Mixed H_2/H_∞ Stochastic Control Problem

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Abstract

In this paper, the mixed H_2/H_∞ control problem for continuous-time Markovian jumping linear systems using state-feedback control is considered. A Nash game approach to this problem is presented, and necessary and sufficient conditions for the solvability of this problem are given in terms of a set of two coupled Riccati equations. For the finite-horizon problem, the Riccati equations can easily be integrated using standard numerical schemes, while for the infinite-horizon, the solutions can be obtained as the limiting values of the finite-horizon case. The robust control problem with norm-bounded uncertainties is also considered. A quadratic stabilization approach to this problem is developed. Sufficient conditions for the solvability of this problem are given. Finally, simulation results are presented to show the usefulness of the results.

Key words: H_2, H_∞-norms, Markov jump process, disturbance attenuation, stochastic stability, quadratic stabilization.

Résumé

Dans cet article, la commande H_2/H_∞ pour la classe des systèmes linéaires à sauts markoviens est étudiée. L'approche basée sur la théorie des jeux est utilisée. Des conditions nécessaire et suffisantes pour la résolution de ce problème sont données en terme d'équations de Riccati couplées. Les deux horizons fini et infini sont considérés. La commande robuste de cette classe de systèmes est aussi étudiée en présence d'incertitudes bornées et des conditions suffisantes pour guarantir la stabilité sont établies. Des simulations utilisant Matlab sont données pour montrer l'utilisabilité de nos résultats.

1 Introduction

The nonuniqueness of the solution of the H_∞ control problem [6] has motivated the development of the mixed H_2/H_∞ control problem and other mixed approaches [18, 17, 19]. In particular, the mixed H_2/H_∞ control problem was considered by Bernstein and Haddad [3] in an LQG setting. Their approach was to minimize an upper bound on the H_2 performance (or auxiliary cost) subject to an H_∞ norm constraint. They gave necessary conditions for the solvability of this problem. Subsequently, the dual case of this problem was considered by Doyle et al. [7]. Mustapha [16] considered the problem of minimizing the entropy of the closed-loop transfer function subject to an H_∞ constraint. His work showed that the entropy of the closed-loop system is closely related to the upper bound on the H_2 performance used by Bernstein and Haddad [3]. Khargonekar and Rotea [12] have also considered the mixed problem in a multi-objective setting. Furthermore, they have given a solution based on convex optimization and a computational algorithm.

The outcome of the above endeavours is a characterization of the solution of the mixed H_2/H_∞ control problem in terms of a pair of cross-coupled Riccati equations and a standard H_∞ Riccati equation for the output feedback problem. However, efficient computational methods for solving these equations are still an on going research problem.

Of all the state-space approaches to the solution of the H_∞ control problem, the game-theoretic approach has proven to be the most transparent [2]. Similarly, the Nash-game approach to the state-feedback mixed H_2/H_∞ control problem of Limebeer et al. [14] has this characteristic. The resulting solution for the finite-horizon problem, in the form of a pair of cross-coupled Riccati equations also lends itself easily to standard numerical integration schemes; while the solution of the infinite-horizon problem can be obtained as the limiting case of the finite-horizon solution.

In this paper, we consider the state-feedback mixed H_2/H_∞ control problem for a class of stochastic linear systems with Markovian Jump parameters (MJLS). This class of systems belongs to the class of hybrid systems with a continuous state dynamics and discrete parameter variation. The H_∞ control problem for this class of systems has been considered in some few references [1, 5, 10] and the discrete-time mixed H_2/H_∞ control problem has also been considered [4]. Therefore, in this paper we shall consider the continuous-time problem for the general time-varying case. The rest of the paper is organized as follows. In section 2, we introduce notations and give problem definition. Then we present the solution of the finite-horizon problem in section 3. In section 4, we discuss the infinite-horizon problem, and in section 5, we discuss the robust control problem. Finally in section 6 we discuss simulation results.

2 Problem Statement

In this section we introduce notations, give problem definition, and define other important concepts related to the solution of the problem. The notation is quite standard except where otherwise specified. Moreover, $\|(.)\|$ will denote the Euclidean norm of a vector or matrix, depending on the context. $L_2([0,T],\Re^n)$ will denote the Lebesgue space of vector functions $[0,T] \to \Re^n$ that are square integrable over $[0,T]$ with the norm $\|(.)\|_2$, while $L_2[0,\infty)$ will denote the same space over the half-line $[0,\infty)$. Similarly, $\mathcal{L}_2([0,T],(\Omega,\mathcal{F},P))$ and $L_2([0,\infty),(\Omega,\mathcal{F},P))$ are the corresponding spaces over the probability space $(\Omega,\mathcal{F},P)$ with the norm $\|(.)\|_{2E}$, in which Ω is the sample space, $\mathcal{F}$ is the σ-algebra generated by Ω and P is a probability measure over $\mathcal{F}$. E denotes the expectation operator of some random function. The notation $M \geq 0$, (respectively $M \leq 0$) will denote a positive semidefinite (respectively, a negative semidefinite) matrix, while the strict inequality will denote positive (respectively, negative) definiteness. Furthermore, for a complex valued matrix M, $M^\star$ will denote the complex conjugate transpose of M.

Now, we shall consider the following linear-time-varying Markovian jump parameter (LTVMJ) system:

$$\dot{x}(t) = A(t,r(t))x(t) + B_1(t,r(t))w(t) + B_2(t,r(t))u(t); x(0) = x_0,\ r(0) = r_0 \tag{1}$$

$$z(t) = C(t,r(t))x(t) + D(t,r(t))u(t) \tag{2}$$

where $x(t) \in \Re^n$ is the state vector, $w(t) \in L_2([0,T],\Re^s)$ is the disturbance (to be rejected) and/or reference to be tracked, $u(t) \in \Re^k$ is the control input, $z(t) \in L_2([0,T],(\Omega,\mathcal{F},\Re^m)$ is the controlled output. Further, the matrices $A(.,.) \in \Re^{n\times n}, B_1(.,.) \in \Re^{n\times s}, B_2(.,.) \in \Re^{n\times k}, C(.,.) \in \Re^{m\times n}, D(.,.) \in \Re^{m\times k}$ are continuous functions of time $t \in [0,T]$ for each value of $r(t) \in \mathcal{S}$. Assume also that the following simplifying assumptions are satisfied

$$C^T(t,r(t))D(t,r(t)) = 0;\ \ D^T(t,r(t))D(t,r(t)) = I > 0 \tag{3}$$

The parameter $r(t)$ is a continuous-time Markov process with finite discrete state-space $\mathcal{S} \stackrel{\triangle}{=} \{1,2,\ldots,l\}$ We assume that the transition probability vector $P_t \stackrel{\triangle}{=} (P_{1t},\ldots,P_{lt})$, with $P_{it} \stackrel{\triangle}{=} P(r(t)=i), i=1,\ldots,l$ satisfies the forward Kolmogorov equation $\frac{dP_t}{dt} = \Lambda P_t;\ P_0 = \bar{P},\ t \in [0,T]$, where the transition matrix $\Lambda = [\lambda_{ij}]_{i,j\in\mathcal{S}}$ and λ_{ij} are real numbers such that for $i \neq j,\ \lambda_{ij} \geq 0$, and for all $i \in \mathcal{S},\ \lambda_{ii} = -\sum_{j=1,j\neq i}^{l} \lambda_{ij}$. In other words, the transition probabilities are given by

$$P[r(t+h)=j|r(t)=i] = \begin{cases} \lambda_{ij}h + o(h) & \text{if } j \neq i \\ 1 + \lambda_{ii}h + o(h) & \text{if } j = i \end{cases} \tag{4}$$

where $o(h)$ are the remainder terms such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$

