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Abstract

The paper answers the three distinct questions of maximizing the perimeter, diameter and area of equilateral unit-width convex polygons. The solution to each of these these problems is trivially unbounded when the number of sides is even. We show that when the number is odd, the optimal solution to these three problems is identical, and arbitrarily close to a trapezoid.

Key Words: Polygon, perimeter, diameter, area, width.

Résumé

Ce papier répond aux trois questions distinctes de maximisation du périmètre, du diamètre et de l'aire d'un polygone équilatéral convexe de largeur unitaire. La solution à ces trois problèmes tend trivialement vers l'infini lorsque le nombre de côté est pair. Nous montrons que lorsque ce nombre est impair, la solution optimale à ces trois problèmes est identique, et est arbitrairement proche d'un trapèze.

Mots clés : Polygone, périmetre, diamètre, aire, largeur.

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1 Introduction

Two families of extremal problems for convex polygons are presented in [1, 3]. For a fixed number of sides, each family consists in maximizing or minimizing an attribute such as the area, perimeter, diameter and width of a polygon, while fixing another attribute. Recall that the diameter of a polygon is the length of its longest line segment joining two vertices, and its width is the minimal distance between two parallel lines enclosing the polygon. In the first family of problems, the length of the sides of the polygon are free, but in the second family they are required to be equilateral. These two papers either give solutions or provide references to solutions of some of these problems, and indicate which ones are trivial, and which ones remain open.

The present paper focuses on the three open questions of maximizing the perimeter, the diameter and the area of unit-width equilateral convex polygons. The paper is divided as follows. The remainder of the introductory section presents the optimal polygons for these three problems, and shows results that are valid for every equilateral unit-width convex polygons. Section 2 gives the proof for the perimeter, and the result for the diameter follows directly as a corollary. Section 3 demonstrates the result for the area.

1.1 Statement of the optimal results

The question of minimizing the perimeter and the diameter of unit-width convex polygons are studied in [4] and [6], respectively. For both problems, as well as for the problem of maximizing the perimeter of unit-diameter convex polygons [5, 2, 7, 8, 10], clipped-Reuleaux regular polygons [9] are optimal when the number of sides contains an odd factor. Clipped-Reuleaux polygons are equilateral, and therefore, are also optimal in the equilateral case. The cases where the number of sides is a power of 2 remain open. The authors are unaware of any published results on the question of minimizing the area of unit-width convex polygons.

In the case where the unit-width polygons are not restricted to be equilateral, the questions of maximizing the perimeter P_n, the area A_n or the diameter D_n are all trivial, since they may be arbitrarily large. In the equilateral case where the number n of sides is even, these attributes can again be arbitrarily large as illustrated in the left part of Figure 1 with a parallelogram.

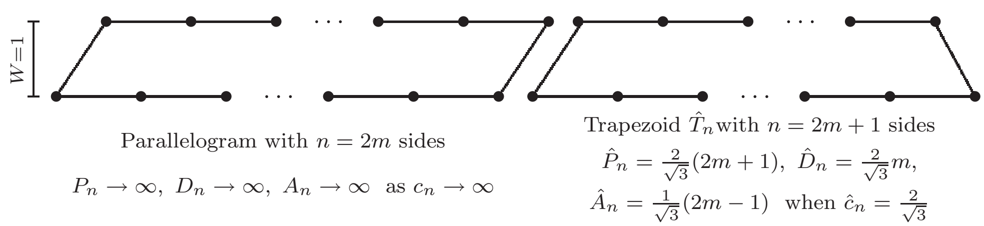

Figure 1: Equilateral n-sided unit-width convex polygons

The paper studies the non-trivial maximization problems, when the number of equilateral sides n is odd. By considering the equivalent problems of minimizing the width of a unit-perimeter, unit-diameter or unit-area convex equilateral polygons, it is trivial that the minimal width is strictly greater than zero when the number of equilateral sides is odd (since the vertices cannot be colinear). Thus, the maximal perimeter, diameter and area of a equilateral unit-width convex polygon with $n = 2m + 1$ sides is bounded.

