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NORTA for portfolio credit risk

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Abstract: We use NORTA (**NOR**mal **T**o **A**nything) to enhance normal credit-risk factor settings in modeling common risk factors and capturing contagion effects. NORTA extends the multivariate Normal distribution in that it enables the simulation of a random vector with arbitrary and known marginals and correlation structure. NORTA can be solved either by numerical integration (Cario and Nelson 1997) or by Monte Carlo simulation (Ilich 2009). The former approach, which is the most efficient, assumes that the marginals' inverse cumulative functions are given, while the latter, which is more flexible but less efficient, does not. We show how to combine both approaches for higher flexibility and efficiency. We solve for NORTA and experiment with Normal, Student, and Asymmetric Exponential Power (AEP) distributions. We match NORTA models to Normal models with the same marginals' first and second moments. Yet, differences in credit-risk measures can be highly significant. This supports NORTA as a viable alternative for credit-risk modeling and analysis.

Keywords: Portfolio credit risk, factor models, NORTA, numerical integration, Monte Carlo simulation

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1 Introduction

The aim of this paper is to use NORTA to enhance Normal credit-risk factor models. NORTA (**NOR**mal **T**o **A**nything) is a Gaussian copula that enables the simulation of random vectors with arbitrary and known marginals and correlation matrices, whose elements are either moment- or rank-based coefficients. Then, Monte Carlo simulation is used to estimate credit-risk measures in factor models, where the set of common risk factors is a NORTA vector.

The Board of Governors of the Federal Reserve System reports that business loans, held by all commercial banks in the US, total $2.1 Trillion as of October 31, 2016. In the absence of regular market signals, credit-risk models provide managers with rational measures for credit-risk monitoring and analysis. In this context, realistic numerical investigations are time consuming, given the high number of commercial and industrial loans, reported as assets on a commercial bank's balance sheet. This partially explains why, despite its simplicity, the Normal vector has been the reference in credit-risk modeling, since CreditMetrics developed by JP Morgan and KMV held by Moody's. NORTA is an acceptable compromise between realism and efficiency for it presents three major advantages. It is simple to implement, flexible enough to accommodate arbitrary marginals, and consistent with almost all the findings on the Normal credit-risk factor model. For these reasons, we propose NORTA as a viable alternative for credit-risk modeling and analysis.

A *factor model* acts as a regression, where the quality of a credit is explained by economic *common risk factors* and the issuer's *specific risk factor*. The so-called *score function* of a credit is a numeric variable that represents the quality of that credit and plays the role of an individual dependent variable. This is a latent variable since the underlying credit is typically a non-traded asset. Depending on the model design, common risk factors can be latent or observable (Egloff et al. 2005, Jiménez and Mencía 2009, and Grundke 2009). Examples include worldwide, country, and industry effects that impact a large class of companies. The specific risk factor of a credit affects the quality of that credit but not the other credits, and plays the role of the error variable. The latter is latent by construction. Factor models assume that risk factors are random, specific risk factors are independent, and the set of common risk factors is independent from the set of specific risk factors. The Normal credit-risk factor model supposes that common and specific factors, all together, form a Normal vector.

Individual score functions give rise to individual loss functions, which in turn result in a *portfolio loss function* (in dollars) whose right-tail parameters are called *credit-risk measures*. Examples include the *value at risk,* the *expected shortfall*, and the *conditional value at risk* at the level $\alpha \in (0, 1)$, indicated by VaR_α, ES_α, and CVaR_α, respectively. The former is the quantile of the portfolio loss function at the level α, the second is the average loss beyond the VaR_α, and the latter is a weighted average of VaR_α and ES_α, for α in the left neighborhood of one. For large portfolios, we instead consider the average loss per credit (in dollars). The marginal contribution of each common risk factor to the overall portfolio credit risk is discussed by Rosen and Saunders (2010). Credit-risk portfolio management consists firstly of minimizing a given credit-risk measure subject to some return constraints (Andersson et al. 2001, Saunders et al. 2007, and Surya and Kurniawan 2014) and secondly of splitting it into individual contributions (Glasserman 2006 and Liu 2015). The optimal solution is then turned into an *economic capital* measure, that is, a risk-based capital requirement for a financial institution to survive under extreme loss events. Finally, credit issuers are charged as a function of their individual risk contributions to the overall credit portfolio.

The credit-risk factor model starts from individual business loans held by a commercial bank and ends up with an overall credit portfolio, loss function, and their credit-risk measures. This "down up" construction is then followed by an "up down" one, which starts from the bank's economic capital, infered from credit-risk measures, and ends up with its associated individual contributions. This approach is consistent with the guidelines of the Basel Committee on Banking Supervision.

The dependence between credits, known as the *contagion effect*, is at the heart of credit-risk modeling and analysis, as it highly inflates credit-risk measures under adverse economic conditions (Schönbucher 2001). Since the contagion effect results from the set of common risk factors in factor models, *copulas* are widely used to describe their dependence structure. A copula is a joint distribution function that results in basic Uniform marginal distributions. Interestingly, it can be used to simulate a vector of dependent random variables given its arbitrary marginal distributions (Nelsen 2006). NORTA is a Gaussian copula that enables the simulation of a random vector with arbitrary and known marginals and correlation structure from a standardized Normal vector with a specific correlation matrix, resulting from a matching procedure solved either by numerical integration and root-finding procedures (Cario and Nelson 1997) or by Monte Carlo simulation, multiple linear regressions, and sorting/permuting procedures (Ilich 2009). The former methodology is less flexible than the latter since it assumes the marginals' inverse cumulative fonctions as given, but more efficient as it inherits only numerical but not statistical errors. Moreover, NORTA à la Cario and Nelson (1997) enables the matching of rank-based correlation matrices, while NORTA à la Ilich (2009) does not. All in all, both methodologies are viable in our context and can be combined for flexibility and efficiency improvement.

We focus herein on a static credit-risk factor model and Gaussian copulas. Despite its simplicity, the Normal copula remains robust to several copulas' miss-specifications (Hamerle and Rösch 2005). Under the Normal assumption, Glasserman (2004) develops numerical approximations for VaR computation in finite portfolios, while Glasserman et al. (2007) derive asymptotics for large loss probabilities and Lucas et al. (2001) derive quasi-closed-form solutions for credit-risk measures in infinite portfolios. Under the same assumption, Morokoff (2004), Egloff et al. (2005), Glasserman and Li (2005), Dunkel and Weber (2007), Glasserman et al. (2008), and Reitan and Aas (2010) use efficient Monte Carlo simulation based on importance sampling to estimate credit-risk measures. The T-copula can produce significantly higher dependence between obligors than would be produced by the Normal vector. In this context, Glasserman et al. (2002), Kang and Shahabuddin (2005), and Bassambo et al. (2008) propose importance sampling to estimate credit-risk measures. Fu et al. (2009) and Chan and Kroese (2010) also use conditional Monte Carlo, the former to estimate VaR sensitivities in finite portfolios and the latter to estimate large-loss probabilities in infinite portfolios. Chan and Kroese (2011) recommend conditional Monte Carlo instead of importance sampling when the likelihood ratio presents severe degeneracy. Frey and McNeil (2003) explore the impact of mixed distributions on credit-risk measures. Fu et al. (2009) and Andersen and Sidenius (2004) consider a model with random factor loadings, which produces higher default correlations in bear markets than in bull markets. He and Gong (2009) use several copulas and Monte Carlo simulation to estimate VaR and CVaR in a mixed setting that combines market and credit risk. For a review on VaR and CVaR estimation, see Hong et al. (2014).

We solve for NORTA and experiment with the Normal, Student, and AEP distributions. We match NORTA to Normal vectors with the same marginals' first and second moments. Yet, differences in credit-risk measures can be highly significant. This supports NORTA as a viable alternative for credit-risk modeling and analysis.

