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A class of interference induced games: Asymptotic Nash equilibria and parameterized cooperative solutions

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Abstract: We consider a multi-agent system with linear stochastic individual dynamics, and individual linear quadratic ergodic cost functions. The agents partially observe their own states. Their cost functions and initial statistics are a priori independent but they are coupled through an interference term (the mean of all agent states), entering each of their individual measurement equations. While in general for a finite number of agents, the resulting optimal control law may be a non linear function of the available observations, we establish that for certain classes of cost and dynamic parameters, optimal separated control laws obtained by ignoring the interference coupling, are asymptotically optimal when the number of agents goes to infinity, thus forming for finite N, an ϵ-Nash equilibrium. More generally though, optimal separated control laws may not be asymptotically optimal, and can in fact result in unstable overall behavior. Thus we consider a class of parameterized decentralized control laws whereby the separated Kalman gain is treated as the arbitrary gain of a Luenberger like observer. System stability regions are characterized and the nature of optimal cooperative control policies within the considered class is explored. Numerical results and an application example for wireless communications are reported.

Key Words: Multi-agent systems, decentralized control, Nash equilibrium, interference induced games.

1 Introduction

There has been a surge of interest in the study and analysis of large population stochastic multi-agent systems due to their wide variety of applications over the past several years. Many practical applications and examples of these systems arise in engineering, biological, social and economic fields, such as wireless sensor networks [6], very large scale robotics [29], controlled charging of a large population of electric vehicles [17], synchronization of coupled oscillators [40], swarm and flocking phenomenon in biological systems [10, 27], evacuation of large crowds in emergency situations [11, 21], sharing and competing for resources on the Internet [2], to cite a few. Large-scale stochastic games with unbounded costs were studied in [1]. Mean field game theory, which addresses a class of dynamic games with a large number of agents in which each agent interacts with the average or so-called mean field effect of other agents via couplings in their individual dynamics and cost functions, was studied in [14, 15, 22, 24, 25, 36, 37]. In [23], the mean field linear quadratic Gaussian (LQG) framework was extended to systems of agents with Long Time Average (LTA) (i.e., ergodic) cost functions such that the set of control laws possesses an almost sure (a.s.) asymptotic Nash equilibrium property.

Stochastic Nash games with partial observation have been of interest since the late 1960s. LQG continuous-time zero-sum stochastic games with output measurements corrupted by additive independent white Gaussian noise were studied in [30, 31] under the constraint that each player is limited to a linear state estimator for generating its optimal controls. These results were extended to nonzero-sum Nash games in [33]. In these works the authors assumed that the separation principle holds. In [18], discrete-time nonzero-sum LQG Nash games with constrained state estimators and two different information structures were investigated, where it is shown that the optimal control laws do not satisfy the separation principle and the estimator characteristics depend on the controller gains.

Distributed decision-making with partial observation for large population stochastic multi-agent systems was studied in [4, 5, 13, 38], where the synthesis of Nash strategies is investigated for the agents that are weakly coupled through either individual dynamics or costs. In [7, 8, 9] the authors studied a somewhat dual situation whereby large populations of partially observed stochastic agents, although a priori individually independent, are coupled only via their observation structure. The latter involves an interference term depending on the empirical mean of all agent states. The study of such measurement-coupled systems is inspired by a variety of applications, including for instance the communications model for power control in cellular telephone systems [12, 28], where any conversation in a cell acts as interference on the other conversations in that cell. Indeed, despite the so-called signal processing gain achieved thanks to a user's specific coding advantage (and considered in our model to be of order $1/N$ where N is the total number of agents), the ability of the base station to correctly decode the signals sent by a given mobile, remains limited by interference formed by the superposition of all other in cell user signals. Viewed in this light, the studied problem can be considered as a game over a noisy channel.