Let $\mathcal{F}_t$ be the σ-algebra generated by the process $r(t), t \in [0,T]$. Then we take the admissible control and disturbance processes $u(.)$ and $w(.)$, respectively, to be $\mathcal{F}_t$-measurable, and piecewise continuous with the corresponding spaces denoted by $\mathcal{U} \subset \Re^k$, $\mathcal{W} \subset \Re^s$. Our objective is to determine an optimal (or suboptimal) state-feedback controller, with the usual smoothness assumptions, of the form:

$$u(t,x,r(t)) = K(t,r(t),x(t)) \tag{5}$$

$$\|K(t,x,r) - K(t,\tilde{x},r)\| < k\|x-\tilde{x}\|;\ \forall x,\tilde{x},r \in \mathcal{S} \tag{6}$$

$$K(t,x,r) < k(1+\|x\|);\ \forall x,\ r \in \mathcal{S} \tag{7}$$

for some constant $k > 0$, when all the state vector $x(t)$ of the system is available and the mode, $r(t)$, of the system is known at each time $t \in [0, T]$ such that:

1. the state $x(t)$ is regulated, while minimizing the output energy in the presence of the worst-case disturbance $w^\star(t)$;

2. the H_∞-norm of the closed-loop system from $w(t)$ to $z(t)$ starting from the state $x_0 = 0$ is less than some positive number $\gamma > 0$ (this is also equivalent to attenuating the effect of the disturbances $w(t)$) and at the same time, guarantee internal stability of the closed-loop system.

Remark 2.1 *In the statement of the problem above, the first objective is associated with an H_2-criterion reminiscent of the LQG problem while the second objective is associated with an H_∞ criterion and guarantees disturbance attenuation.*

In the next section, we discuss a Nash game approach to the state-feedback mixed H_2/H_∞ control problem for the system (1)-(2).

3 Finite Horizon H_2/H_∞-Control Problem

In this section, we consider the finite-horizon H_2/H_∞-control problem for the Markovian jump linear system (MJLS) (1), (2). We give necessary and sufficient conditions for the solvability of this problem in terms of a set of two cross-coupled Riccati equations. For this purpose, define the map from $w(t)$ to $z(t)$ by T_{zw}. Then the 2-norm of the system is defined as

$$\|T_{zw}\|_2^2 = \frac{1}{2\pi}\mathsf{E}\left[\int_{-\infty}^{\infty} Trace[T_{zw}^\star(j\omega)T_{zw}(j\omega)]d\omega\right] \tag{8}$$

while the requirement that the H_∞-norm of the closed-loop system from w to z be less than γ can be expressed as

$$\|T_{zw}\|_\infty \stackrel{\triangle}{=} \sup_{x(0)=0,\ 0\neq w\in L_2[0,T]} \frac{\|z(t)\|_{2E}}{\|w(t)\|_2} < \gamma \tag{9}$$

This requirement can be associated with the following criterion function:

$$J_1(u,w) = \mathsf{E}\left[\int_0^T (\gamma^2\|w(t)\|^2 - \|z(t)\|^2)dt | x_0, r_0\right] \tag{10}$$

Hence by making $J_1(.,.)$ nonnegative, the H_∞-norm will be rendered less than γ.

Remark 3.1 *The problem of minimizing $J_1(.,.)$ alone over the set $\mathcal{U}$, subject to the dynamical equations (1), (2) such that the above condition (9) is satisfied is the pure H_∞ control problem.*

Now the second objective of minimizing the output energy while regulating $x(t)$ to zero is easily associated with the following LQ criterion:

$$J_2(u, w) = \mathsf{E}\left[\int_0^T \|z(t)\|^2 dt | x_0, r_0\right] \tag{11}$$

Remark 3.2 *The problem of minimizing $J_2(.,.)$ alone over the set $\mathcal{U}$, subject to the dynamical equations (1), (2) is the pure H_2 control problem.*

The problem of minimizing the above two criteria $J_1(.,.)$, $J_2(.,.)$ to find equilibrium strategies $u^\star(t)$, and $w^\star(t)$ such that

$$J_1(u^\star, w^\star) \leq J_1(u^\star, w); \;\; \forall w \in L_2[0, T] \tag{12}$$

$$J_2(u^\star, w^\star) \leq J_2(u, w^\star); \; \forall u \in L_2[0, T] \tag{13}$$

represents a two-player nonzero sum Nash game with equilibrium strategies $u^\star, w^\star$ respectively. The solution to this problem is summarized in the following theorem (see also [14] for the deterministic case).

Theorem 3.1 *Consider the MJLTV system (1), (2) with parameter $r(t) \in \mathcal{S}$, and consider the problem of minimizing the criteria functions $J_1(.,.)$, $J_2(.,.)$ such that the Nash equilibrium conditions (12), (13) are satisfied for $u^\star(.,.), w^\star(.,.) \in L_2[0, T]$. Then, this problem is solvable if and only if, there exist two sets of solutions $P_1(t, r(t)) \leq 0$, $P_2(t, r(t)) \geq 0$, $r(t) \in \mathcal{S}$ to the following coupled Riccati differential equations:*

$$\begin{aligned}
-\dot{P}_1(t, r(t)) \;=\;& A^T(t, r(t))P_1(t, r(t)) + P_1(t, r(t))A(t, r(t)) \\
& -\gamma^{-2}P_1(t, r(t))B_1(t, r(t))B_1^T(t, r(t))P_1(t, r(t)) \\
& -P_1(t, r(t))B_2(t, r(t))B_2^T(t, r(t))P_2(t, r(t)) \\
& -P_2(t, r(t))B_2(t, r(t))B_2^T(t, r(t))P_1(t, r(t)) \\
& P_2(t, r(t))B_2(t, r(t))B_2^T(t, r(t))P_2(t, r(t)) \\
& -C^T(t, r(t))C(t, r(t)) + \sum_{j\in\mathcal{S}} \lambda_{ij}P_1(t, j); \;\; P_1(T, r(t)) = 0 \; \forall r(t) \in \mathcal{S}
\end{aligned} \tag{14}$$

$$\begin{aligned}
-\dot{P}_2(t, r(t)) \;=\;& A^T(t, r(t))P_2(t, r(t)) + P_2(t, r(t))A(t, r(t)) \\
& -\gamma^{-2}P_2(t, r(t))B_1(t, r(t))B_1^T(t, r(t))P_1(t, r(t)) \\
& -\gamma^{-2}P_1(t, r(t))B_1(t, r(t))B_1^T(t, r(t))P_2(t, r(t)) \\
& -P_2(r(t))B_2(t, r(t))B_2^T(t, r(t))P_2(t, r(t)) \\
& +C^T(t, r(t))C(t, r(t)) + \sum_{j\in\mathcal{S}} \lambda_{ij}P_2(t, j); \;\; P_2(T, r(t)) = 0 \; \forall r(t) \in \mathcal{S}
\end{aligned} \tag{15}$$