The rest of the paper is organized as follows. Section 2 presents a credit-risk factor model and main credit-risk measures. While Section 3 solves for NORTA à la Cario and Nelson (1997), Section 4 combines the latter approach with NORTA à la Ilich (2009) for higher flexibility and efficiency. Section 5 reports a numerical investigation and Section 6 concludes.

2 Model and notation

We consider a static credit-risk factor model and a portfolio of n zero-coupon corporate bonds, all maturing in one year, as in (Schönbucher 2001 and Glasserman and Li 2005). There are d common risk factors indicated by X_k, for $k = 1, \ldots, d$, n specific risk factors indicated by ϵ_i, for $i = 1, \ldots, n$, and two credit classes for individual bonds (default or survival in one year). The score function of bond i is

$$S_i = \sum_{k=1}^{d} a_k X_k + b\epsilon_i, \quad \text{for } i = 1, \ldots, n, \tag{1}$$

where the *loadings* a_k and b are known constants of the same sign. This is to say that score functions are monotone functions of risk factors, which disable undesired diversification based on risk factors' movements. The loadings can either be constant or variable. A compromise is probably a model where the coefficients are constant by class of individual credits or random with specific prior distributions. Factor models suppose that risk factors are random, specific risk factors are independent, and the set of common risk factors is independent from the set of specific risk factors. Equation (1) clearly shows that common risk factors are responsible for the contagion effect. Alternative constructions for credit loss (Klugman et al. 2008) and credit contagion (Davis and Lo 2001 and Egloff et al. 2007) do exist.

We use NORTA to simulate the random vector of common risk factors $X = (X_1, \ldots, X_d)^{\mathrm{T}}$ with arbitrary and known marginal distributions and correlation structure. The specific risk factors are produced individually and independently from their marginal distribution $\mathcal{N}(0,1)$. This construction results in the Normal setting when NORTA's marginals are Normal.

The *default indicator function* of bond i is

$$Y_i = \begin{cases} 1, & \text{if bond } i \text{ defaults in one year with a probability } p_i \\ 0, & \text{elsewhere with a probability } 1-p_i \end{cases},$$

where p_i is the *default probability* of bond i, a known constant in $(0,1)$. The *score function* S_i is related to bond i as follows:

$$Y_i = 1 \quad \text{if, and only if,} \quad P\left(S_i \leq K_i\right) = F_{S_i}\left(K_i\right) = p_i, \tag{2}$$

where K_i is the *default threshold* of bond i and F_{S_i} the cumulative density function of S_i. The default thresholds K_i, for $i = 1, \ldots, n$, can be computed 1- in closed form under the Normal vector, 2- by Fourier approximations under independent risk factors (Abate and Whitt 1992), and 3- by (efficient) Monte Carlo simulation under the NORTA assumption.

The *portfolio loss function* is

$$L = \sum_{i=1}^{n} c_i P_i Y_i = \sum_{i=1}^{n} L_i,$$

where L_i is the individual *loss function* of bond i, c_i its *loss given default rate*, and P_i its *principal amount*. The coefficients c_i and P_i are known positive constants. Following Rockafellar and Uryasev (2002), main credit-risk measures are defined as follows:

1. VaR_α is the quantile of L at level $\alpha \in (0,1)$, that is,

$$\mathrm{VaR}_\alpha = \inf\left\{l \in \mathbb{R} \text{ such that } F_L(l) \geq \alpha\right\},$$

where F_L is the cumulative density function of L;

2. CVaR_α^-, also referred to as the *tail* VaR, is the average loss for $L \geq \mathrm{VaR}_\alpha$, that is,

$$\mathrm{CVaR}_\alpha^- = E\left[L \mid L \geq \mathrm{VaR}_\alpha\right];$$

3. CVaR_α^+, also referred to as the *expected shortfall*, is the average loss for $L > \mathrm{VaR}_\alpha$, that is,

$$\mathrm{CVaR}_\alpha^+ = E\left[L \mid L > \mathrm{VaR}_\alpha\right];$$

4. CVaR is a weighted average of VaR_α and CVaR_α^+, that is,

$$\mathrm{CVaR}_\alpha = \begin{cases} \lambda_\alpha \times \mathrm{VaR}_\alpha + (1-\lambda_\alpha) \times \mathrm{CVaR}_\alpha^+, & \text{if } F_L\left(\mathrm{VaR}_\alpha\right) < 1 \\ \mathrm{VaR}_\alpha, \quad \text{if } F_L\left(\mathrm{VaR}_\alpha\right) = 1 \end{cases},$$

where $\lambda_\alpha = \left(F_L\left(\mathrm{VaR}_\alpha\right) - \alpha\right) / (1-\alpha) \in [0,1]$.

These credit-risk measures verify

$$\mathrm{VaR}_\alpha \leq \mathrm{CVaR}_\alpha^- \leq \mathrm{CVaR}_\alpha \leq \mathrm{CVaR}_\alpha^+,$$

and

$$\mathrm{VaR}_\alpha \leq \mathrm{CVaR}_\alpha^- = \mathrm{CVaR} = \mathrm{CVaR}_\alpha^+,$$

when $P\left(L = \mathrm{VaR}_\alpha\right) = 0$ (no jump in F_L at VaR_α) and $F_L\left(\mathrm{VaR}_\alpha\right) < 1$ (relevance of L beyond VaR_α)

$$\mathrm{VaR}_\alpha = \mathrm{CVaR}_\alpha^- = \mathrm{CVaR} = L_{\max},$$

when $F_L(\mathrm{VaR}_\alpha) = 1$, while CVaR^+_α is not defined anymore.

Only CVaR is coherent in the sense of Artzner et al. (1999) and verifies the following four properties all together: 1– monotonicity: $\mathrm{CVaR}^L_\alpha \le \mathrm{CVaR}^{L'}_\alpha$, for $L \le L'$, that is, a higher loss results in a higher CVaR, 2– translation equivalence: $\mathrm{CVaR}^{c+L}_\alpha = c + \mathrm{CVaR}^L_\alpha$, for $c \in \mathbb{R}$, that is, a change in the loss function by a constant results in a change in CVaR by the same constant, 3– positive homogeneity: $\mathrm{CVaR}^{\lambda L}_\alpha = \lambda \times \mathrm{CVaR}^L_\alpha$, for $\lambda \in \mathbb{R}^*_+$, that is, doubling the portfolio size doubles CVaR, and 4– subadditivity: $\mathrm{CVaR}^{L+L'}_\alpha \le \mathrm{CVaR}^L_\alpha + \mathrm{CVaR}^{L'}_\alpha$, that is, CVaR is reduced by diversification, while VaR is not. Additional properties of CVaR are given by Pflug (2000). Rockafellar and Uryasev (2002) and Mansini et al. (2007) propose CVaR for credit-risk monitoring for its economic relevance, computational efficiency, and superior mathematical properties. Despite its computational complexity, VaR is still recommended when models for tail loss distributions are unavailable and used for robust portfolio optimization (Romanko and Mausser 2016).

Since F_L and F_L^{-1} are usually unknown, (efficient) Monte Carlo simulation is used to estimate these credit-risk measues. Each NORTA model is run twice. While the first run is done under the NORTA assumption for credit-risk factors, the second run is done under an associated Normal vector with the same risk factors' first moments (means, variances, correlation coefficients, and possibly skewness parameters).

Reduction-of-variance techniques can be used to accelerate Monte Carlo estimates of the individual default thresholds and the portfolio credit-risk measures. For example, one can compute the default thresholds in closed form under the "matched" Normal assumption, make use of correlation induction techniques, and improve their crude Monte Carlo estimates in the more general NORTA model. More interestingly, as NORTA is a Gaussian copula, almost all efficient Monte Carlo estimates of credit-risk measures and their sensitivities, which have been already developed in the literature under the Normal vector, remain valid. Examples include stratification and importance sampling of Glasserman and Li (2005) and conditional Monte Carlo of Fu et al. (2009). Bounds on values at risk can be obtained consistently to Mesfioui and Quessy (2005), since NORTA's marginal distributions and their correlation coefficients are known. These relevant properties are further reasons to use NORTA for credit-risk modeling and analysis.