The case where $n = 3$ is trivial as the equilateral triangle of unit-width is unique. For $n \geq 5$, we show that for the three considered problems, the optimal solutions are arbitrarily close to the trapezoid whose non-parallel sides have length equal to $\hat{c}_n = \frac{2}{\sqrt{3}}$, and the parallel ones have length $\frac{2}{\sqrt{3}}m$ and $\frac{2}{\sqrt{3}}(m-1)$. The words *arbitrarily close* are used to indicate that, technically, the trapezoid has four sides, not $2m + 1$. This trapezoid, denoted by $\hat{T}_n$, is illustrated in the right part of Figure 1, together with its perimeter $\hat{P}_n$, diameter $\hat{D}_n$ and area $\hat{A}_n$.

1.2 General results for equilateral unit-width convex polygons

Consider an equilateral n-sided convex polygon having a strictly positive width W_n. Coordinates are represented in the standard xy plane, and all distances are Euclidean. Let $y = 0$ and $y = W_n$ be the two parallel lines enclosing the polygon, and defining its width. By taking a vertical symmetry if necessary, assume that the line $y = 0$ contains at least one side of the polygon, and the line $y = W_n$ contains at least one vertex.

Let A denote the vertex on the line $y = 0$ that has the least value of x, and B the vertex on the line $y = W_n$ with the least value of x. Let C denote the vertex of the polygon that has the largest value of x (if there are more than one such vertex, the one with the largest value of y is chosen). The coordinates of A, B and C are denoted by (x_A, y_A), (x_B, y_B) and (x_C, y_C), respectively. Let B^- denote the vertex adjacent to B, obtained in an anti-clockwise way. These four vertices are represented in Figure 2.

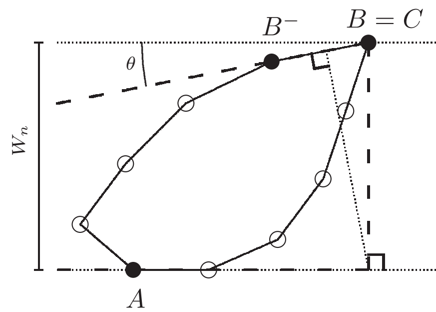

Figure 2: The case where the vertex with $y = W_n$ and the least value of x coincides with the vertex having the largest value of x.

We first demonstrate that $A \neq B \neq C \neq A$. By construction, $A \neq C$ since there are at least vertices on the line $y = 0$ adjacent to A with a larger value of x. Moreover, $A \neq B$ since $y_B = W_n > 0 = y_A$.

Now, suppose by contradiction that $B = C$. This implies that there is a unique vertex on the line $y = W_n$ as illustrated in Figure 2. Let $\theta \in]0, \frac{\pi}{2}]$ be the strictly positive angle between the line $y = W_n$ and the line L supporting B and B^-. By convexity of the polygon, all vertices of the polygon are contained in the triangle delimited by the lines $y = 0$, $x = x_C$ and L, represented by dashes in Figure 2. The width of this triangle is equal to $W_n \cos\theta$ which is stricty less than W_n. This contradicts the fact that W_n is the minimal width of the polygon which shows that $B \neq C$.

These three distinct points A, B and C are used in the proofs of the next sections.

2 Maximizing the perimeter and the diameter

This section demonstrates the results announced in Section 1.1, that the equilateral unit-width convex polygons which maximize the perimeter or the diameter are identical and arbitrarily close to the trapezoid $\hat{T}_n$. Theorem 2.2 demonstrates the result for the maximization of the perimeter, and Corollary 2.3 proves it for the diameter.

All results presented below rely on the notation presented in Section 1.2.

Lemma 2.1 *In any $(2m+1)$-unit-sided convex polygon with minimum width, the vertex with $y > 0$ adjacent to* A *is* B.

Proof. Consider a $(2m+1)$-unit-sided convex polygon with minimum width W_n^*, and let A^+ be the vertex with $y > 0$ adjacent to A.

Suppose by contradiction that $A^+ \neq B$. Denote the interior angles of the polygon associated to the vertices A, B, C and A^+ by α, β, γ and α^+, respectively. The definition of the vertices ensure strict bounds on the angles: $\alpha < \pi$, $\beta < \pi$, $\gamma > 0$ and $\alpha^+ > 0$. The quadrilateral AA^+BC and the four angles are illustrated in Figure 3.