Moreover, the Nash equilibrium strategies are given by

$$u^\star(t, x, r(t)) = -B_2^T(t, r(t))P_2(t, r(t))x(t) \tag{16}$$

$$w^\star(t, x, r(t)) = -\gamma^{-2}B_1^T(t, r(t))P_1(t, r(t))x(t) \tag{17}$$

Proof: *Sufficiency.* We use standard completion of squares for $J_1(.,.)$ and $J_2(.,.)$ respectively. Therefore, consider

$$\begin{aligned}
&J_1(u,w) - x_0^T P_1(0,r_0)x_0 \\
&\quad = \mathsf{E}\left[\int_0^T \left(\gamma^2\|w(t)\|^2(t) - \|z(t)\|^2 + \mathcal{L}x^T(t)P(t,r(t))x(t)\right) dt\right] \\
&\quad = \mathsf{E}[\int_0^T (\gamma^2\|w(t)\|^2 - \|z(t)\|^2 + x^T(t)\dot{P}_1(t,r(t))x(t) + \dot{x}^T(t)P_1(t,r(t))x(t) + \\
&\quad x^T(t)P_1(t,r(t))\dot{x}(t) + \sum_{j\in\mathcal{S}} \lambda_{ij}x^T(t)P_1(t,j)x(t)dt]
\end{aligned} \tag{18}$$

where $\mathcal{L}$ is the infinitesimal generator of the process $\{x(t), r(t), t \geq 0\}$ and is defined by:

$$\mathcal{L}V(t,x(t),r(t)) \triangleq \frac{\partial V(t,x,r(t))}{\partial t} + \frac{\partial V(t,x,r(t))}{\partial x} + \sum_{j\in\mathcal{S}} \lambda_{ij}V(t,x,j)$$

Moreover, in the above equation (18), we have made use of the Dynkin's formula [15, 5]

$$\mathsf{E}V(T,x(T),r(T)) - V(0,x_0,r_0) = \mathsf{E}\left[\int_0^T \mathcal{L}V(t,x(t),r(t))dt\right]; \;\; V(T,x(T),r(T)) = 0; \;\; \forall T > 0$$

Now substituting for $\dot{x}(t)$ and $\dot{P}_1(t,r(t))$ from (1) and (14) respectively, in (18) and rearranging, we get,

$$\begin{aligned}
&J_1(u,w) - x_0^T P_1(0,r_0)x_0 = \\
&\mathsf{E}\left[\int_0^T \left(\gamma^2\|w - w^\star\|^2 - \|u\|^2 + \|u^\star\|^2 + 2x^T(t)P_1(t,r(t))B_2(t,r(t))(u - u^\star)\right) dt\right]
\end{aligned}$$

where $u^\star(t,x,r(t))$ and $w^\star(t,x,r(t))$ are given by (16), (17) respectively. Setting $u(t) = u^\star(.,.,.)$ gives

$$J_1(u^\star,w) - x_0^T P_1(0,r_0)x_0 = \mathsf{E}\left[\int_0^T \gamma^2\|w(t) - w^\star(t,x,r(t))\|^2 dt\right] \tag{19}$$

which implies that

$$J_1(u^\star,w^\star) \leq J_1(u^\star,w); \;\; \forall w(t) \in L_2[0,T] \tag{20}$$

Similarly, it can be shown that using $J_2(u,w)$ and (1), (15), we get

$$J_2(u,w) - x_0^T P_2(0,r_0)x_0 = \mathsf{E}\left[\int_0^T \left(\|u - u^\star\|^2 + 2x^T(t)P_2(t,r(t))B_1(t,r(t))(w - w^\star)\right) dt\right] \tag{21}$$

and setting $w = w^\star$ results in

$$J_2(u^\star, w^\star) \leq J_2(u, w^\star); \;\; \forall u(t) \in L_2[0,T] \tag{22}$$

To determine the signs of $P_1(.,.)$, $P_2(.,.)$, we again resort to the completion of squares for $J_1(.,.)$, $J_2(.,.)$. Hence consider the following value function for $J_1(.,.)$

$$\begin{aligned}
&\mathsf{E}\left[\int_t^T \left(\gamma^2\|w(t)\|^2 - \|z(t)\|^2\right) d\tau\right] - \mathsf{E}x^T(t)P_1(t,r(t))x(t) = \\
&\quad \mathsf{E}\left[\int_t^T \left(\gamma^2\|w - w^\star\|^2 - \|u\|^2 + \|u^\star\|^2 + 2x^T(t)P_1(t,r(t))B_2(t,r(t))(u - u^\star)\right) d\tau\right]
\end{aligned}$$

substituting $u = u^\star$ and $w = 0$, we get

$$\mathsf{E}\left[\int_t^T \left(-\gamma^2\|w^\star(t)\|^2 - \|z(t)\|^2\right) d\tau\right] = \mathsf{E}x^T(t)P_1(t,r(t))x(t); \;\; \forall\, x(t),\, r(t) \in \mathcal{S}$$

and thus, $P_1(t,r(t)) \leq 0 \; \forall r(t) \in \mathcal{S}$.

Similarly, we have the following identity for $J_2(.,.)$:

$$\begin{aligned}
0 \leq \mathsf{E}&\left[\int_t^T (x^T C^T(t,r(t))C(t,r(t))x + u^T u) d\tau\right] \\
&= \mathsf{E}\left[x^T(t)P_2(t,r(t))x(t) + \int_t^T \|u - u^\star\|^2 d\tau\right] \\
&\quad + \mathsf{E}\left[2\int_t^T x^T(t)P_2(t,r(t))B_1(t,r(t))(w - w^\star) d\tau\right].
\end{aligned}$$

and setting $u = u^\star,\; w = w^\star$, we get

$$\mathsf{E}x^T(t)P_2(t,r(t))x(t) \geq 0; \;\; \forall x(t),\; r(t) \in \mathcal{S}$$

and therefore, $P_2(.,.) \geq 0 \; \forall r(t) \in \mathcal{S}$.

Necessity: The necessity part of the theorem can be proven using standard techniques from dynamic programming or minimum principle. However, we need to show that the Riccati equations (14), (15) have no conjugate points (or finite escape time) [2] over $[0,T]$. A proof of this can be found in [14]. To complete the proof, we first assume that there exist Nash equilibrium strategies of the form :

$$\begin{aligned}
u^\star(t,x,r(t)) &= K(t,r(t))x(t) \\
w^\star(t,x,r(t)) &= F(t,r(t))x(t)
\end{aligned}$$

and consider the control problem:

$$\begin{aligned}
&\min_{w \in L_2[0,T]} J_1(u^\star, w) \\
&\quad s.t.\;\; \dot{x}(t) = [A(t,r(t)) + B_2(t,r(t))K(t,r(t))]x(t) + B_1(t,r(t))w(t) \\
&\quad z(t) = [C(t,r(t)) + D(t,r(t))K(t,r(t))]x(t)
\end{aligned}$$