3 NORTA à la Cario and Nelson (1997)

Let $X = (X_1, \ldots, X_d)^{\mathrm{T}}$ be a random vector with given and known marginals and moment-based correlation coefficients $\rho^X_{k,l}$, for k and $l = 1, \ldots, d$. The expected value of X_k is indicated by $\mu_k = E[X_k]$, its standard deviation by $\sigma_k = \sigma[X_k]$, its cumulative distribution function by F_k, and its inverse function by F_k^{-1} when available, for $k = 1, \ldots, d$. The cumulative of the standard Normal distribution and its inverse function are indicated by Φ and Φ^{-1}, respectively.

To start with, we solve for NORTA when the matching procedure relies on moment-based correlation coefficients. Then, we address the rank-based matching procedure. We synthesize and address Avramidis et al. (2009) and Channouf and L'Écuyer (2009) within a slightly different construction.

3.1 The continuous case

Let $(X_k, X_l)^{\mathrm{T}}$ be a continuous random couple. One has

$$X_k = F_k^{-1}(U_k) = F_k^{-1}(\Phi(Z_k)) \quad \text{and} \quad X_l = F_l^{-1}(U_l) = F_l^{-1}(\Phi(Z_l)), \tag{3}$$

where $(Z_k, Z_l)^{\mathrm{T}}$ and $(U_k, U_l)^{\mathrm{T}}$ are random couples of correlated standard Normals and correlated Uniforms$-[0,1]$, respectively. The correlation coefficient of the random couple $(Z_k, Z_l)^{\mathrm{T}}$ is indicated by $\rho \in [-1, 1]$, its probability density function by $\phi_\rho : \mathbb{R}^2 \to \mathbb{R}^*_+$, and its cumulative distribution function by $\Phi_\rho : \mathbb{R}^2 \to (0, 1)$. It's worth noticing that only Normal couples result in correlation coefficients that span the entire interval $[-1, 1]$. To determine the range of feasible correlation coefficients for a non-Gaussian random couple, simulate independent large samples from the marginals, sort each component alone in ascending order, and estimate the resulting correlation coefficient. This is the maximum correlation between the marginals, while the minimum is obtained via a double sort in ascending/descending order of the first/second marginal as indicated by Whitt (1976).

Equation (3) clearly shows that $E\left[X_k X_l\right] = g_{k,l}\left(\rho\right)$ can be seen as a function of $\rho \in [-1, 1]$. Solving for NORTA means to searching for $\rho = \rho^Z_{k,l}$ that verifies

$$\rho^X_{k,l} = \frac{g_{k,l}\left(\rho\right) - \mu_k \mu_l}{\sigma_k \sigma_l},$$

for each couple $1 \leq k < l \leq d$, or finding the unique root of the non-decreasing function

$$\begin{array}{llll} f_{k,l}: & [-1,1] & \rightarrow & \mathbb{R} \\ & \rho & \rightarrow & f_{k,l}\left(\rho\right) = g_{k,l}\left(\rho\right) - \mu_k \mu_l - \sigma_k \sigma_l \rho^X_{k,l} \end{array}, \tag{4}$$

which results in a system of $d\left(d-1\right)/2$ nonlinear equations to be solved numerically. The matched correlation matrices of X and Z are indicated by $\rho^X = \rho^X_{k,l}$ and $\rho^Z = \rho^Z_{k,l}$, for k and $l = 1, \ldots, d$, respectively.

An integral representation of the function $g_{k,l}$ is

$$\begin{aligned} g_{k,l}\left(\rho\right) &= E\left[F_k^{-1}\left(\Phi\left(Z_k\right)\right) F_l^{-1}\left(\Phi\left(Z_l\right)\right)\right] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_k^{-1}\left(\Phi\left(z_k\right)\right) F_l^{-1}\left(\Phi\left(z_l\right)\right) \phi_\rho\left(z_k, z_l\right) dz_k dz_l. \end{aligned} \tag{5}$$

Since the improper integrals in Equation (5) are not supported by the libraries GSL and CUBATURE, which have been used for computational purposes, a transformation is required. Cholesky decomposition for a Normal couple gives

$$\left\{ \begin{array}{l} Z_k = Z = \Phi^{-1}\left(U\right) \\ Z_l = \rho Z + \sqrt{1-\rho^2} Z' = \rho \Phi^{-1}\left(U\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(U'\right) \end{array} \right., \tag{6}$$

where $\left(Z, Z'\right)^{\mathrm{T}}$ and $\left(U, U'\right)^{\mathrm{T}}$ are random couples of independent standard Normals and independent Uniforms$-\left(0,1\right)$, respectively. The integral representation of the function $g_{k,l}$ becomes

$$\begin{aligned} g_{k,l}\left(\rho\right) &= E\left[F_k^{-1}\left(U\right) F_l^{-1}\left(\Phi\left(\rho\Phi^{-1}\left(U\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(U'\right)\right)\right)\right] \\ &= \int_0^1\int_0^1 F_k^{-1}\left(u\right) F_l^{-1}\left(\Phi\left(\rho\Phi^{-1}\left(u\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(u'\right)\right)\right) du du'. \end{aligned} \tag{7}$$

We use GSL to compute the inverse cumulative functions, whenever they are available, and Brown's quadrature of CUBATURE to compute the integral in Equation (7). For root finding, we use Brent's procedure of GSL, based on bisection and false position (Gerald and Wheatley 1999). The following values, that is, ρ^X_{kl} and $2\sin\left(\frac{\pi}{6}\rho^X_{kl}\right)$, often help to envelop the target $\rho^Z_{k,l}$, where ρ^X_{kl} and $2\sin\left(\frac{\pi}{6}\rho^X_{kl}\right)$ are the matched closed-form solutions when $\left(X_k, X_l\right)^{\mathrm{T}}$ is a couple of correlated Normals and Uniforms$-\left(0,1\right)$ (Li and Hammond 1975), respectively.

Finally, to simulate a continuous NORTA vector, the procedure acts as follows:

1. Approximate the moment-based correlation matrix ρ^Z;
2. Use Cholesky decomposition and simulate a standardized Normal vector $Z = \left(Z_1, \ldots, Z_d\right)^{\mathrm{T}}$ with a correlation matrix ρ^Z from a standard Normal vector (centered, reduced, and non-correlated marginals);
3. Use Equation (3) and simulate the NORTA random vector $X = \left(X_1, \ldots, X_d\right)^{\mathrm{T}}$.