Individual agent dynamics are assumed to be linear, stochastic, with linear local state measurements, and in the current paper, we focus on the case where the measurements interaction model is assumed to depend only on the empirical mean of agents states in a purely additive manner. In general, in such decentralized control problems, the measurement system could be used for some sort of signalling, and control and estimation are typically coupled [39]. We assume that each agent is constrained to use a linear Kalman filter-like state estimator to generate its optimal strategies. For a finite number of agents, we establish that for certain classes of cost and dynamic parameters, optimal separated control laws obtained by ignoring the interference coupling, are asymptotically optimal when the number of agents goes to infinity, thus forming for finite N, an ϵ-Nash equilibrium. More generally though, optimal separated control laws may not be asymptotically optimal, and can in fact result in unstable overall behavior. Thus we consider a class of parameterized decentralized control laws whereby the separated Kalman gain is treated as the arbitrary gain of a Luenberger like observer. System stability regions are characterized and the nature of optimal cooperative control policies within the considered class is explored.

The rest of the paper is organized as follows. The problem is defined and formulated in Section 2. Section 3 presents the closed-loop dynamics model. In Section 4, a decentralized control and state estimation algorithm via stability analysis is described and a characterization of its optimality properties is given. Section 5 presents parameterized cooperative solutions. Also, both Section 4 and Section 5 provide some numerical simulation results. Section 6 presents an application example for wireless communications. Concluding remarks are stated in Section 7.

2 Problem formulation

Consider a system of N agents, with individual scalar dynamics for simplicity of computations. The evolution of the state component is described by

$$x_{k+1,i} = ax_{k,i} + bu_{k,i} + w_{k,i} \tag{1}$$

with partial scalar state observations given by:

$$y_{k,i} = cx_{k,i} + h\left(\frac{1}{N}\sum_{j=1}^{N} x_{k,j}\right) + v_{k,i} \tag{2}$$

for $k \geq 0$ and $1 \leq i \leq N$, where $x_{k,i}, u_{k,i}, y_{k,i} \in \mathbb{R}$ are the state, the control input and the measured output of the i^{th} agent, respectively. The random variables $w_{k,i} \sim \mathcal{N}(0, \sigma_w^2)$ and $v_{k,i} \sim \mathcal{N}(0, \sigma_v^2)$ represent independent Gaussian white noises at different times k and at different agents i. The Gaussian initial conditions $x_{0,i} \sim \mathcal{N}(\bar{x}_0, \sigma_0^2)$ are mutually independent and are also independent of $\{w_{k,i}, v_{k,i}, 1 \leq i \leq N,\ k \geq 0\}$. σ_w^2, σ_v^2 and σ_0^2 denote the variance of $w_{k,i}$, $v_{k,i}$ and $x_{0,i}$, respectively. Moreover, a is a scalar parameter and $b, c, h > 0$ are positive scalar parameters.

The problem to be considered is to synthesize the linear time invariant decentralized separated policies such that each agent is stabilized by a feedback control of the form

$$u_{k,i} = -f\hat{x}_{k,i}, \tag{3}$$

where $\hat{x}_{k,i}$ is an estimator of $x_{k,i}$ based only on local observations of the i^{th} agent, and f is a constant scalar gain. For the purposes of the paper, the class of *decentralized separated policies* (3) includes all control policies satisfying the following three conditions: (i) they are defined by two time invariant feedback gains K and f, (ii) they are separated in that the control is a linear feedback $-f\hat{x}_{k,i}$ on the state estimate of $x_{k,i}$, while the state estimate $\hat{x}_{k,i}$ is obtained from a Luenberger like observer equation under the assumed state estimate feedback structure, i.e., it evolves according to:

$$\hat{x}_{k+1,i} = (a - bf)\hat{x}_{k,i} + K(y_{k+1,i} - c(a - bf)\hat{x}_{k,i}), \tag{4}$$

(iii) they are decentralized in that the state estimate is based solely on agent based observations $y_{k,i}$.

Furthermore, when the gain K is the Kalman gain as obtained when assuming zero interference in the local measurements (setting $h = 0$ in (2)), the resulting estimator (4) will be called the *naive* Kalman filter. Moreover, the individual cost function for each agent is given by

$$J_i \triangleq \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(x_{k,i}^2 + ru_{k,i}^2). \tag{5}$$

where $r > 0$ is a positive scalar parameter.