This problem has a unique solution [9, 11] $w^\star(t) = -\gamma^{-2}B_1^T(t,r(t))P_1(t,r(t))x(t)$ where $P_1(t,r(t)) \le 0,\ r(t) \in \mathcal{S}$ satisfies the following coupled Riccati equations:

$$\begin{aligned}-\dot{P}_1(t,r(t)) \;&=\; [A(t,r(t)) + B_2(t,r(t))K(t,r(t))]^T P_1(t,r(t))\\ &\quad +P_1(t,r(t))[A(t,r(t)) + B_2(t,r(t))K(t,r(t))]\\ &\quad -\gamma^{-2}P_1(t,r(t))B_1(t,r(t))B_1^T(t,r(t))P_1(t,r(t))\\ &\quad -C^T(t,r(t))C(t,r(t)) - K^T(t,r(t))K(t,r(t))\\ &\quad +\sum_{j\in\mathcal{S}}\lambda_{ij}P_1(t,j);\quad P_1(T,r(t)) = 0\ \forall r(t)\in\mathcal{S}\end{aligned} \tag{23}$$

Therefore, $F(t,r(t)) = -\gamma^{-2}B_1^T(t,r(t))P_1(t,r(t))$. Similarly, applying $w^\star(t)$ in (1) and minimizing $J_2(.,.)$ subject to the resulting dynamic equations results in the following standard optimal control problem:

$$\begin{aligned}&\min_{u\in L_2[0,T]} J_2(u,w^\star)\\ &s.t.\ \ \dot{x}(t) = [A(t,r(t)) - \gamma^{-2}B_1(t,r(t))B_1^T(t,r(t)P_1(t,r(t))]x(t) + B_2(t,r(t))u(t)\\ &z(t) = C(t,r(t))x(t) + D(t,r(t))u(t)\end{aligned}$$

whose solution is well known [9], and is given by

$$u^\star(t,x,r(t)) = -B_2^T(t,r(t))P_2(t,r(t))x(t);\ \ \Rightarrow K(t,r(t)) = -B_2^T(t,r(t))P_2(t,r(t))$$

where $P_2(t,r(t)) \ge 0$ satisfies the following coupled Riccati equations:

$$\begin{aligned}-\dot{P}_2(t,r(t)) \;&=\; [A(t,r(t)) - \gamma^{-2}B_1(t,r(t))B_1^T(t,r(t))P_1(t,r(t))]^T P_2(t,r(t))\\ &\quad +P_2(t,r(t))[A(r(t)) - \gamma^{-2}B_1(t,r(t))B_1^T(r(t))P_1(t,r(t))]\\ &\quad -P_2(t,r(t))B_2(t,r(t))B_2^T(t,r(t))P_2(t,r(t)) + C^T(t,r(t))C(t,r(t))\\ &\quad +\sum_{j\in\mathcal{S}}\lambda_{ij}P_2(t,j);\quad P_2(T,r(t)) = 0\ \forall r(t)\in\mathcal{S}\end{aligned}$$

which are the coupled Riccati equations (15). Furthermore, substituting for $K(t,r(t))$ in (23) results in the coupled Riccati equations (14). This completes the proof of the theorem. $\Box$

Regarding the requirement $\|T_{zw}\|_\infty < \gamma$, we have the following corollary.

Corollary 3.1 *For the mixed H_2/H_∞ problem with cost functions $J_1(.,.)$, $J_2(.,.)$ subject to the state equations (1), let the conditions of Theorem 3.1 hold. Then for any $x_0 \ne 0$, $r_0 \in \mathcal{S}$*

$$\begin{aligned}J_1(u^\star,w^\star) &= x_0^T P_1(0,r_0)x_0\\ J_2(u^\star,w^\star) &= x_0^T P_2(0,r_0)x_0\end{aligned}$$

Furthermore, $\|T_{zw}\|_\infty < \gamma$ with $x_0 = 0$.

Proof: The optimal costs follow from (19), (21). From (19), $J_1(u^\star,w) \ge 0\ x_0 = 0,\ r_0 \in \mathcal{S},\ w \in L_2[0,T]$; which implies that $\|T_{zw}\|_\infty \le \gamma$. However, $J_1(u^\star,w) > 0\ \forall x_0 = 0, r_0 \in \mathcal{S},\ w \ne w^\star, w \in L_2[0,T]$. Using the definition of $\|T_{zw}\|_\infty$-norm in (9), the result follows.$\Box$

4 Infinite Horizon Case

In this section, we discuss the mixed H_2/H_∞ control problem on the infinite-time horizon as $T \to \infty$, and we give sufficient conditions for the solvability of this problem. Of most paramount importance, we are interested in the existence of the solutions to the resulting coupled algebraic Riccati equations, and the boundedness/stability of the trajectories of the closed-loop system. To guarantee these, it is necessary to have a certain form of observability (or detectability) of the system, otherwise, if the unobservable part of the system is unstable, then stability cannot be achieved. We therefore begin with some definitions of stochastic observability and stability. Furthermore, for the sake of simplicity, we shall assume in the sequel that the matrices $A(.,.)$, $B_1(.,.)$, $B_2(.,.)$, $C(.,.)$, $D(.,.)$ are time invariant and constant for all $r(t) \in \mathcal{S}$.

Definition 4.1 *The jump linear time-invariant (JLTI) system (1), (2) is said to be stochastically observable if and only if for each $r(t) \in \mathcal{S}$, the pair $(A(r(t)), C(r(t))$ is observable.*

Definition 4.2 *[8] The equilibrium point $x = 0$ of the JLTI system (1) with $u(t) \equiv 0$, $w(t) \equiv 0$ is asymptotically mean square stable, if for any initial $x_0 \in \Re^n$ and $r_0 \in \mathcal{S}$,*

$$\lim_{T\to\infty} \mathsf{E}\{\|x(t, x_0, r_0)\|^2\} = 0$$

A necessary and sufficient condition for the jump linear system to be stochastically stable (equivalently, mean square asymptotically stable) is given by the following lemma [8].

Lemma 4.1 *A necessary and sufficient condition for the JLTI system (1) with $u(t) = 0$, $w(t) = 0$ to be stochastically (equivalently, mean square asymptotically) stable, is that there exist positive definite matrices $M(r(t)), r(t) \in \mathcal{S}$ such that*

$$\lambda_{ii} M(r(t)) + A(r(t))^T M(r(t)) + M(r(t))A(r(t) + \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij} M(j) = -I; \ \ \forall r(t) \in \mathcal{S}$$

Definition 4.3 *[11] The JLTI system (1) is stochastically stabilizable, if for any initial $x_0 \in \Re^n$ and $r_0 \in \mathcal{S}$, there exists a linear feedback gain $L(r(t))$ that is constant for each $r(t) \in \mathcal{S}$:*

$$u(t) = -L(r(t))x(t)$$

with $\|L(r(t))\| < \infty$ such that there exists a symmetric positive definite matrix M satisfying

$$\lim_{T\to\infty} \mathsf{E}\left[\int_0^T \|x(T, x_0, r_0, u)\|^2 dt | x_0, r_0\right] \leq x_0^T M x_0$$

A sufficient condition for stochastic stabilizability can also be given in terms of a set of cross-coupled Lyapunov equations in the following theorem [11, 5].