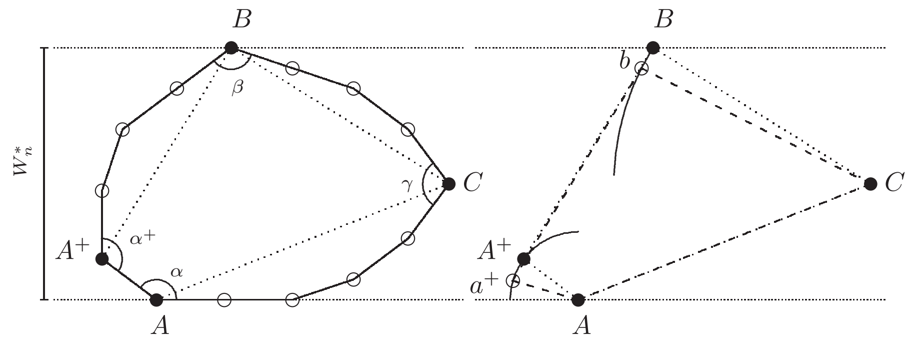

Figure 3: The quadrilateral AA^+BC inscribed in the polygon

The proof consists in constructing a new $(2m+1)$-unit-sided convex polygon with the same perimeter but with a smaller width than the original one. The construction is based on a transformation of the quadrilateral AA^+BC into Aa^+bC. In order to do so, let b be a point in the xy plane satisfying $\|bC\| = \|BC\|$ and where the y component of b is slightly less than that of B, and let a^+ be another point close to A^+ satisfying $\|Aa^+\| = \|AA^+\|$ and $\|a^+b\| = \|A^+B\|$. The points b and a^+ are illustrated in the right part of Figure 3, and are located on the circular arcs centered at C and A.

The new polygon is created by copying in a clockwise way all vertices between (and including) C and A. Then, we consider the distance preserving transformation that maps A^+ to a^+ and B to b, and add the image under this transformation of all vertices between (and including) A^+ and B. Finally, consider the rotation centered at C that maps B to b and add the image under this rotation of all vertices between B and C. Observe that the y coordinates of all vertices of the new polygon are strictly less than W_n^*, except possibly y_C, the y coordinate of the vertex C.

By construction, the new polygon is equilateral, with the same perimeter as the original one. The construction decreased the value of the angles $\gamma > 0$ and $\alpha^+ > 0$ and increased $\alpha < \pi$ and $\beta < \pi$. All other angles in the polygon are unchanged. By taking the vertex b sufficiently close to B, these strict inequalities on the angles remain valid for the new polygon, which implies that it is convex.

Recall that the width of the new polygon cannot be less than the minimal value W_n^*, which implies that $y_C = W_n^*$. But the results derived in Section 1.2 and illustrated in Figure 2 ensure that the width of the new polygon is strictly less than W_n^*. This contradiction implies that $A^+ = B$. ■

The previous lemma ensures that in an optimal polygon, there is a vertex adjacent to A with $y = W_n^*$. This results reduces the number of possible configurations, and allow the proof of the main result.

Theorem 2.2 *The perimeter of any unit-width equilateral $(2m+1)$-sided convex polygon is bounded above by $\frac{2}{\sqrt{3}}(2m+1)$. This bound is attained in the limit by polygons arbitrarily close to the trapezoid $\hat{T}_n$.*

Proof. Instead of maximizing the perimeter over unit-width polygons, the proof addresses the equivalent question of showing that the width of any equilateral $(2m+1)$-sided convex polygon with perimeter $\hat{P}_n = \frac{2n}{\sqrt{3}}$ is bounded below by 1, and this lower bound is attained in the limit by polygons arbitrarily close to the trapezoid $\hat{T}_n$.