Assumption 1 *To simplify the synthesis procedure we assume zero mean for initial conditions of all agents, i.e.,* $\mathbb{E}x_{0,i} = \bar{x}_0 = 0, i \geq 1$.

Remark 1 *To show that the decentralized control problem formulated here is a game, let us assume for the sake of discussion that the original agent dynamics is unstable. Then it suffices to observe that, for finite N at least, the inability of a single agent to stabilize its own dynamics would have direct consequences on the ability of other agents to stabilize their own, hence demonstrating the impact of that agent on other agents' individual costs.*

3 Closed-loop dynamics model

3.1 Closed-loop agent dynamics

In this section first we obtain the 4^{th} order model of the closed-loop agent dynamics. In particular, when local state estimate feedback (3) is included in the i^{th} agent state equation (1), the result is as follows:

$$x_{k+1,i} = ax_{k,i} - bf\hat{x}_{k,i} + w_{k,i}. \tag{6}$$

In addition, anticipating the need to account for the influence of average states in the dynamics through the measurement equation, and letting a tilde $(\tilde{\cdot})$ indicate an averaging over agents operation, we define:

$$m_k = \frac{1}{N}\sum_{j=1}^{N} x_{k,j}, \quad \tilde{m}_k = \frac{1}{N}\sum_{j=1}^{N} \hat{x}_{k,j}, \tag{7}$$

$$\tilde{w}_k = \frac{1}{N}\sum_{j=1}^{N} w_{k,j}, \quad \tilde{v}_k = \frac{1}{N}\sum_{j=1}^{N} v_{k,j}. \tag{8}$$

Now, combining (6), (7), (8), we obtain the population average state evolution:

$$m_{k+1} = am_k - bf\tilde{m}_k + \tilde{w}_k. \tag{9}$$

Also averaging the estimate $\hat{x}_{k+1,i}$ given by (4), yields the population average state estimate dynamics:

$$\tilde{m}_{k+1} = (a - bf)\tilde{m}_k + K((c+h)m_{k+1} - c(a-bf)\tilde{m}_k + \tilde{v}_{k+1}). \tag{10}$$

Thus, combining (4) and (6) with (9) and (10) yields

$$X_{k+1,i} = AX_{k,i} + DW_{k,i}, \tag{11}$$

where the augmented state is

$$X_{k,i} = [x_{k,i}, \hat{x}_{k,i}, m_k, \tilde{m}_k]^T, \tag{12}$$

and matrix A is given by

$$A = \begin{bmatrix} a & -bf & 0 & 0 \\ acK & a(1-cK) - bf & ahK & -bfhK \\ 0 & 0 & a & -bf \\ 0 & 0 & a(c+h)K & a_{4,4} \end{bmatrix}, \tag{13}$$

with

$$a_{4,4} = a(1-cK) - bf(1+hK), \tag{14}$$

and also we have:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ cK & hK & K & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (c+h)K & 0 & K \end{bmatrix}, \quad W_{k,i} = \begin{bmatrix} w_{k,i} \\ \tilde{w}_k \\ v_{k+1,i} \\ \tilde{v}_{k+1} \end{bmatrix}. \tag{15}$$

Furthermore, the covariance matrix of $W_{k,i}$ is given by:

$$\Sigma_w = \begin{bmatrix} \sigma_w^2 & \frac{\sigma_w^2}{N} & 0 & 0 \\ \frac{\sigma_w^2}{N} & \frac{\sigma_w^2}{N} & 0 & 0 \\ 0 & 0 & \sigma_v^2 & \frac{\sigma_v^2}{N} \\ 0 & 0 & \frac{\sigma_v^2}{N} & \frac{\sigma_v^2}{N} \end{bmatrix}. \tag{16}$$