Theorem 4.1 *The JLTI system (1) with $w = 0$, is stochastically stabilizable if and only if, for each mode $r(t) \in \mathcal{S}$, there exists a control $u(t) = -L(r(t))x(t)$ such that for any given positive-definite symmetric matrix $N(r(t))$, the unique set of symmetric solutions, $M(r(t))$, of the l-coupled equations:*

$$[A(r(t)) - B_2(r(t))L(r(t))]^T M(r(t)) + M(r(t))[A(r(t)) - B_2(r(t))L(r(t))] + \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij} M(j) = -N(r(t)); \ \ \forall r(t) \in \mathcal{S}$$

are positive definite for each $r(t)) \in \mathcal{S}$

Now on the infinite-horizon, the set of coupled Riccati differential equations (14), (15), take the following form of algebraic equations:

$$\begin{aligned} &A^T(r(t))P_1(r(t)) + P_1(r(t))A(r(t)) - \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) - \\ &\quad P_1(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - P_2(r(t))B_2(r(t))B_2^T(r(t))P_1(r(t)) - \\ &\quad P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - C^T(r(t))C(r(t)) + \sum_{j\in\mathcal{S}} \lambda_{ij} P_1(j) = 0; \ \ \forall r(t) \in \mathcal{S} \end{aligned} \tag{24}$$

$$\begin{aligned} &A^T(r(t))P_2(r(t)) + P_2(r(t))A(r(t)) - \gamma^{-2}P_2(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) - \\ &\quad \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_2(r(t)) - P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) + \\ &\quad C^T(r(t))C(r(t)) + \sum_{j\in\mathcal{S}} \lambda_{ij} P_2(j) = 0; \ \ \forall r(t) \in \mathcal{S} \end{aligned} \tag{25}$$

We now present our main result for this section.

Theorem 4.2 *Consider the JLTI system (1), (2) and the infinite horizon mixed H_2/H_∞ control problem with criteria functions $J_1^\infty = \lim_{T\to\infty} J_1(u, w)$, $J_2^\infty = \lim_{T\to\infty} J_2(u, w)$. Assume that $(A(r(t), C(r(t))$ is stochastically observable and there exist two sets of solutions $P_1(r(t)) < 0$, $P_2(r(t)) > 0$; $\forall r(t) \in \mathcal{S}$ to the coupled algebraic Riccati equations (ARE) (24), (25), then the system is stochastically stabilizable. Moreover, the closed-loop system (1), (2) with the feedback strategies*

$$u^\star(x, r(t)) = -B_2^T(r(t))P_2(r(t))x(t) \tag{26}$$

$$w^\star(x, r(t)) = -\gamma^{-2}B_1^T(r(t))P_1(r(t))x(t) \tag{27}$$

is stochastically stable.

Proof: The Riccati equation (24) can be written as

$$\begin{aligned} &[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t))]^T P_1(r(t)) \\ &\quad + P_1(r(t))[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t))] \\ &\quad - C^T(r(t))C(r(t)) - \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\ &\quad - P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) + \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij} P_1(j) = 0; \ \ \forall r(t) \in \mathcal{S} \end{aligned} \tag{28}$$

The above equation can further be rearranged as

$$\begin{aligned}
&[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t)) + \frac{1}{2}\lambda_{ii}I]^T P_1(r(t)) \\
&+P_1(r(t))[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t)) + \frac{1}{2}\lambda_{ii}I] \\
&+ \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij}P_1(j) = C^T(r(t))C(r(t)) \\
&+\gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\
&+P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)); \;\; \forall r(t) \in \mathcal{S} \qquad (29)
\end{aligned}$$

Now write $L(r(t)) \stackrel{\triangle}{=} B_2^T(r(t))P_2(r(t))$ and

$$\begin{aligned}
N(r(t)) \stackrel{\triangle}{=} & C^T(r(t))C(r(t)) + \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) + \\
& P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)); \; \forall r(t) \in \mathcal{S} \qquad (30)
\end{aligned}$$

Then, $N(r(t)) > 0$ which implies that $-N(r(t)) < 0$. By assumption, $P_1(r(t)) < 0$, which implies that there exists a matrix $P_1^{'}(r(t)) = -P_1(r(t)) > 0$, and hence the system is stochastically stabilizable. Regarding the stochastic stability, note that, by the observability assumption, $(A(r(t)), C(r(t)),\ r(t) \in \mathcal{S}$ observable, implies

$$\begin{aligned}
&\left(A(r(t)), \begin{bmatrix} C(r(t)) \\ B_2(r(t))P_2(r(t)) \end{bmatrix}\right) \; \textit{observable} \Rightarrow \\
&\left([A(r(t) - B_2(r(t))B_2^T(r(t))P_2(r(t))], \begin{bmatrix} C(r(t)) \\ B_2(r(t))P_2(r(t)) \end{bmatrix}\right) \; \textit{observable}
\end{aligned}$$

Again by assumption, $P_1(r(t)) < 0$ and $-N(r(t)) < 0$, therefore, from (28),

$$[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t)) < 0$$

and the closed-loop system is stochastically stable.□

Remark 4.1 *Notice that in the proof of the above theorem, we have used the assumption that the JLTI system (1) is observable rather than the weaker form of detectability. This is however not necessary for the stability of the closed-loop system. It is possible to use a detectability assumption and modify the definition of stabilizability to allow for a positive semidefinite* $N(r(t))$*. In this case also, it is sufficient to have* $P_1(r(t)) \leq 0,\ \forall r(t) \in \mathcal{S}$*. See also [11]*

Remark 4.2 *Notice also that in the proof of the above theorem, it is tempting to conclude stabilizability from* $[A(r(t)) - B_2(r(t))B_2^T(r(t))P_2(r(t)] < 0$*. This has however been shown to be insufficient [11]*

Corollary 4.1 *Let all the conditions of theorem 4.1 hold, then for* $u(t) = u^\star(t,x),\ x_0 = 0$, $\|T_{zw}\|_\infty < \gamma$ *and* $J_2^\infty(u^\star, w^\star) \leq J_2^\infty(u, w^\star);\ \forall u \in L_2[0,\infty)$

Proof: Again from (19), we have:

$$J_1^\infty(u^\star, w) - x_0^T P_1(0, r_0)x_0 = \mathsf{E}\left[\int_0^\infty \gamma^2 \|w(t) - w^\star(t, x, r(t))\|^2 dt\right]$$

since $(w - w^\star) \in L_2[0, \infty)$, then for $x(0) = 0$, we have

$$J_1^\infty(u^\star, w) = \mathsf{E}\left[\int_0^\infty \gamma^2\|w(t)\|^2 - \|z\|^2 dt\right] = \mathsf{E}\left[\int_0^\infty \gamma^2 \|w(t) - w^\star(t, x, r(t))\|^2 dt\right] \geq 0$$

Now applying the definition of the H_∞-norm (9), the result follows. Similarly, by considering (21), it can be shown that the inequality $J_2^\infty(u^\star, w^\star) \leq J_2^\infty(u, w^\star)$; $\forall u \in L_2[0, \infty)$. □

Remark 4.3 *Conversely, by assuming the results of the above Theorem 4.2 and Corollary 4.1, it is possible to prove the existence of the solutions to the coupled Riccati equations (24) and (25); thereby, establishing necessary and sufficient conditions for the solution of the mixed H_2/H_∞ control problem for MJLTI systems on the infinite horizon (see also [14]).*

We now turn our attention to the robust control problem in which the system model or data is assumed to be imprecise or uncertain, and we wish to develop a robust control policy that will guarantee the stability and performance of the system for all possible uncertainties.