Consider an equilateral $(2m+1)$-sided convex polygon with perimeter $\hat{P}_n$ and minimal width $W_n^* > 0$. Lemma 2.1 ensures that the vertex adjacent to A with $y > 0$ is B. Let Z denote the vertex of the polygon with the largest value of x on the line $y = 0$. By symmetry with the vertex A, Lemma 2.1 ensures that the vertex adjacent to Z with $y > 0$ lies on the line $y = W_n^*$. Thus, by convexity of the polygon, the equilateral $(2m+1)$-sided convex polygon with perimeter $\hat{P}_n$ and minimal width W_n^* is a trapezoid, a parallelogram or a triangle.

Let p denote the number of sides of the polygon on the line $y = W_n^*$ and q the number of sides on the support line $y = 0$. By taking a vertical symmetry if necessary, assume that $p \leq q$. The two paths of consecutive vertices joining A and Z satisfy $q \leq p+2$. Therefore, the number of sides q may only take one of the values $p, p+1$ or $p+2$. But since there are a total of $p+q+2$ sides, and that this number is odd, it follows that the only possibility is $q = p+1$. Therefore, the equilateral $(2m+1)$-sided convex polygon with perimeter $\hat{P}_n$ and minimal width W_n^* is the trapezoid $\hat{T}_n$ and thus $W_n^* = 1$. ∎

The last theorem implies that the maximal perimeter of unit-width equilateral n-sided convex polygons is $\hat{P}_n = \frac{2}{\sqrt{3}}(2m+1)$, when $n = 2m+1$. The similar result for the diameter follows directly as a corollary.

Corollary 2.3 *The diameter of any unit-width equilateral $(2m+1)$-sided convex polygon is bounded above by $\frac{2m}{\sqrt{3}}$. This bound is attained in the limit by polygons arbitrarily close to the trapezoid $\hat{T}_n$.*

Proof. Let P_n, for $n = 2m+1$ be the perimeter of an equilateral unit-width convex polygon. The endpoints of any diagonal joining two vertices is necessarily joined by a path of at most $\lfloor \frac{n}{2} \rfloor = m$ consecutive vertices. Combining this with the fact that the length of each side is $\frac{P_n}{n}$, and with Theorem 2.2 provides an upper bound of $\frac{\hat{P}_n}{n} \times m = \frac{2}{\sqrt{3}}m$ on the value of the diameter of the polygon. This bound is attained by the trapezoid $\hat{T}_n$ presented in the introduction. ∎

3 Maximizing the area

Throughout this section, let A_n^* denote the maximal area of unit-width equilateral convex n-sided polygon. Let P_n and D_n denote the perimeter and diameter of an optimal solution, and set $c_n = \frac{P_n}{n}$ be the length of its sides. The results of the previous section ensure that

$$P_n \leq \hat{P}_n, \quad D_n \leq \hat{D}_n \quad \text{and} \quad \hat{A}_n \leq A_n^*.$$

Lemma 3.1 *Let $n = 2m+1 \geq 5$ be an odd number and consider a unit-width equilateral n-sided convex polygon with maximal area A_n^*, and let c_n be the length of each side. The lenght satisfies $c_n > 1$, and if $n \geq 7$ then $c_n \geq \frac{1}{14}\left(\frac{20}{\sqrt{3}} + \pi\right)$.*

Proof. Jaglom and Boltianskii [11, Problem 6.11] show that the maximal area of an unit-width convex figure of perimeter P_n is $\frac{2P_n - \pi}{4}$. In particular, this provides an upper bound on A_n^*. Combining this with the lower bound $\hat{A}_n$ yields

$$\frac{2m-1}{\sqrt{3}} = \hat{A}_n \leq A_n^* \leq \frac{2P_n - \pi}{4} = \frac{2(2m+1)c_n - \pi}{4}.$$

Solving for c_n leads to the following inequality:

$$c_n \geq \frac{8m + \sqrt{3}\pi - 4}{2\sqrt{3}(2m+1)}.$$

This lower bound on c_n is monotone increasing with respect to m. The least value occurs when $n = 2m+1 = 5$, and gives a lower bound slightly exceeding 1. When $n = 2m+1 \geq 7$ the lower bound is $\frac{1}{14}\left(\frac{20}{\sqrt{3}} + \pi\right)$. ∎

The proof that the trapezoid $\hat{T}_n$ has maximal area is done in two parts. The pentagonal case is done in the following lemma, and the remaining cases $(n \geq 7)$ are covered by Theorem 3.3.