3.2 Population average dynamics

The mean state and mean state estimate equation can be isolated from (11) as:

$$\begin{bmatrix} m_{k+1} \\ \tilde{m}_{k+1} \end{bmatrix} = A_p \begin{bmatrix} m_k \\ \tilde{m}_k \end{bmatrix} + D_p \begin{bmatrix} \tilde{w}_k \\ \tilde{v}_{k+1} \end{bmatrix}, \tag{17}$$

where

$$A_p = \begin{bmatrix} a & -bf \\ a(c+h)K & a(1-cK) - bf(1+hK) \end{bmatrix}, \tag{18}$$

and

$$D_p = \begin{bmatrix} 1 & 0 \\ (c+h)K & K \end{bmatrix}. \tag{19}$$

Also, the covariance matrix of $[\tilde{w}_k, \tilde{v}_{k+1}]^T$ is given by:

$$\Sigma_{w,p} = \begin{bmatrix} \frac{\sigma_w^2}{N} & 0 \\ 0 & \frac{\sigma_v^2}{N} \end{bmatrix}. \tag{20}$$

4 Decentralized controller and state estimator

4.1 The race between N and T

It may appear obvious that as N goes to infinity, from Assumption 1 and (7) we have $\mathbb{E}[m_k] = 0$, and as a result at least asymptotically, the agent systems become essentially independent and individually *optimal control laws* are obtained via a Kalman filter K^* coupled with a gain f^* obtained from a Riccati equation. However, it turns out that while this is indeed correct if N is allowed to go to infinity before the length of the control horizon T is, *it is no longer always true if instead T is allowed to go to infinity first.* Theorem 1 establishes the separation result when N goes to infinity before T. Thus, in general, interchanging the orders of limits in N and T does not produce the same results.

Theorem 1 *For $N = \infty$, the separated optimal policies consisting of the naive Kalman filter (4) denoted as $K^*(a)$, and the control gain obtained by the appropriate algebraic Riccati equation denoted as $f^*(a,r)$, define the optimal solution.*

Proof. If $N = \infty$, the agent i observes that given independence and $\mathbb{E}[x_{0,j}] = 0$ for $j \geq 1$, $\frac{1}{N}\sum_{j=1}^{N} x_{1,j} \sim 0$ a.e. (almost everywhere). The agent makes the assumption that this situation will persist in the future and under this most optimistic assumption computes its optimal K^* and f^* based control law. At step 2, because the applied control inputs are independent from one agent to the other, the agents states remain independent, and since the optimal control law is stabilizing, the individual state variance remains bounded while the mean is still zero. Once again then $\frac{1}{N}\sum_{j=1}^{N} x_{2,j} \sim 0$, and the initial i^{th} agent optimistic assumption is validated. In general, by assuming that the separated optimal control law is applied and that at step k the agents have zero mean independent states, one can establish that the property of zero mean state independence persists at step $(k+1)$, thus yielding $\frac{1}{N}\sum_{j=1}^{N} x_{(k+1),j} \sim 0$. As a result, one can recursively establish that $\forall k$, under the K^* and f^* based control law, the optimistic assumption $\frac{1}{N}\sum_{j=1}^{N} x_{k,j} \sim 0$ holds, while under that assumption K^* and f^* would indeed be parameters of the optimal control law. Since no improvement to estimation can occur for N infinite, if $\frac{1}{N}\sum_{j=1}^{N} x_{k,j} \neq 0$, then K^* and f^* will indeed define the optimal control law. □

While one can suspect that K^* and f^* will be asymptotically optimal in a number of situations, it is not always so because as it turns out, stability is at the heart of the question. In order to illustrate this point, we show a simulation of the behavior of an agent sample path and the agents mean sample path for increasing values of N when $a = 2.5$, $r = 1$, and the optimal separated gains $K^*(a)$, $f^*(a,r)$ are applied.

4.1.1 Numerical results I

The numerical results reported in this paper are obtained considering the following parameter setting: $b = 1$, $c = 1$, $h = 1$, $\sigma_v = 1$, $\sigma_w = 1$ and initial standard deviation $\sigma_0 = 1$. The value of a and f (or $a_f = a - bf$) will be specified in the different simulations. Figures 1–2 which are obtained using the optimal Kalman-Riccati pair, show a simulation case where the cost runs to infinity for $N = 1000$ and $N = 1000000$, respectively.