5 Robust H_2/H_∞ Control Problem

Our aim in this section is to derive a robust control law that will guarantee the robust performance of the system in the presence of modelling errors and/or parameter uncertainties. In the previous sections, we have concentrated on developing optimal control laws that achieve a disturbance attenuation requirement and minimize the output energy of the system while at the same time regulating the states of the system to zero. This has been achieved on the finite and infinite-time horizons under the assumption that the system model is precise and the states and mode of the system are known exactly at each time $t \in [0, \infty)$. This is practically unrealistic as model uncertainties can never be dispelled. Therefore, we begin by considering the following uncertain model of the MJLTI system (1), (2) corresponding to the model (1):

$$\dot{x}(t) = [A(r(t)) + \Delta A(t, r(t))]x(t) + B_1(r(t))w(t) + [B_2(r(t)) + \Delta B_2(t, r(t))]u(t) \tag{31}$$

$$z(t) = C(r(t))x(t) + D(r(t))u(t) \quad C^T(r(t))D(r(t)) = 0, \;\; D^T(t, r(t))D(t, r(t)) = I \tag{32}$$

where all the variables have their previous meanings and dimensions, while $\Delta A(.,.)$, $\Delta B(.,.)$ are the uncertainties in the model which may possibly be time-varying and $r(t)$-dependent. The dependence on $r(t)$ is two-fold; first, in each mode of the system, the system matrices $A(.,.)$, $B_2(.,.)$ are not known exactly, and secondly, the transition rates of the system defined by the matrix $\Lambda = [\lambda_{ij}]_{i,j\in\mathcal{S}}$ are not also known exactly. In this case, we can represent Λ as a convex combination of some known extreme values $\Lambda_k,\ k = 1,\dots,\nu$ as:

$$\Lambda = \sum_{k=1}^{\nu} \alpha_k \Lambda_k; \ \ \sum_{k=1}^{\nu} \alpha_k = 1, \ \ \alpha_k \ge 0\ \forall k \tag{33}$$

For the sake of brevity, let us assume that the uncertainties in the jump parameter $r(t) \in \mathcal{S}$ can be lumped in the the system uncertainties $\Delta A(.,.)$ and $\Delta B(.,.)$, and further, let us assume that the uncertainties satisfy the following.

Assumption 5.1 *The uncertainties are such that*

$$[\Delta A(t) \ \ \Delta B_2(t)] = [H_1(r(t))F(t,r(t))E_1(r(t)) \ \ H_2(r(t))F(t,r(t))E_2(r(t))] \tag{34}$$

$$F^T(t,r(t))F(t,r(t)) \le 1; \ \ \forall r(t) \in \mathcal{S} \tag{35}$$

We now define the following concept of stochastic robust stability which is an extension of the definition of quadratic stability used in [21, 13, 23].

Definition 5.1 *The uncertain system (31), (32) is mixed stochastically quadratically stabilizable with disturbance attenuation γ, if there exists a fixed linear time-invariant proper state-feedback $u(t) = K(r(t))x(t)$ and a set of two real symmetric matrices $P_1(r(t)) < 0$ and $P_2(r(t)) > 0$, $r(t) \in \mathcal{S}$ such that, the following inequalities are satisfied for all admissible uncertainties $\Delta A(t,r(t))$ and $\Delta B_2(t,r(t))$.*

$$\begin{aligned}&\{[(A_\Delta(t,r(t)) + B_{2\Delta}(t,r(t))K(r(t))]x + B_1(r(t))w\}^T P_1(r(t))x + \\ &\quad x^T P_1(r(t))\{[(A_\Delta(r(t)) + B_{2\Delta}(r(t))K(r(t))]x + B_1(r(t))w\} - \|z\|^2 + \gamma^2\|w\|^2 + \\ &\quad x^T \sum_{j\in\mathcal{S},j\ne i} \lambda_{ij} P_1(j)x \le 0; \ \ \forall x, \ r(t) \in \mathcal{S}, \ w \in L_2[0,\infty)\end{aligned} \tag{36}$$

$$\begin{aligned}&\{[(A_\Delta(r(t)) + B_{2\Delta}(r(t))K(r(t)))]x + B_1(r(t))w\}^T P_2(r(t))x + \\ &\quad x^T P_2(r(t))\{[(A_\Delta(t,r(t)) + B_{2\Delta}(t,r(t))K(r(t)))]x + B_1(r(t))w\} + \|z\|^2 + \\ &\quad x^T \sum_{j\in\mathcal{S},j\ne i} \lambda_{ij} P_2(j)x \le 0; \ \ \forall\ x, \ r(t) \in \mathcal{S}, \ w \in L_2[0,\infty)\end{aligned} \tag{37}$$

where $A_\Delta(t,r(t)) = A(r(t)) + \Delta A(t,r(t))$ and $B_{2\Delta}(t,r(t)) = B_2(r(t)) + \Delta B_2(t,r(t))$

Before we present our main result in this section, we first present the following lemma which is a variation of Petersen's [21, 13]

Lemma 5.1 *Given any constant $\epsilon > 0$, and any matrix $F \in \Re^{p\times q}$ such that $F^T F \le I$,*

$$2x^T PHFEx \ \ \le \ \ \epsilon x^T PHH^T Px + \frac{1}{\epsilon} x^T E^T Ex; \ \ \forall x \tag{38}$$

$$2x^T PHFEKx \ \ \le \ \ \epsilon x^T PHH^T Px + \frac{1}{\epsilon} x^T K^T E^T EKx; \ \ \forall x \tag{39}$$

We now state our main result in this section.

Theorem 5.1 *Consider the Uncertain LTI system (31), (32), (34),(35) and suppose for some sequence of numbers $\epsilon_r > 0,\ r \in \mathcal{S}$, there exist two sets of symmetric matrices $P_1(r(t)) < 0,\ P_2(r(t)) > 0,\ \forall r(t) \in \mathcal{S}$ satisfying the following coupled AREs*

$$\begin{aligned}
&A^T(r(t))P_1(r(t)) + P_1(r(t))A(r(t)) - \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\
&\quad -P_1(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - P_2(r(t))B_2(r(t))B_2^T(r(t))P_1(r(t)) \\
&\quad -P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - C^T(t)C(r(t)) + \\
&\quad \epsilon_r P_1(r(t))[H_1(r(t))H_1^T(r(t)) + H_2(r(t))H_2^T(r(t))]P_1(r(t)) + \\
&\quad \frac{1}{\epsilon_r}[P_2(r(t))B_2(r(t))E_2^T(r(t))E_2(r(t))B_2^T(r(t))P_2(r(t)) + E_1^T(r(t))E_1(r(t))] \\
&\quad + \sum_{j\in\mathcal{S},j\neq i} \lambda_{ij}P_1(j) = 0;\ \ \forall r(t) \in \mathcal{S}
\end{aligned} \tag{40}$$

$$\begin{aligned}
&A^T(r(t))P_2(r(t)) + P_2(r(t))A(r(t)) - \gamma^{-2}P_2(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\
&\quad -\gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_2(r(t)) - P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) \\
&\quad +C^T(r(t))C(r(t)) + \epsilon_r P_2(r(t))[H_1(r(t))H_1^T(r(t)) + H_2^T(r(t))H_2(r(t))]P_2(r(t)) \\
&\quad +\frac{1}{\epsilon_r}[E_1^T(r(t))E_1(r(t)) + P_2(r(t))B_2(r(t))E_2^T(r(t))E_2(r(t))B_2^T(r(t))P_2(r(t))] \\
&\quad + \sum_{j\in\mathcal{S},j\neq i} \lambda_{ij}P_2(j) = 0;\ \ \forall r(t) \in \mathcal{S}
\end{aligned} \tag{41}$$

Then the closed-loop system is mixed stochastically quadratically stabilizable with disturbance attenuation γ.