Lemma 3.2 *The area of any unit-width equilateral convex pentagon is bounded above by $\sqrt{3}$. This bound is attained in the limit by pentagons arbitrarily close to the trapezoid $\hat{T}_5$.*

Proof. Let A and B be the vertices defined in Section 1.2 on an optimal unit-width convex pentagon with side length c_5. The four possible configurations of the relative positions of A and B are represented in Figure 4: In both figures to the left, the x coordinate of B is between x_A and $x_A + c_5$, and in the figures to the right,

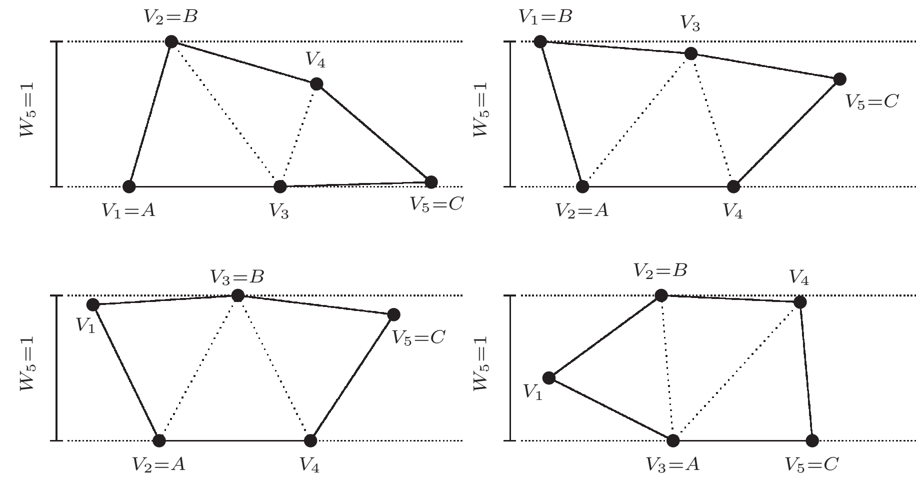

Figure 4: Four possible configurations for the pentagon

the coordinate is less than x_A. In the two top figures, the vertex B is adjacent to A, and not in the bottom figures.

In the four configurations represented in the figure, the vertices are denoted by $V_i = (x_i, y_i)$ with $i = 1, 2, \ldots, 5$ and are ordered in such a way that $x_1 < x_2 < \cdots < x_5$. This ordering with strict inequalities is possible since Lemma 3.1 ensures that the length c_5 of the sides of the pentagon is strictly greater than one. Observe that for $i \in \{2, 3, 4\}$, each triangle $V_{i-1}V_iV_{i+1}$ in every configuration has an area equal to $\frac{h_i c_5}{2}$ where h_i is the height associated to the vertex V_i. However, since $h_i \leq \max(|y_i - y_{i-1}|, |y_i - y_{i+1}|) \leq 1$ for every $i \in \{2, 3, 4\}$, it follows that the area of the pentagon satisfies

$$A_5^* \;=\; \frac{c_5}{2}(h_2 + h_3 + h_4) \;\leq\; \frac{3c_5}{2} \;\leq\; \frac{3\hat{c}_5}{2} \;=\; \sqrt{3}.$$

This upper bound is achieved by the trapezoid $\hat{T}_5$. ∎

The next theorem closes the general case with an odd number of vertices.

Theorem 3.3 *The area of any unit-width equilateral* $(2m + 1)$*-sided convex polygon is bounded above by* $\frac{2m-1}{\sqrt{3}}$*. This bound is attained in the limit by polygons arbitrarily close to the trapezoid* $\hat{T}_n$*.*

Proof. The case where $n = 3$ is trivial, and the one where $n = 5$ is shown in Lemma 3.2. Consider a unit-width convex polygon with maximal area A_n^* and $n = 2m + 1 \geq 7$ sides. Using the notation presented in Section 2, let (x_C, y_C) denote the coordinates of the vertices having the largest value of x. Denote by C^- and C^+ the vertices adjacent to C.

