7	7	0.8	0.8	0	0
2	301	10	101	1	10	10	5	10	0	50
3	10	2.5	3	4	0.8	3.2	0.8	0.8	1	0
4	10	2	3	4	0.1	0.1	0.1	0.1	6.4	5.4
5	1	10	1	1	0.1	0.2	0	0	0	0
6	19	10	20	1	1	1	0	0.9	0	0
7	4	2	5	1	2	0.1	0	0.1	0	2.6
8	5	2	3	4	8	0.1	0.5	0	0	0
9	4	2	3	4	0.1	8	0.1	0	0.6	0

Below, three of these cases are discussed in more detail, specifically, the first two rows of numbers of this table, respectively, Examples #3a, #3b and the fourth row of numbers, Example #3c.

Remarks

1. The above theorem provides insight into the difficulty of finding integer-valued solutions to MCPs in general. Even when all the data are integers, the solution may not be integer-valued itself. To see this compare the first row of numbers (case 1) where the solution is integer production levels $q_i, i = 1, 2$ as well as zero for the dual variables $\lambda_i, i = 1$ with the last row of numbers (case 9) which despite having integer data has non-integer solutions.
2. In the case of linear programs such as the classical transportation problem, the total unimodularity property of the coefficient matrix is enough to ensure integer-valued answers as long as the data in the problem are integers.
3. The equivalent result to ensure that the capacitated duopoly problem has integer MCP solutions follows from the second part of the results of the theorem above. If we assume $\beta = \frac{1}{6}$ and since β is the only parameter in the denominator of the solutions for some of the cases, we can simplify the conditions to ensure integer solutions. Namely, since case 1 has in the denominator 3β, cases 2, 3, 6, and 8 have 2β, we see that if $\beta = \frac{1}{6}$, then the solutions are all integer as long all the other data $(\alpha, \gamma_1, \gamma_2, q_1^{\max}, q_2^{\max})$ are integer-valued, $\beta q_1^{\max}, \beta q_2^{\max}$ are integers (from cases 2, 3, 4, 7, and 9), $\{\gamma_1 - 2\gamma_2 + \alpha - 3\beta q_2^{\max}, -2\gamma_1 + \gamma_2 + \alpha - 3\beta q_1^{\max}\}$ are even numbers (from cases 2 and 3). These are only sufficient conditions. For example let $\gamma_1 = \gamma_2 = 1, q_1^{\max} = q_2^{\max} = 10$ and α any integer at least 6. The table below verifies that the solutions are integer-valued but that $\beta q_1^{\max}, \beta q_2^{\max}$ are not integers (they are $\frac{10}{6}$). The reason the solutions are integers is that it is case 4 each time and a weaker set of conditions are needed, namely that $-\beta(2q_1^{\max} + q_2^{\max})$ and $-\beta(q_1^{\max} + 2q_2^{\max})$ are integers. These values are, respectively, $-\beta(2q_1^{\max} + q_2^{\max}) = -\frac{1}{6}(30) = -5 = -\beta(q_1^{\max} + 2q_2^{\max})$.

Table 2: Example of Theorem 5, case 4 integer solutions

#	Theorem 5 case	α	β	γ_1	γ_2	$q_1^{\max}$	$q_2^{\max}$	q_1^*	q_2^*	λ_1^*	λ_2^*
1	4	6	$\frac{1}{6}$	1	1	10	10	10	10	0	0
2	4	7	$\frac{1}{6}$	1	1	10	10	10	10	1	1
3	4	8	$\frac{1}{6}$	1	1	10	10	10	10	2	2
4	4	9	$\frac{1}{6}$	1	1	10	10	10	10	3	3
5	4	10	$\frac{1}{6}$	1	1	10	10	10	10	4	4

Example #3a

For a first example, the following parameters were used:

$$\alpha = 10, \beta = 2, \gamma_1 = \gamma_2 = 4, q_1^{\max} = q_2^{\max} = 7$$

The continuous solution is

$$q_1 = 1, q_2 = 1, \lambda_1 = 0, \lambda_2 = 0$$

which is by construction integer-valued. The DC-MCP formulation also achieves this solution for a range of M values from $\{7, 17, 117, 217, 317, \ldots, 4917\}$ noting that $M = \alpha + \max\{q_1^{\max} q_2^{\max}\} = 17$. It does not achieve this solution for $M = 2$ as that value is too low. However, from Theorem 4, the upper bound of $M = 17$ is not tight as demonstrated by the value of $M = 7$ that produces the same solution.