3.2 The discrete case

Let $\left(X_k, X_l\right)^{\mathrm{T}}$ be a discrete random couple with given and known marginals, defined by $p_{k,i} = P\left(X_k = x_{k,i}\right)$, for $i = 0, \ldots, m \in \overline{\mathbb{N}}$, and $p_{l,j} = P\left(X_l = x_{l,j}\right)$, for $j = 0, \ldots, n \in \overline{\mathbb{N}}$, and a correlation coefficient $\rho^X_{k,l}$. One can construct the couple $(X_k, X_l)^{\mathrm{T}}$ from the Normal couple $(Z_k, Z_l)^{\mathrm{T}}$ introduced above as follows:

$$X_k \overset{\mathcal{D}}{=} \begin{cases} x_{k,0}, & \text{if} \quad z_{k,0} < Z_k \leq z_{k,1} \quad \text{with} \quad \Phi(z_{k,1}) = p_{k,0} \\ & \vdots \\ x_{k,i}, & \text{else if} \quad z_{k,i} < Z_k \leq z_{k,i+1} \quad \text{with} \quad \Phi(z_{k,i+1}) - \Phi(z_{k,i}) = p_{k,i} \\ & \vdots \\ x_{k,m}, & \text{else} \quad z_{k,m} < Z_k \leq z_{k,m+1} \quad \text{with} \quad 1 - \Phi(z_{k,m}) = p_{k,m} \end{cases}, \tag{8}$$

which results in the expression

$$X_k = \sum_{i=0}^{m} x_{k,i} \times \mathbb{I}(z_{k,i} < Z_k \leq z_{k,i+1}) \text{ and } X_l = \sum_{j=0}^{n} x_{l,j} \times \mathbb{I}(z_{l,j} < Z_l \leq z_{l,j+1}),$$

where $\mathbb{I}$ is the indicator function, $z_{k,0} = z_{l,0} = -\infty$, and $z_{k,m+1} = z_{l,n+1} = +\infty$. The thresholds $z_{k,i}$, for $i = 1, \ldots, m$, are obtained from the first-order linear difference equation embedded in Equation (8). The thresholds $z_{l,j}$, for $j = 1, \ldots, n$, are obtained as well.

One has

$$\begin{aligned} g_{k,l}(\rho) &= \sum_{i=0}^{m} \sum_{j=0}^{n} x_{k,i} x_{l,j} E\left[\mathbb{I}(z_{k,i} < Z_k \leq z_{k,i+1} \text{ and } z_{l,j} < Z_l \leq z_{l,j+1})\right] \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} x_{k,i} x_{l,j} \pi_{i,j}(\rho), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \pi_{i,j}(\rho) &= P(z_{k,i} < Z_k \leq z_{k,i+1} \text{ and } z_{l,j} < Z_l \leq z_{l,j+1}) \\ &= \Phi_\rho(z_{k,i+1}, z_{l,j+1}) + \Phi_\rho(z_{k,i}, z_{l,j}) - \left[\Phi_\rho(z_{k,i}, z_{l,j+1}) + \Phi_\rho(z_{k,i+1}, z_{l,j})\right], \end{aligned}$$

where the function Φ_ρ is computed following Genz (2004). See Genz and Malik (1980) and Berntsen et al. (1991) for further details. It's worth noticing that, for a given ρ, the computation of $\pi_{i,j}(\rho)$ requires the valuation of Φ_ρ only once. See Barbiero and Ferrari (2015) for a couple of Poisson random variables.

All in all, to simulate a discrete NORTA vector X, given its marginal distributions and correlation matrix ρ^X, one has to find the matched semidefinite positive matrix ρ^Z, simulate the associated standardized Normal vector Z, and use Equation (8) to simulate the marginals of X consistent with ρ^X.

3.3 The mixed case

Consider now the mixed case, where $(X_k, X_l)^{\mathrm{T}}$ is a random couple characterized by a continuous first marginal and a discrete second marginal. Given Equation (3), Equation (6), and Equation (8), one has

$$\begin{aligned} g_{k,l}(\rho) &= E\left[F_k^{-1}(\Phi(Z_k)) \sum_{j=0}^{n} x_{l,j} \mathbb{I}(z_{l,j} < Z_l \leq z_{l,j+1})\right] \\ &= \sum_{j=0}^{n} x_{l,j} E\left[F_k^{-1}(\Phi(Z))\, \mathbb{I}\left(z_{l,j} < \rho Z + \sqrt{1-\rho^2} Z' \leq z_{l,j+1}\right)\right] \\ &= \sum_{j=0}^{n} x_{l,j} E\left[F_k^{-1}(U)\, \mathbb{I}\left(z_{lj} < \rho \Phi^{-1}(U) + \sqrt{1-\rho^2}\, \Phi^{-1}(U') \leq z_{l,j+1}\right)\right] \\ &= \sum_{j=0}^{n} x_{l,j} \int_0^1 F_k^{-1}(u) \left[\Phi\left(\frac{z_{l,j+1} - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{z_{l,j} - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right] du, \end{aligned} \tag{10}$$

where the integrals in Equation (10) are computed using the Gauss-Legendre quadrature of GSL.

3.4 Matching based on rank correlation coefficients

The (*Pearson*) moment-based correlation coefficient measures the amplitude of the (potential) linear relationship between two random variables, while the (*Spearman*) rank-based correlation coefficient measures the amplitude of their (potential) monotone relationship. Both correlation coefficients belong to $[-1, 1]$. A Pearson (Spearman) correlation coefficient of ± 1 indicates a perfect increasing/decreasing linear (monotone) relationship between the two random variables, while a Pearson (Spearman) correlation of 0 indicates the absence of a linear (monotone) relationship between them. Rank-based correlation coefficients are known to be more robust to extreme values than moment-based correlation coefficients.

The rank-based correlation coefficient $r^X_{k,l}$ of $(X_k, X_l)^{\mathrm{T}}$ is defined as the moment-based correlation coefficient of $(F_k(X_k), F_l(X_l))^{\mathrm{T}}$, that is,

$$r^X_{k,l} = \frac{E\left[F_k(X_k) F_l(X_l)\right] - E\left[F_k(X_k)\right] E\left[F_l(X_l)\right]}{\sigma\left[F_k(X_k)\right] \sigma\left[F_l(X_l)\right]}.$$

Consequently, we redefine the function $f_{k,l}$ in Equation (4) as

$$f_{k,l}(\rho) = g_{k,l}(\rho) - \mu_k \mu_l - \sigma_k \sigma_l r^X_{k,l},$$

where $g_{k,l}(\rho) = E\left[F_k(X_k) F_l(X_l)\right]$, $\mu_k = E\left[F_k(X_k)\right]$, $\mu_l = E\left[F_l(X_l)\right]$, $\sigma_k = \sigma\left[F_k(X_k)\right]$, and $\sigma_l = \sigma\left[F_l(X_l)\right]$.

Solving for NORTA in the continuous case is done in closed form (Li and Hammond 1975), that is,

$$\rho^Z_{k,l} = 2 \sin\left(\frac{\pi}{6} r^X_{k,l}\right),$$

since $(F_k(X_k), F_l(X_l))^{\mathrm{T}}$ is a random couple of Uniforms$-(0,1)$ with a moment-based correlation coefficient r^X_{kl}. For the discrete case, the random variables $F_k(X_k)$ and $F_l(X_l)$ are discrete as well, which results in

$$g_{k,l}(\rho) = \sum_{i=0}^{m} \sum_{j=0}^{n} F_k(x_{k,i}) F_l(x_{l,j}) \pi_{i,j}(\rho),$$

as indicated in Equation (8). Finally, the mixed case, that is, $F_k(X_k) = U_k$ is Uniform$-(0,1)$ and $F_l(X_l)$ is discrete, results in

$$g_{k,l}(\rho) = \sum_{j=0}^{n} F_l(x_{l,j}) \int_0^1 u \left[\Phi\left(\frac{z_{l,j+1} - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{z_{l,j} - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right] du,$$

as indicated in Equation (10). Except for the continuous case, solving rank-based matching is similar to solving moment-based matching.