Let α be the area of the region comprised within the band of width 1 enclosing the polygon composed of the part above the side delimited by C^- and C and the part below the side delimited by C and C^+, as represented in Figure 5 by the shaded region.

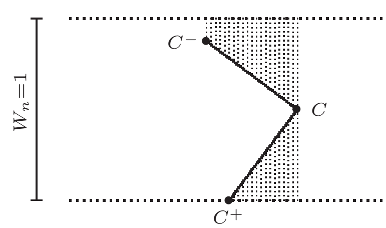

Figure 5: Correction α to the upper bound of the area.

Since Lemma 3.1 ensures that $c_n > 1$ when $n = 2m+1 \geq 7$, the value of α is bounded below by the sum of the area of the triangle of height y_C and hypothenuse c_n with the area of the triangle of height $(1 - y_C)$ and hypothenuse c_n:

$$\alpha \geq \frac{1}{2}\left(y_C\sqrt{c_n^2 - y_C^2} + (1-y_C)\sqrt{c_n^2 - (1-y_C)^2}\right).$$

This lower bound on α is a concave function with respect to the variable y_C and achieves its minimal value when $y_C = 0$ or 1. In both cases, the inequality becomes $\alpha \geq \frac{1}{2}\sqrt{c_n^2 - 1}$. Therefore, the upper bound $D_nW_n \leq mc_n$ on the value of A_n^* may be improved by removing a first value of α associated to the vertex C and another value of α for the vertex with the least value of x. This leads to the bounds:

$$\frac{2m-1}{\sqrt{3}} \leq A_n^* \leq mc_n - 2\alpha \leq mc_n - \sqrt{c_n^2 - 1}.$$

Recall that for any $n = 2m+1 \geq 7$, Theorem 2.2 and Lemma 3.1 provide bounds on the length of the sides:

$$\frac{1}{14}\left(\frac{20}{\sqrt{3}} + \pi\right) \;\leq\; c_n \;\leq\; \frac{2}{\sqrt{3}}.$$

However, since $mc_n - \sqrt{c_n^2 - 1}$ is a convex function with respect to c_n, it follows that the largest upper bound is achieved at either $c_n = \frac{1}{14}\left(\frac{20}{\sqrt{3}} + \pi\right)$ or $c_n = \frac{2}{\sqrt{3}}$. It can be verified that for $m \geq 3$, the largest value occurs in the latter case, and therefore

$$\frac{2m-1}{\sqrt{3}} \;=\; \hat{A}_n \;\leq\; A_n^* \;\leq\; \frac{2}{\sqrt{3}}m - \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - 1} \;=\; \frac{2m-1}{\sqrt{3}}.$$

This shows that the optimal area is $\frac{2m-1}{\sqrt{3}}$, and is achieved by the trapezoid $\hat{T}_n$ illustrated in Figure 1. ■

Notice that the last part of the proof does not hold for $m = 2$, and this is why Lemma 3.2 is necessary for the pentagon.

4 Conclusion

The optimal solutions to the equivalent problems consisting of minimizing the width of equilateral convex polygon with unit-perimeter, unit-diameter or unit-area follow by applying homotheties to the trapezoid $\hat{T}_n$.

Table 1 summarizes the solutions of the non-trivial cases of the equivalent problems. The values represent the upper and lower bounds of the width of an equilateral convex polygon with unit-perimeter, unit-diameter or unit-area.

Table 1: Equilateral convex polygons with unit-perimeter, unit-diameter or unit-area which minimize or maximize the width.

	$P_n = 1$	$D_n = 1$	$A_n = 1$
$\max W_n$	$\frac{1}{2n}\cot(\frac{\pi}{2n})$ Clipped-Reuleaux for n with odd factor	$\cos(\frac{\pi}{2n})$ Clipped-Reuleaux for n with odd factor	open
$\min W_n$	Trapezoid with odd n $\frac{\sqrt{3}}{2n}$	Trapezoid with odd n $\frac{\sqrt{3}}{n-1}$	Trapezoid with odd n $\sqrt{\frac{\sqrt{3}}{n-2}}$

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