Proof: Multiplying the Riccati equation (40) from the left by x^T and right by x and substituting $u^\star = -B_2^T(r(t))P_2(r(t))x$, $w^\star = -\gamma^{-2}B_1^T(r(t))P_1(r(t))x$, we get

$$\begin{aligned}
&x^T[A^T(r(t))P_1(r(t)) + P_1(r(t))A(r(t))]x - \gamma^2\|w^\star\|^2 + x^TP_1(r(t))B_2(r(t))u^\star + (u^\star)^TB_2^T(r(t))P_1(r(t))x - \\
&\quad \|u^\star\|^2 - x^TC^T(t)C(r(t))x + \epsilon_r x^TP_1(r(t))[H_1(r(t))H_1^T(r(t)) + H_2(r(t))H_2^T(r(t))]P_1(r(t))x + \\
&\quad \frac{1}{\epsilon_r}[x^TE_1^T(r(t))E_1(r(t))x + (u^\star)^TE_2^T(r(t))E_2(r(t))u^\star] + x^T\sum_{j\in\mathcal{S},j\neq i}\lambda_{ij}P_1(j)x = 0;\ \ \forall x,\ r(t) \in \mathcal{S}
\end{aligned} \tag{42}$$

Rearranging further we get,

$$\begin{aligned}
&[x^TA^T(r(t)) + (u^\star)^TB_2^T(r(t))]P_1(r(t))x + x^TP_1(r(t))[A(r(t))x + B_2^T(r(t))u^\star] - \gamma^2\|w^\star\|^2 - \|z\|^2 + \\
&\quad \epsilon_r x^TP_1(r(t))[H_1(r(t))H_1^T(r(t)) + H_2(r(t))H_2^T(r(t))]P_1(r(t))x + \frac{1}{\epsilon_r}[x^TE_1^T(r(t))E_1(r(t))x + \\
&\quad (u^\star)^TE_2^T(r(t))E_2(r(t))u^\star] + x^T\sum_{j\in\mathcal{S},j\neq i}\lambda_{ij}P_1(j)x = 0;\ \ \forall x,\ r(t) \in \mathcal{S}
\end{aligned} \tag{43}$$

Now substituting $u^\star = K(r(t))x$ and using lemma 5.1, the condition (36) follows. Similarly, the same procedure can be applied to the Riccati equation (41) to show that the complementary condition (37) is also satisfied. This completes the proof of the theorem. □

The following corollary is also a consequence of the definition of stochastic quadratic stability.

Corollary 5.1 *Suppose that there exist two sets of symmetric matrices $P_1(r(t)) < 0,\ P_2(r(t)) > 0,\ \forall r(t) \in \mathcal{S}$ satisfying the coupled AREs (40), (41), and the pair $(A(r(t), C(r(t)), r(t) \in \mathcal{S}$ is stochastically observable, then the closed-loop system with the control law*

$$u(t) = -B_2(r(t))P_2(r(t))x(t); \quad r(t) \in \mathcal{S} \tag{44}$$

is stochastically (equivalently mean square asymptotically) stable.

Proof: Substituting the control law $u = K(r(t))x(t) = -B_2^T(r(t))P_2(r(t))x(t)$ in (36) and simplifying as in Theorem 5.1 above, results in the following Riccati inequality:

$$\begin{aligned}
&A^T(r(t))P_1(r(t)) + P_1(r(t))A(r(t)) - \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\
&\quad -P_1(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - P_2(r(t))B_2(r(t))B_2^T(r(t))P_1(r(t)) \\
&\quad -P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) - C^T(t))C(r(t)) \\
&\quad +\epsilon_r P_1(r(t))[H_1(r(t))H_1^T(r(t)) + H_2(r(t))H_2^T(r(t))]P_1(r(t)) \\
&\quad +\frac{1}{\epsilon_r}[P_2(r(t))B_2(r(t))E_2^T(r(t))E_2(r(t))B_2^T(r(t))P_2(r(t)) + E_1^T(r(t))E_1(r(t))] \\
&\quad + \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij}P_1(j) \leq 0; \quad \forall r(t) \in \mathcal{S}
\end{aligned} \tag{45}$$

The above inequality can be rearranged as in Theorem 4.1 and using lemma 5.1, to get,

$$\begin{aligned}
&[A_\Delta(r(t)) - B_{2\Delta}(r(t))B_2^T(r(t))P_2(r(t)) + \frac{1}{2}\lambda_{ii}I]^T P_1(r(t)) + \\
&\quad P_1(r(t))[A_\Delta(r(t) - B_{2\Delta}(r(t))B_2^T(r(t))P_2(r(t)) + \frac{1}{2}\lambda_{ii}I] + \\
&\quad \sum_{j\in\mathcal{S}, j\neq i} \lambda_{ij}P_1(j) \leq N(r(t)) \quad \forall r(t) \in \mathcal{S}
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&N(r(t)) \stackrel{\triangle}{=} C^T(r(t))C(r(t)) + \gamma^{-2}P_1(r(t))B_1(r(t))B_1^T(r(t))P_1(r(t)) \\
&\quad +P_2(r(t))B_2(r(t))B_2^T(r(t))P_2(r(t)) > 0 \\
&\quad A_\Delta(r(t)) = A(r(t)) + \Delta A(r(t)); \quad B_{2\Delta}(r(t)) = B_2(r(t)) + \Delta B_2(r(t))
\end{aligned}$$

The result now follows as in Theorem 4.1 □.

6 Simulation Results

In this section, we show some computational results for the mixed H_2/H_∞ control of a scalar jump system to illustrate its performance. We also compare these results with those of the pure H_2 and H_∞ cases. For simplicity, we use the finite-horizon solution, as the resulting coupled differential Riccati equations can easily be integrated with standard Runge-Kutta routine of order four. We consider the following scalar system:

$$\dot{x}(t) = \begin{cases} \frac{1}{3}x(t) + u(t) + w(t) & \text{if } r(t) = 1 \\ -\frac{4}{3}x(t) + u(t) + w(t) & \text{if } r(t) = 2 \end{cases} \tag{47}$$

$$y(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}; \;\; \Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{48}$$

The corresponding Riccati equations for the finite-horizon mixed H_2/H_∞, H_∞ and H_2 control problems are respectively,

H_2/H_∞ Control

$$\begin{aligned}
-\dot{p}_1(t,1) &= \frac{2}{3}p_1(t,1) - (1.16)^{-2}p_1^2(t,1) - 2p_1(t,1)p_2(t,1) - p_2^2(t,1) - p_1(t,1) + p_1(t,2) - 1; \;\; p_1(T,1) = 0 \\
-\dot{p}_1(t,2) &= -\frac{8}{3}p_1(t,2) - (1.16)^{-2}p_1^2(t,2) - 2p_1(t,2)p_2(t,2) - p_2^2(t,2) + p_1(t,1) - p_1(t,2) - 1; \;\; p_1(T,2) = 0 \\
-\dot{p}_2(t,1) &= \frac{2}{3}p_2(t,2) - 2(1.16)^{-2}p_2^2(t,1)p_1(t,2) - p_2^2(t,1) - p_2(t,1) + p_2(t,2) + 1; \;\; p_2(T,2) = 0 \\
-\dot{p}_2(t,2) &= -\frac{8}{3}p_2(t,2) - 2(1.16)^{-2}p_2(t,2)p_1(t,2) - p_2^2(t,2) + p_2(t,1) - p_2(t,2) + 1; \;\; p_2(T,2) = 0
\end{aligned}$$