3.5 Examples from Cario and Nelson (1997)

The first example considers a 4-dimensional continuous random vector with an identical marginal Gamma $(14.4, 0.03424)$, with a shape parameter of 14.4 and a scale parameter of 0.03424, $\mu = 0.493$, $\sigma = 0.130$, and a moment-based correlation matrix

$$\rho^X = \begin{bmatrix} 1.0 & 0.7 & 0.5 & -0.9 \\ & 1.0 & 0.7 & -0.6 \\ & & 1.0 & -0.3 \\ & & & 1.0 \end{bmatrix}.$$

The minimum/maximum correlation coefficients are estimated at $-0.970/1.000$. The matrix ρ^Z, which requires a CPU time of 6.94 seconds to be produced, is

$$\rho^Z = \begin{bmatrix} 1.000 & 0.703 & 0.504 & -0.927 \\ & 1.000 & 0.703 & -0.615 \\ & & 1.000 & -0.306 \\ & & & 1.000 \end{bmatrix},$$

while Cario and Nelson (1997) report

$$\rho^Z_{\mathrm{CN}} = \begin{bmatrix} 1.000 & 0.704 & 0.504 & -0.920 \\ & 1.000 & 0.704 & -0.616 \\ & & 1.000 & -0.304 \\ & & & 1.000 \end{bmatrix}.$$

Monte Carlo estimate of ρ^X is

$$\widehat{\rho}^X = \begin{bmatrix} 1.000 & 0.700 & 0.499 & -0.900 \\ & 1.000 & 0.700 & -0.600 \\ & & 1.000 & -0.300 \\ & & & 1.000 \end{bmatrix}.$$

The second example considers a 3-dimensional discrete random vector with an identical Binomial$(3, 0.5)$ marginal with 3 independent draws and a success probability of 0.5 on each draw, $\mu = 1.500$, $\sigma = 0.866$, and a moment-based correlation matrix

$$\rho^X = \begin{bmatrix} 1.0 & 0.2 & -0.8 \\ & 1.0 & 0.2 \\ & & 1.0 \end{bmatrix}.$$

The minimum/maximum correlation coefficients are estimated at $-0.999/0.999$. The matrix ρ^Z, which requires a CPU time less than 0.01 seconds to be produced, is

$$\rho^Z = \begin{bmatrix} 1.000 & 0.228 & -0.895 \\ & 1.000 & 0.228 \\ & & 1.000 \end{bmatrix},$$

while

$$\rho^Z_{\mathrm{CN}} = \begin{bmatrix} 1.000 & 0.229 & -0.896 \\ & 1.000 & 0.229 \\ & & 1.000 \end{bmatrix}.$$

Monte Carlo estimate of ρ^X is

$$\widehat{\rho}^X = \begin{bmatrix} 1.000 & 0.198 & -0.800 \\ & 1.000 & 0.201 \\ & & 1.000 \end{bmatrix}.$$

The third example considers a random couple with a continuous first marginal, an Exponential(10) with $\mu = 10$ and $\sigma = 10$, a discrete second marginal, a Uniform$(1, \ldots, 10)$ with $\mu = 5.5$ and $\sigma = 2.872$, and a moment-based correlation matrix

$$\rho^X = \begin{bmatrix} 1.0 & -0.5 \\ & 1.0 \end{bmatrix}.$$

The minimum/maximum correlation coefficients between these two random variables are estimated at $-0.856/0.856$. The matrix ρ^Z, which requires a CPU time less than 0.01 seconds to be produced, is

$$\rho^Z = \begin{bmatrix} 1.000 & -0.576 \\ & 1.000 \end{bmatrix},$$

and ρ^Z_{CN} is

$$\rho^Z_{\mathrm{CN}} = \begin{bmatrix} 1.000 & -0.576 \\ & 1.000 \end{bmatrix}.$$

Monte Carlo estimate of ρ^X is

$$\widehat{\rho}^X = \begin{bmatrix} 1.000 & -0.501 \\ & 1.000 \end{bmatrix}.$$

4 Combination of NORTA à la Cario and Nelson (1997) and à la Ilich (2009)

NORTA à la Cario and Nelson (1997) faces some concerns, essentially for high-dimensional vectors. To start with, the matched matrix ρ^Z can be non-semidefinite positive. A first remedy, pointed out by Brigo (2002) and Channouf and L'Écuyer (2012), consists of relating the elements of ρ^Z to a few parameters that ensure the desired property. This limits the set of attainable semidefinite positive matrices. A second remedy, discussed by Higham (2002), finds the nearest correlation matrix to ρ^Z with the desired property. This slows down the resolution procedure. Next, Ghosh and Henderson (2002 and 2003) show that NORTA cannot reach the set of all feasible semidefinite positive matrices, as the dimension of the random vector increases. They identify the set of "NORTA-deficient matrices," whose elements belong to the right/left neigborhood of $\rho^X_{k,l}\,(\min)\,/\rho^X_{k,l}\,(\max)$, respectively. They propose a solution that alters ρ^X before searching for ρ^Z. This issue is not relevant in our context since the number of common risk factors is moderate. Finally, when some marginals' inverse cumulative functions are not available, we propose to use numerical integration whenever possible, then Monte Carlo simulation to complete the NORTA vector as follows:

1. Split the random vector X into two components, where the marginals of the first component have interior correlation coefficients and known inverse cumulative functions;
2. Simulate the first component X_{CN} consistently with Cario and Nelson (1997);
3. Simulate the second component $X^{\mathrm{T}}_{\mathrm{IL}}$, given the first X_{CN}, consistently with Ilich (2009).

This combination overcomes the above-mentioned drawbacks, while a large part of the overall simulation experiment is still worked out via numerical integration, which ensures higher flexibility and efficiency.

We rework the 8-dimensional NORTA vector in Ilich's (2009) first example as a combination of NORTA à la Cario and Nelson (1997) and à la Ilich (2009). We write the code lines in C, compile them under GCC, and use the GSL and CUBATURE libraries to achieve specific computational tasks. We run our experiments with a laptop computer running with a speed of 2.5 Ghz under Windows 10.

We split the random vector $X = (X_1, \ldots, X_8)^{\mathrm{T}}$ into two parts, that is, $X_{\mathrm{CN}} = (X_1, \ldots, X_7)^{\mathrm{T}}$ and $X_{\mathrm{IL}} = X_8$, since F_k and F_k^{-1}, for $k = 1, \ldots, 7$, are known, but not F_8 and its inverse F_8^{-1}. Table 1 presents eight NORTA marginals with their parameters. We also report their means and standard deviations, which are required for the implementation. These distributions, used by Ilich (2009), include Weibull, Extreme Value, Log-Normal, Binomial, Gamma, Poisson, Chi-Square, and Pearson Type V.

Table 1: NORTA's marginals

Distribution	Parameters	μ	σ
Weibull	Shape: 2.65 − Scale: 10.33	9.181	3.729
Extreme Value	Location: 7.65 − Shape: 2.76	9.243	3.540
Log-Normal	Mean: 13.26 − STD: 4.53	13.260	4.530
Binomial	No draws: 19 − Success prob: 0.46	8.740	2.172
Gamma	Shape: 4.48 − Scale: 1.24	5.555	2.625
Poisson	Intensity: 8.26	8.260	2.874
Chi-Square	Degrees of freedom: 10	10.000	4.472
Pearson V	Shape: 7.45 − Scale: 60.15	9.326	3.995

The 8×8 target moment-based correlation matrix is

$$\rho^X = \begin{bmatrix} 1.000 & 0.901 & 0.684 & 0.567 & -0.521 & 0.487 & 0.393 & 0.418 \\ & 1.000 & 0.838 & 0.648 & -0.570 & 0.577 & 0.483 & 0.519 \\ & & 1.000 & 0.866 & -0.738 & 0.800 & 0.734 & 0.770 \\ & & & 1.000 & -0.910 & 0.938 & 0.877 & 0.857 \\ & & & & 1.000 & -0.919 & -0.822 & -0.788 \\ & & & & & 1.000 & 0.940 & 0.926 \\ & & & & & & 1.000 & 0.942 \\ & & & & & & & 1.000 \end{bmatrix}.$$

Extreme Value and Pearson V distributions (marginals no 2 and no 8) are also known as Gumbel and Inverse Gamma, respectively.