Example #3b

This is the same as Example #3a except that $\beta = 2.5$ instead of 2. The resulting continuous solution is $q_1 = 0.8, q_2 = 0.8, \lambda_1 = 0, \lambda_2 = 0$ and the corresponding DC-LCP solution is again $q_1 = 1, q_2 = 1, \lambda_1 = 0, \lambda_2 = 0$ with an objective function value of 2. The optimal solution and objective function value is stable over the same range of M.

Example #3c

In this example the data were changed slightly to provide a more interesting case to consider relative to weighted relaxing of complementarity. In particular,

$$\alpha = 10, \beta = 2.5, \gamma_1 = 3, \gamma_2 = 4, q_1^{\max} = 0.8, q_2^{\max} = 3.2$$

and try different variations on the weights ω_p in $\sum_{i=1,4} \omega_i |H_i(q_1, q_2, \lambda_1, \lambda_2)|$. The table below shows a summary of the results with * denoting optimal values.[3]

Table 3: Results of varying the weights for near complementarity

Case	ω_{q_1}	ω_{q_2}	ω_{λ_1}	ω_{λ_2}	q_1^*	q_2^*	λ_1^*	λ_2^*	optimal obj. function
continuous LCP	n/a	n/a	n/a	n/a	0.8	0.8	1	0	n/a
1	1	1	1	1	0	1	7	1	1.8
2	1	1	1000	1	0	3	0	0	3.0

The first row of numbers corresponds to the continuous-valued LCP with no integer restrictions. The solution reported by GAMS is to take both energy production levels the same, at 0.8 which is the capacity for player 1 but under the capacity for player 2. Consequently, player 1's capacity price (dual variable) is positive

[3]The integer-constrained solutions were stable for $\{13.2, 113.2, 213.2, \ldots, 4913.2\}$ where $M = \alpha + \max\{q_1^{\max}, q_2^{\max}\} = 13.2$ for the last two rows of this table.

and has a value of 1 whereas for the second player the value is 0. When equally weighting $|H_i(q_1, q_2, \lambda_1, \lambda_2)|$, the solution in the next row (case 1) is not simply just rounding the production values q nor the capacity prices λ but rather a combination of adjusting all these values (relating to solving the associated LCP) to achieve a lowest objective function value of 1.8. This solution shows that the integer constraints on an LCP solution can induce non-obvious answers.

In the second case, it is desired to get the first capacity price λ_1^* much closer to its continuous value of 1 so ω_{λ_1} is set equal to 1000 with the other weights unchanged. The result is that the DC-LCP solution for λ_1 is now much closer to the continuous solution (0 vs. 5) but this weight changes both q_2^* and λ_2^* as well to satisfy the complementarity conditions or come close to doing this.

3.4 An energy-related spatial price equilibrium with if-then logic on the flows (SG: Energy angle)

The spatial price equilibrium problem (SPE) is a generalization of the classical linear programming transportation problem [13, 14, 18]. In the SPE, given a bipartite network of spatially dispersed supply nodes $i \in I$ and demand nodes $j \in J$ and set of connecting arcs $a \in A = \{(i,j) : i \in I, j \in J\}$ for the resulting complete network, the objective is to determine the vector of nonnegative flows $x = \{x_{ij} : i \in I, j \in J\}$ such that

$$0 \leq \Psi_i \left(\sum_j x_{ij} \right) + c_{ij}\left(y\right) - \theta_j \left(\sum_i x_{ij} \right) \perp x_{ij} \geq 0 \tag{25}$$

where $\Psi_i \left(\sum_j x_{ij} \right)$, $c_{ij}\left(y\right), \theta_j \left(\sum_i x_{ij} \right)$ are respectively, the inverse supply function (i.e., supply price), the transportation cost function, and the inverse demand function (i.e., demand price) as a function of the total supply at node i and total demand at node j, respectively, $\sum_j x_{ij}$ and $\sum_i x_{ij}$. Here

$$F_{ij}\left(x\right) \triangleq \Psi_i \left(\sum_j x_{ij} \right) + c_{ij}\left(x\right) - \theta_j \left(\sum_i x_{ij} \right) \tag{26}$$