H_∞ Control

$$\begin{aligned}
-\dot{p}(t,1) &= \frac{2}{3}p(t,1) + [(1.16)^{-2} - 1]p^2(t,1) - p(t,1) + p(t,2) + 1; \;\; p(T,1) = 0 \\
-\dot{p}(t,2) &= -\frac{8}{3}p(t,2) + [(1.16)^{-2} - 1]p^2(t,2) + p(t,1) - p(t,2) + 1; \;\; p(T,2) = 0
\end{aligned}$$

H_2 Control

$$\begin{aligned}
-\dot{p}(t,1) &= \frac{2}{3}p(t,1) - p^2(t,1) - p(t,1) + p(t,2) + 1; \;\; p(T,1) = 0 \\
-\dot{p}(t,2) &= -\frac{8}{3}p(t,2) - p^2(t,2) + p(t,1) - p(t,2) + 1; \;\; p(T,2) = 0
\end{aligned}$$

The results of integrating the above coupled Riccati equations corresponding to the above system are shown in Fig. 1. below. Similarly, the resulting solutions for the pure H_∞ and H_2 control problems are shown in Fig. 2 and Fig. 3 respectively. Comparing the mixed H_2/H_∞ results with the H_∞ results for the same value of $\gamma = 1.16$, it is clear from the Figs. 1 and 2, that the resulting control magnitude (which in this case is proportional to the magnitude of the solutions of the Riccati equations) is higher for the mixed case for both the modes of the system. This is to be expected as we are adding more constraints to the system in the mixed case. Furthermore, the resulting control magnitude for the pure H_2 case is less than for both the pure H_∞ and the mixed H_2/H_∞ cases. This is also as expected, since in this case, there is no additional constraints on the system except for the minimization of the input energy. Moreover, the disturbance is assumed here to be a zero-mean white noise process.

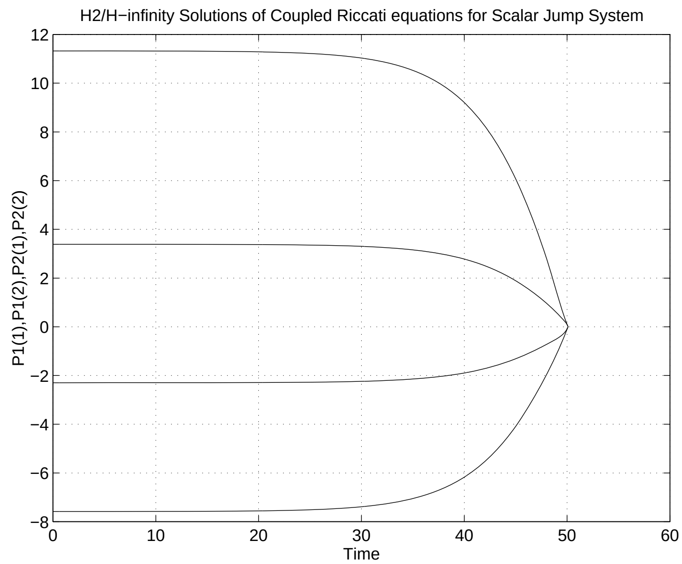

Figure 1: Mixed H_2/H_∞ Control with $\gamma = 1.16$

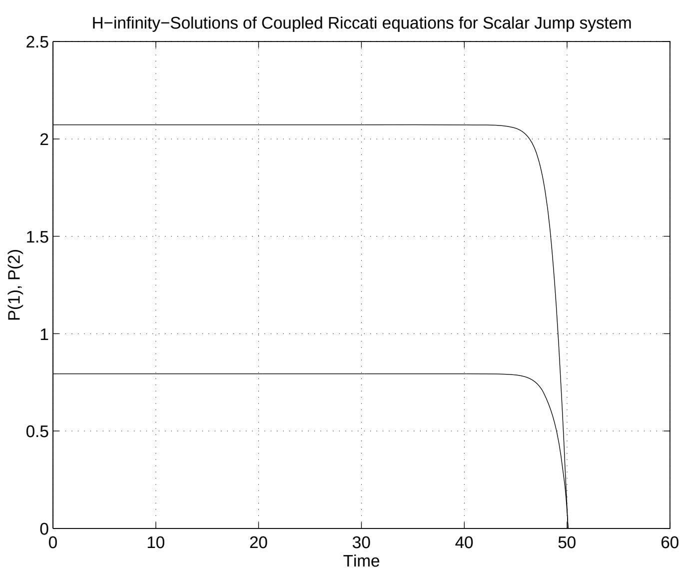

Figure 2: H_∞ Control with $\gamma = 1.16$

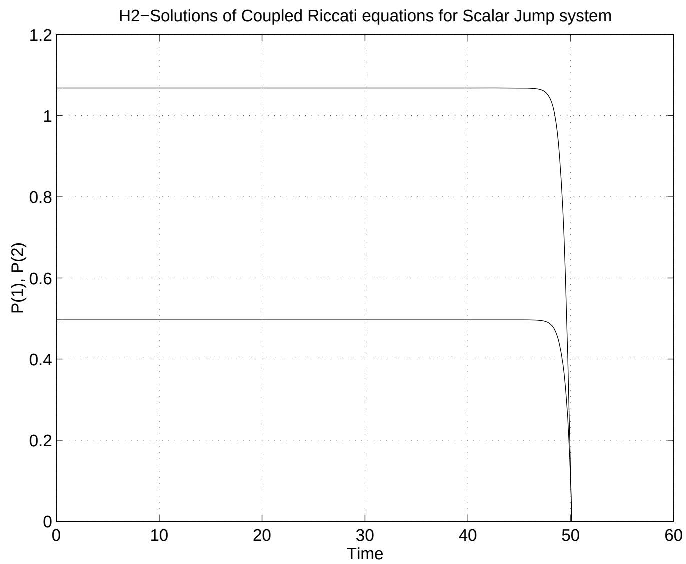

Figure 3: H_2 Control

7 Conclusion

In this paper, we have considered the mixed H_2/H_∞ control problem for Markovian Jump linear systems using state-feedback. A Nash-game approach to this problem has been presented. Both the finite and infinite-horizon problems have been considered. In the case of the finite-horizon problem, we have given necessary and sufficient conditions for the solvability of the problem in the form of two sets of cross-coupled Riccati differential equations, while for the infinite-horizon problem, we have given sufficient conditions in terms of two sets of cross-coupled algebraic Riccati equations.

Furthermore, we have considered the state-feedback robust control problem in the presence of system uncertainties. For this problem, we have also given sufficient conditions for its solution. Finally, we have shown simulation results to compare the mixed H_2/H_∞ approach with the pure H_2 and H_∞ approaches.

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