The minimum/maximum correlation coefficients between the marginals are estimated at

$$\widehat{\rho}^X_{\min} = -\begin{bmatrix} 0.988 & 0.946 & 0.950 & 0.987 & 0.955 & 0.981 & 0.958 & 0.891 \\ & 0.886 & 0.891 & 0.957 & 0.894 & 0.941 & 0.900 & 0.824 \\ & & 0.895 & 0.962 & 0.900 & 0.945 & 0.905 & 0.828 \\ & & & 0.974 & 0.965 & 0.981 & 0.968 & 0.915 \\ & & & & 0.904 & 0.949 & 0.910 & 0.831 \\ & & & & & 0.974 & 0.953 & 0.890 \\ & & & & & & 0.915 & 0.836 \\ & & & & & & & 0.753 \end{bmatrix}$$

and

$$\widehat{\rho}^X_{\max} = \begin{bmatrix} 1.000 & 0.982 & 0.985 & 0.988 & 0.989 & 0.994 & 0.990 & 0.945 \\ & 1.000 & 1.000 & 0.962 & 0.999 & 0.981 & 0.999 & 0.990 \\ & & 1.000 & 0.966 & 1.000 & 0.983 & 0.999 & 0.987 \\ & & & 1.000 & 0.969 & 0.984 & 0.971 & 0.921 \\ & & & & 1.000 & 0.986 & 1.000 & 0.982 \\ & & & & & 1.000 & 0.987 & 0.948 \\ & & & & & & 1.000 & 0.980 \\ & & & & & & & 1.000 \end{bmatrix}$$

To start with, we solve for the NORTA vector $X_{\mathrm{CN}} = (X_1, \ldots, X_7)^{\mathrm{T}}$, as explained in Section 3, which results in the 7×7 matched correlation matrix

$$\rho^Z_{\mathrm{CN}} = \begin{bmatrix} 1.000 & 0.918 & 0.698 & 0.574 & -0.541 & 0.492 & 0.401 \\ & 1.000 & 0.846 & 0.674 & -0.624 & 0.593 & 0.496 \\ & & 1.000 & 0.898 & -0.812 & 0.817 & 0.744 \\ & & & 1.000 & -0.943 & 0.954 & 0.903 \\ & & & & 1.000 & -0.968 & -0.899 \\ & & & & & 1.000 & 0.953 \\ & & & & & & 1.000 \end{bmatrix}.$$

The matrix ρ^Z_{CN} takes a CPU time of 4.83 seconds to be produced. Next, following Ilich (2009), we simulate X_8, given X_{CN}, as follows:

1. Simulate a random sample of size N (large) from the 1×7 NORTA vector X_{CN} according to the procedure discussed in Section 3, which results in a $N \times 7$ matrix $X_{\mathrm{CN},i}$, for $i = 1, \ldots, N$;
2. Simulate a random sample of size N from the marginal distribution of X_8, independent of X_{CN}, which results in an $N \times 1$ column vector $Y_i \overset{\mathcal{D}}{=} X_{8,i}$, for $i = 1, \ldots, N$;
3. Solve the equation $R_{11} b = R_{12}$ on b, where b is an 7×1 column vector, while R_{11} and R_{12} are the 7×7 and 7×1 submatrices of
$$\rho^X = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 1 \end{bmatrix};$$
4. Compute the squared multiple correlation coefficient of the regression of X_8 on X_{CN} as follows:
$$R^2 = R_{21} R_{11}^{-1} R_{12};$$
5. Compute the coefficients of the regression of $Y \overset{\mathcal{D}}{=} X_8$ on X_{CN} as follows:
$$\beta_k = \frac{\sigma_8}{\sigma_k} b_k, \quad \text{for } k = 1, \ldots, 7;$$
6. Compute the intercept of the regression of X_8 on X_{CN} as follows:
$$\beta_0 = \mu_8 - \sum_{k=1}^{7} \beta_k \mu_k;$$

7. Simulate a random sample of size N from

$$\widehat{Y} = \beta_0 + \sum_{k=1}^{7} \beta_k X_k,$$

where X_k, for $k = 1, \ldots, 7$, are taken from step 1, which results in an $N \times 1$ column vector $\widehat{Y}_i$, for $i = 1, \ldots, N$;

8. Set $\delta^{(0)} = 1$ and compute

$$e^{(0)} = \sigma_8 \sqrt{1 - R^2};$$

9. Set $j = 0$;

10. Simulate a random sample of size N from

$$Y^{(j)} = \widehat{Y} + \delta^{(j)} \tau,$$

where τ follows the Normal distribution $\mathcal{N}\left(0, e^{(0)}\right)$ independent of X_{CN} and Y obtained at step 1 and step 2, which results in an $N \times 1$ column vector $Y_i^{(j)}$, for $i = 1, \ldots, N$;

11. Sort in ascending order both of the $N \times 1$ column vectors Y_i and $Y_i^{(j)}$, for $i = 1, \ldots, N$, and output the ordered vectors $Y_{(i)} = Y_{f(i)}$ and $Y_{(i)}^{(j)} = Y_{g(i)}^{(j)}$, where f and g are two permutations on $\{1, \ldots, N\}$, which ensures the highest possible correlation between Y and $Y^{(j)}$;

12. Output the $N \times 1$ column vector $X_{8,i}^{(j)} = Y_{f \circ g^{-1}(i)}$, for $i = 1, \ldots, N$, where $f \circ g^{-1}$ is the permutation on $\{1, \ldots, N\}$ defined by $f \circ g^{-1}(i) = f\left(g^{-1}(i)\right)$ and $X_8^{(j)}$ is the copy of $Y \overset{\mathcal{D}}{=} X_8$, obtained at iteration j, which better matches the desired statistical dependence with X_{CN};

13. Compute

$$R_{(j)}^2 = 1 - \frac{\sum_{i=1}^{N} \left(X_{8,i}^{(j)} - \widehat{Y}_i\right)^2}{\sum_{i=1}^{N} \left(Y_i - \overline{Y}\right)^2} \in [0, 1];$$

14. If $R_{(j)}$ is close to R, stop and consider $X_8^{(j)}$ a copy from X_8 consistent with the dependence structure of X; else, compute

$$e^{(j+1)} = \sqrt{\frac{\sum_{i=1}^{N} \left(X_{8,i}^{(j)} - \widehat{Y}_i\right)^2}{N - (\text{size}(X_{\text{CN}}) + 1)}} > 0 \quad \text{and} \quad \delta^{(j+1)} = \frac{e^{(j+1)}}{e^{(j)}},$$

set $j \leftarrow j + 1$, go to step 10, and repeat.

Ilich (2009) can be seen as an extension of Whitt (1976) as it enables the simulation of X_8, given X_{CN}, which does respect the known moment-based correlation coefficients between X_8 and the components of X_{CN}. These coefficients are within their associated minimum and maximum counterparts.

The simulation of 10^6 copies of the NORTA vector X à la Ilich (2009) takes a CPU time of 332 seconds, while the combination of Cario and Nelson (1997) and Ilich (2009) takes only a CPU time of 104 seconds and yet is more accurate. Monte Carlo estimates of the components of ρ^X result in absolute errors of

$$\left|\widehat{\rho}_{\text{IL}}^X - \rho^X\right| = \begin{bmatrix} .000 & .007 & .010 & .017 & .046 & .008 & .005 & .007 \\ & .000 & .002 & .020 & .066 & .003 & .020 & .034 \\ & & .000 & .020 & .067 & .003 & .012 & .008 \\ & & & .000 & .014 & .000 & .014 & .053 \\ & & & & .000 & .037 & .068 & .127 \\ & & & & & .000 & .004 & .035 \\ & & & & & & .000 & .009 \\ & & & & & & & .000 \end{bmatrix}$$

and

$$\left|\widehat{\rho}^{X}_{\mathrm{CN+IL}} - \rho^{X}\right| = \begin{bmatrix} .000 & .000 & .000 & .000 & .000 & .000 & .000 & .014 \\ & .000 & .000 & .001 & .000 & .000 & .000 & .008 \\ & & .000 & .000 & .001 & .000 & .000 & .006 \\ & & & .000 & .000 & .000 & .000 & .034 \\ & & & & .000 & .000 & .001 & .057 \\ & & & & & .000 & .000 & .028 \\ & & & & & & .000 & .010 \\ & & & & & & & .000 \end{bmatrix}.$$

The combination of Cario and Nelson (1997) and Ilich (2009) is relevant since a large part of the experiment is done in numerical integration, which is more efficient than Monte Carlo simulation.