is the resulting MCP function where $F : R^{|A|} \to R^{|A|}$. Thus, SPE in the form given above is an instance of a MCP with only nonnegative variables x_{ij}. In a spatially diverse set of regions, an equilibrium in the flows between these regions means that the delivered price (supply plus transportation costs) must be greater than or equal to the selling price with equality holding if there is positive flow on that arc. Conversely, if the delivered price is strictly greater than the selling price, no flow results along arc (i,j). For volumetric flows such as electricity, natural gas, or water where individual quanta such as electrons, methane or water molecules are not tracked, the flow can reasonably be taken to be continuous-valued. However, a flow of goods that is measured in discrete units (e.g., chairs), needs to be integer-valued. Mixtures of continuous and discrete variables are also possible as might arise with a multi-commodity version of the problem. Thus, (25) with integer restrictions on a subset of the flows x_{ij} is an instance of a DC-MCP. The following sample SPE with $i = 1, \ldots, 4$ supply nodes and $j = 1, \ldots, 5$ demand nodes is taken from Chapter 4 of [13]. The various functions and data are the following:

$$\Psi_i \left(\sum_j x_{ij} \right) = \kappa_i^S + \rho_i^S \left(\sum_j x_{ij} \right) \tag{27}$$

$$\theta_j \left(\sum_i x_{ij} \right) = \kappa_j^D - \rho_j^D \left(\sum_j x_{ij} \right) \tag{28}$$

with $\rho_i^S, \rho_j^D > 0, c_{ij}(x) = c_{ij} x_{ij}$ and

$$c_{ij} = \begin{array}{cc} \mathbf{i/j} & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 10 & 20 & 30 & 5 & 6 \\ 3 & 15 & 12 & 5 & 6 \\ 17 & 40 & 5 & 3 & 7 \\ 50 & 23 & 28 & 12 & 4 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \kappa_1^S + \rho_1^S \left(\sum_j x_{1j} \right) \\ \kappa_2^S + \rho_2^S \left(\sum_j x_{1j} \right) \\ \kappa_3^S + \rho_3^S \left(\sum_j x_{1j} \right) \\ \kappa_4^S + \rho_4^S \left(\sum_j x_{1j} \right) \end{pmatrix} = \begin{pmatrix} 0 + \frac{1}{20} \left(\sum_j x_{1j} \right) \\ 0 + \frac{1}{5} \left(\sum_j x_{2j} \right) \\ -7 + \frac{1}{5} \left(\sum_j x_{3j} \right) \\ 0 + \frac{1}{5} \left(\sum_j x_{4j} \right) \end{pmatrix}$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{pmatrix} = \begin{pmatrix} \kappa_1^D - \rho_1^D \left(\sum_i x_{i1} \right) \\ \kappa_2^D - \rho_2^D \left(\sum_i x_{i2} \right) \\ \kappa_3^D - \rho_3^D \left(\sum_i x_{i3} \right) \\ \kappa_4^D - \rho_4^D \left(\sum_i x_{i4} \right) \\ \kappa_5^D - \rho_5^D \left(\sum_i x_{i5} \right) \end{pmatrix} = \begin{pmatrix} 29 - 1 \left(\sum_i x_{i1} \right) \\ 46 - 1 \left(\sum_i x_{i2} \right) \\ 15 - 1 \left(\sum_i x_{i3} \right) \\ 13 - 1 \left(\sum_i x_{i4} \right) \\ 42 - 1 \left(\sum_i x_{i5} \right) \end{pmatrix}$$

As given in [13], the (rounded) reported solution is

$$x_{ij} = \begin{array}{cc} \mathbf{i/j} & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 0 & 15 & 0 & 0 & 5 \\ 20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 10 & 15 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix} \end{array}$$

These values when non-rounded are actually slightly different and are the following with an associated complementarity sum of $-7.72501E-6$:

$$x_{ij} = \begin{array}{cc} \mathbf{i/j} & \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 0 & 14.9999940350495 & 0 \\ 19.9999990411316 & 10.0000054324367 & 0 \\ 0 & 0 & 10.0000007797335 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} \mathbf{i/j} & \begin{array}{cc} \mathbf{4} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 0 & 5.00000867749416 \\ 0 & 0 \\ 10.0000007798807 & 14.9999947807364 \\ 0 & 14.9999969524887 \end{bmatrix} \end{array}$$

For the three examples reported below, an M value of 1000 was used and tested to see that it produced stable answers.

Example #4a

While it is fine to report the rounded values given the precision used, a natural question is what happens if we want *exact* integers flow values? For example, suppose we wanted to take exactly 15 for the value of

x_{12} and so on. Clearly, one way would be to take the rounded valuesto check to see if they are solutions to the SPE. A check using the DC-LCP MIP formulation shows that in fact in this case at least, rounding the values leads to the correct, integer-only solution as reported above.