5 Normal- versus NORTA-based credit-risk models

We use the Normal, Student, and Asymmetric Exponential Power (AEP) distributions. The last one can be controlled for its asymmetry (skewness) parameter $\eta \in (0,1)$ and its left and right tail (kurtosis) parameters $(p_1, p_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$. The triplet $(\eta, p_1, p_2)^{\mathrm{T}}$ represents the shape parameters of an AEP distribution, where $\eta = 0.5$ indicates symmetry, $\eta < 0.5$ right skewness, $\eta > 0.5$ left skewness, $p_1 = p_2 = 2$ Normal kurtosis, $p_1 < 2$ left fat tail, $p_1 > 2$ left thin tail, $p_2 < 2$ right fat tail, and $p_2 > 2$ right thin tail. Zhu and Zinde-Walsh (2009) provide estimation methods for these parameters. The standard Normal distribution $\mathcal{N}(0,1)$ is a particular AEP with $\eta = 0.5$, $p_1 = 2$, and $p_2 = 2$.

To assess NORTA for credit-risk analysis, we twice simulate the credit-risk model of Section 2 and estimate its credit-risk measures, once under a NORTA vector and then under a Normal vector with the same marginals' first and second moments. Yet, gaps in credit-risk measures can be significant, which challenges Hamerle and Rösch's (2005) conclusion and supports NORTA as a viable alternative to the Normal credit-risk model.

Set $d = 5$, $a_1 = 0.3$, $a_2 = 0.15$, $a_3 = 0.2$, $a_4 = 0.45$, $a_5 = 0.25$, $b^2 = 0.5825$, so that $\sum_{k=1}^{5} a_k^2 + b^2 = 1$, $n = 1000$, $p_i = 2\% + 1\% \times \sin\left(-\frac{\pi}{2} + i\frac{\pi}{n}\right)$, $c_i = 1$, and $P_i = 1$, for $i = 1, \ldots, n$. The specific risk factors ϵ_i are independent $\mathcal{N}(0,1)$, for $i = 1, \ldots, n$. The set of common risk factors $X = (X_1, \ldots, X_d)^{\mathrm{T}}$ is a standardized NORTA vector with a correlation matrix

$$\rho^X = \begin{bmatrix} 1 & \rho & \rho & \rho & \rho \\ & 1 & \rho & \rho & \rho \\ & & 1 & \rho & \rho \\ & & & 1 & \rho \\ & & & & 1 \end{bmatrix},$$

where ρ is the sole contagion parameter. The vector X is independent of all specific risk factors ϵ_i, for $i = 1, \ldots, n$.

The first experiment considers a NORTA vector made up of five standardized Student marginals, $\mathcal{T}(5)$, each with five degrees of freedom, whose probability density function is depicted in Figure 1. From Table 2, CVaR shows contagion effects at all levels of risk $\alpha \in \{0.950, 0.990, 0.999\}$, while VaR illustrates the same effects only for high risk levels $\alpha \in \{0.990, 0.999\}$, as $\rho \in \{0.0, 0.2, 0.4, 0.6\}$ increases.

NORTA effects are confirmed, as the difference between associated credit-risk measures expands. This result is explained by differences beyond the marginals' third moments. In all cases, CVaR^-, CVaR, and CVaR^+ are close to each others since the loss function L behaves as if it were continuous around VaR_α. For each level of risk $\alpha \in \{0.950, 0.990, 0.999\}$ and of contagion $\rho \in \{0.0, 0.2, 0.4, 0.6\}$, the highest CPU time is 473 seconds for NORTA versus 408 seconds for its matched Normal model to say that NORTA requires only a few additional effort to be implemented.

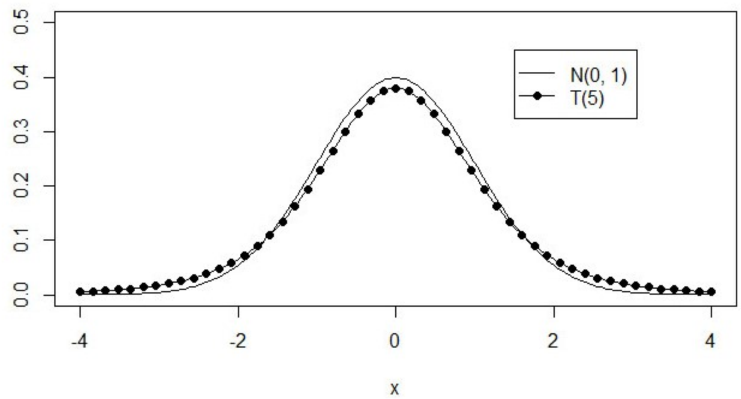

Figure 1: PDF of T(5) vs N(0,1)

Table 2: NORTA with $\mathcal{T}(5)$ versus Normal marginals

Risk measure	level ρ	$\alpha = 0.950$ Normal	$\alpha = 0.950$ NORTA	$\alpha = 0.990$ Normal	$\alpha = 0.990$ NORTA	$\alpha = 0.999$ Normal	$\alpha = 0.999$ NORTA
VaR	0.0	97.00	90.00	232.00	263.00	461.00	689.00
CVaR$^-$	0.0	180.44	201.56	329.75	432.53	548.46	837.96
CVaR	0.0	180.83	202.90	330.16	433.22	549.42	838.41
CVaR$^+$	0.0	181.65	203.16	330.92	433.65	549.69	839.01
VaR	0.2	106.00	96.00	304.00	354.00	616.00	861.00
CVaR$^-$	0.2	225.25	254.02	437.66	566.29	710.34	941.64
CVaR	0.2	226.23	254.35	438.08	566.36	710.43	941.72
CVaR$^+$	0.2	226.60	255.42	438.62	567.51	711.29	942.04
VaR	0.4	110.00	94.00	356.00	430.00	716.00	949.00
CVaR$^-$	0.4	256.63	289.28	514.64	668.47	806.98	982.94
CVaR	0.4	256.92	290.21	515.08	668.54	807.53	982.98
CVaR$^+$	0.4	257.99	290.82	515.69	669.11	807.53	983.21
VaR	0.6	110.00	89.00	397.00	492.00	788.00	986.00
CVaR$^-$	0.6	278.69	315.36	572.45	742.46	867.85	996.53
CVaR	0.6	279.10	316.05	572.79	743.03	868.17	996.59
CVaR$^+$	0.6	279.95	317.19	573.25	743.13	868.25	996.77

The second experiment considers a NORTA made up of five standardized marginals AEP$(0.5, 1, 1)$, known as the Laplace distribution, whose probability density function is depicted in Figure 2.

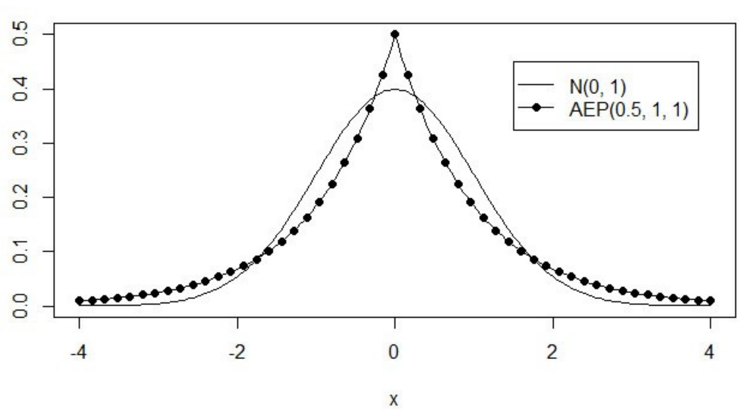

Figure 2: PDF of AEP(0.5, 1, 1) vs N(0,1)

Table 3 exhibits similar results than Table 2. The highest CPU time is 455 seconds for NORTA and 408 seconds for its matched Normal model.

Table 3: NORTA with AEP$(0.5, 1, 1)$ versus Normal marginals

Risk measure	level ρ	$\alpha = 0.950$ Normal	$\alpha = 0.950$ NORTA	$\alpha = 0.990$ Normal	$\alpha = 0.990$ NORTA	$\alpha = 0.999$ Normal	$\alpha = 0.999$ NORTA
VaR	0.0	97.00	92.00	232.00	273.00	461.00	641.00
CVaR$^-$	0.0	180.44	204.74	329.75	426.64	548.46	768.25
CVaR	0.0	180.83	205.72	330.16	426.96	549.42	768.76
CVaR$^+$	0.0	181.65	206.28	330.92	427.90	549.69	769.02
VaR	0.2	106.00	97.00	304.00	367.00	616.00	827.00
CVaR$^-$	0.2	225.25	258.54	437.66	566.56	710.34	909.82
CVaR	0.2	226.23	259.65	438.08	566.92	710.43	910.16
CVaR$^+$	0.2	226.60	260.02	438.62	567.36	711.29	910.24
VaR	0.4	110.00	95.00	356.00	450.00	716.00	930.00
CVaR$^-$	0.4	256.63	298.03	514.64	674.71	806.98	970.88
CVaR	0.4	256.92	298.31	515.08	675.22	807.53	971.17
CVaR$^+$	0.4	257.99	299.58	515.69	675.52	807.53	971.29
VaR	0.6	110.00	88.00	397.00	521.00	788.00	977.00
CVaR$^-$	0.6	278.69	326.38	572.45	754.56	867.85	992.56
CVaR	0.6	279.10	326.59	572.79	754.82	868.17	992.62
CVaR$^+$	0.6	279.95	327.97	573.25	755.22	868.25	992.97

The third experiment considers a NORTA vector made of five standardized AEP$(0.8, 0.5, 4)$ marginals whose probability density function is depicted in Figure 3.

As expected, Table 4 shows higher contagion and NORTA effects, since AEP$(0.8, 0.5, 4)$ is skewed left. The highest CPU time is 533 seconds for NORTA and for 408 seconds for its matched Normal model.

CVaR is more consistent than VaR for it shows contagion and NORTA effects at all levels of risk $\alpha \in \{0.950, 0.990, 0.999\}$ and of contagion $\rho \in \{0.0, 0.2, 0.4, 0.6\}$. Even though the matching procedure is based on the first three marginals' moments, gaps between credit-risk measures under the NORTA versus the Normal vector can be significant, as shown in Table 2 and Table 3. With a matching procedure based only on the first two marginals' moments, the gaps can be remarkably high, as reported in Table 4. This supports NORTA for credit-risk modeling and analysis.

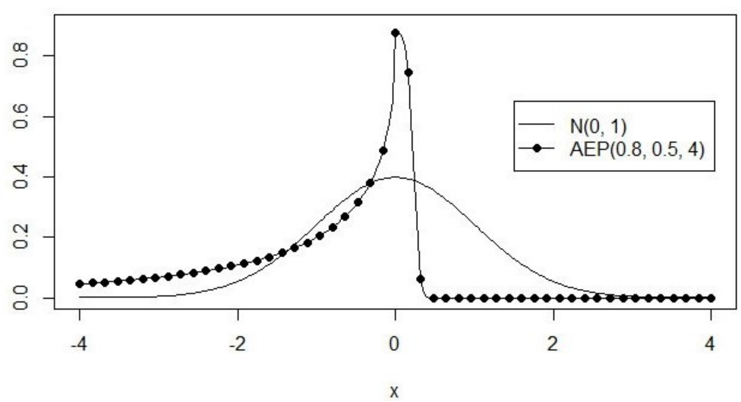

Figure 3: PDF of AEP(0.8, 0.5, 4) vs N(0,1)

Table 4: NORTA with AEP$(0.8, 0.5, 4)$ versus Normal marginals

Risk	level	$\alpha = 0.950$		$\alpha = 0.990$		$\alpha = 0.999$	
measure	ρ	Normal	NORTA	Normal	NORTA	Normal	NORTA
VaR	0.0	97.00	79.00	232.00	507.00	461.00	995.00
CVaR$^-$	0.0	180.44	311.49	329.75	764.78	548.46	999.00
CVaR	0.0	180.83	312.25	330.16	765.32	549.42	999.07
CVaR$^+$	0.0	181.65	313.48	330.92	765.45	549.69	999.25
VaR	0.2	106.00	66.00	304.00	648.00	616.00	1000.00
CVaR$^-$	0.2	225.25	358.36	437.66	872.13	710.34	1000.00
CVaR	0.2	226.23	358.55	438.08	872.34	710.43	1000.00
CVaR$^+$	0.2	226.60	360.29	438.62	872.58	711.29	1000.00
VaR	0.4	110.00	50.00	356.00	737.00	716.00	1000.00
CVaR$^-$	0.4	256.63	377.46	514.64	924.06	806.98	1000.00
CVaR	0.4	256.92	378.91	515.08	924.29	807.53	1000.00
CVaR$^+$	0.4	257.99	379.85	515.69	924.42	807.53	1000.00
VaR	0.6	110.00	38.00	397.00	790.00	788.00	1000.00
CVaR$^-$	0.6	278.69	388.34	572.45	948.00	867.85	1000.00
CVaR	0.6	279.10	388.41	572.79	948.05	868.17	1000.00
CVaR$^+$	0.6	279.95	391.16	573.25	948.21	868.25	1000.00

6 Conclusion

We use NORTA (**NOR**mal **T**o **A**nything) to simulate the set of common risk factors of a credit-risk factor model and estimate its main credit-risk measures. NORTA is a Gaussian copula that enables the simulation of a random vector with arbitrary and known marginal distributions and moment- or rank-based correlation coefficients. NORTA can be solved either by numerical integration (Cario and Nelson 1997) or by Monte Carlo simulation (Ilich 2009). The former approach, which is the most efficient, assumes that the marginals' inverse cumulative functions are given, while the latter, which is more flexible but less efficient, does not. We show how to combine both approaches for higher flexibility and efficiency.

We experiment with Normal, Student, and AEP distributions. We match each NORTA vector to a Normal vector with the same marginals' first and second moments. Yet, credit-risk measures can be highly different. All in all, NORTA presents three advantages. NORTA is simple to implement, flexible enough to accommodate arbitrary marginals, and remains consistent with almost all the findings on the Normal credit-risk factor model.

An extension to this work consists of building a dynamic credit-risk model based on ARTA (Cario and Nelson 1998) and VARTA (Biller and Nelson 2003). We believe that NORTA-based dynamic frameworks can achieve an acceptable compromise between realism and efficiency.

Another extension builds on Markov-Lévy processes, which extend the well-known pure-diffusion lognormal process. Examples include Gaussian and double-exponentiel jump-diffusions as well as variance-gamma. They are usually used to model the dynamics of quoted financial assets, such as stocks and stock indices. Multivariate Lévy processes cannot be used to describe the behavior of the non-quoted commercial loans underlying credit portfolios, but, in contrast, they can be used to model the common factors that affect the credit quality of these commercial loans.

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