Example #4b

Changing c_{45} from a value of 4 to 100 is the next case considered. The flow pattern for the continuous LCP to this problem dramatically changes as shown next.

$$x_{ij} = \begin{bmatrix} \textbf{i/j} & \textbf{1} & \textbf{2} & \textbf{3} \\ \textbf{1} & 0 & 12 & 0 \\ \textbf{2} & 19.5714286 & 12.5714286 & 0 \\ \textbf{3} & 0 & 0 & 9.5714286 \\ \textbf{4} & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \textbf{i/j} & \textbf{4} & \textbf{5} \\ \textbf{1} & 0 & 16.5714286 \\ \textbf{2} & 0 & 0 \\ \textbf{3} & 9.5714286 & 18 \\ \textbf{4} & 0 & 0 \end{bmatrix}$$

The solution taking all flows to be integer-valued is the following:

$$x_{ij} = \begin{bmatrix} \textbf{i/j} & \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \\ \textbf{1} & 0 & 15 & 0 & 0 & 20 \\ \textbf{2} & 20 & 10 & 0 & 0 & 0 \\ \textbf{3} & 0 & 0 & 10 & 10 & 15 \\ \textbf{4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which fairly closely matches the solution from Example #4a with the exception of $x_{15} = 20$ (was 5 before) and $x_{44} = 0$ (was 15 before). The total deviation from complementarity as reported via the objective function was 1.5. This is different than just looking at the total complementarity between the reported solution above and the associated F values. That is $\sum_i \sum_j x_{ij} \cdot F_{ij}(x_{ij}) = 26.25$.

Example #4c

In this third example, the data from Example #4b are used but an additional constraint of the if-then type is used to demonstrate the flexibility of the proposed DC-MCP approach. Since the solution to Example #4b shows that the energy supply node 4 has no flow from it, a supply planner trying to better balance the supply-demand network could add constraints on top of the equilibrium conditions for better equity between the supply nodes. Consider the following logic that such an energy planner might use to enforce some kind of equity in the network:

$$\text{if } \sum_j x_{ij} \le \delta_i \text{ then } \sum_j x_{ij} \ge 0.25 \sum_i \sum_j x_{ij}, \forall i$$

where δ_i is some minimum contractual threshold for supply guranteed to supply node i. This if-then condition says that if the SPE flow is less than the contractual minimum, then the ith energy supply node gets at least $\frac{1}{4} = 25\%$ of the total flows. Such conditions are implemented by adding the following constraints where the M_i are positive constants to be chosen:

$$\delta_i - \sum_j x_{ij} \le \widehat{b}_i M_i, \in \{0,1\}, i = 1,2,3,4 \tag{29}$$

$$-\sum_j x_{ij} + 0.25 \sum_i \sum_j x_{ij} \le M_i \left(1 - \widehat{b}_i\right), i = 1,2,3,4 \tag{30}$$

Clearly many other if-then type constraints on the flows could also be tried. Using $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 3$, the following is the DC-LCP solution reported by GAMS.

$$x_{ij} = \begin{bmatrix} \textbf{i/j} & \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \\ \textbf{1} & 0 & 12 \text{ (was 15)} & 0 & 0 & 20 \\ \textbf{2} & 20 & 10 & 0 & 0 & 0 \\ \textbf{3} & 0 & 0 & 10 & 10 & 15 \\ \textbf{4} & 0 & 3\text{(was 0)} & 0 & 0 & 0 \end{bmatrix}$$

This output shows that only two flows were minimally affected: x_{12} and x_{42} to enforce these equity constraints while at the same time minimizng the deviation from complementarity and preserving integer flows.

4 Conclusions

In this paper we have presented a novel formulation based on a certain median function to solve discretely constrained mixed complementarity problems in both general form (arbitrary upper and lower bounds) as well the more traditional approach where the lower bounds are zero and the upper bounds are infinity. Such problems seek to combine integer (discrete) solutions that are also equilibrium ones and have applications in engineering and economics.

The median function is transformed via binary variables and appropriate constraints to arrive at an easier-to-solve, mixed-integer, nonlinear program (MINLP) in general. When the complementarity functions F_i are linear (affine), the resulting formulation is a mixed-integer linear program (MILP). Several theoretical results show the correspondence between the formulations presented and the original discretely constrained MCP. Lastly, the approach is tested on a variety of illustrative examples.

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