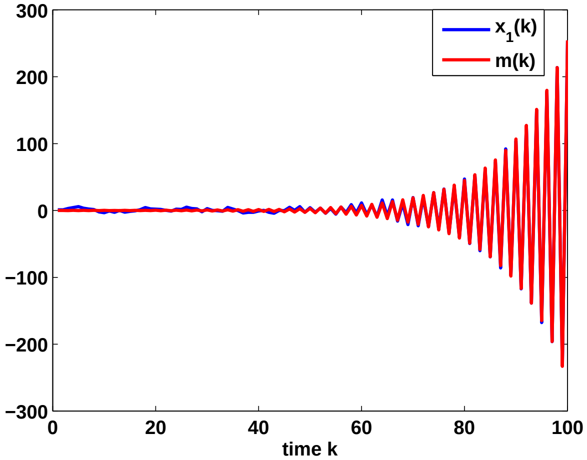

Figure 1: Unstable behavior of agent 1 and of the average of all the population when $a = 2.5$, $r = 1$, and $N = 1000$.

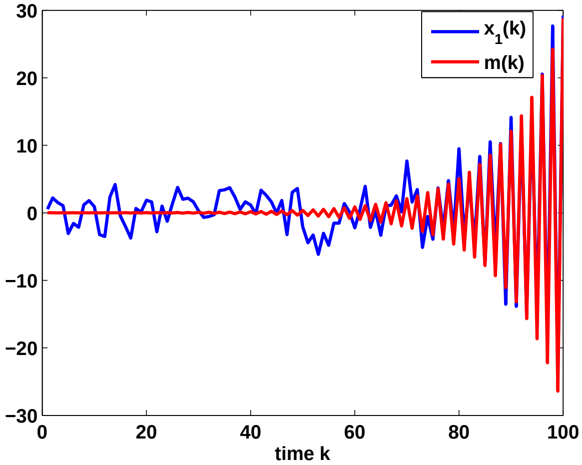

Figure 2: Unstable behavior of agent 1 and of the average of all the population when $a = 2.5$, $r = 1$, and $N = 1000000$.

4.2 Stability analysis

It is desirable to investigate the necessary and sufficient conditions for the closed-loop individual systems to be stable. By applying the Jury's stability criterion [26] to a second order polynomial we firstly note that the following lemma holds.

Lemma 1 *[32] A second-order discrete-time linear system having the following characteristic polynomial*

$$a_0 z^2 + a_1 z + a_2, \tag{21}$$

with real coefficients a_0*,* a_1 *and* a_2*, is stable iff*

$$a_0 + a_1 + a_2 > 0, \tag{22}$$

$$a_0 - a_1 + a_2 > 0, \tag{23}$$

$$a_0 - a_2 > 0. \tag{24}$$

Also, we have the following Lemma.

Lemma 2 *The population average dynamics (17) is such that the pair* $(A_p, D_p\sqrt{\Sigma_{w,p}})$ *is controllable.*

Proof. It is not difficult to show that $D_p\sqrt{\Sigma_{w,p}}$ is invertible. Therefore, the controllability matrix associated with the pair $(A_p, D_p\sqrt{\Sigma_{w,p}})$ is full rank. □

Now the mean system mean square stability conditions are given by the following theorem.

Theorem 2 *Mean system (17) is mean square stable (MSS) iff the following inequalities are satisfied:*

$$K(ac(1-a) + bf(ac+h)) > bf(a-1) - (1-a)^2, \tag{25}$$

$$K(-ac(1+a) + bf(ac-h)) > bf(a+1) - (1+a)^2, \tag{26}$$

$$|a(a-bf)(1-cK)| < 1. \tag{27}$$

Proof. First note that since by Lemma 2, system (17) is controllable by noise, a necessary and sufficient condition for the mean square stability of $[m_k, \tilde{m}_k]^T$ is the stability of matrix A_p (see Theorem 3.13, p. 31 in [20]). Now A_p will be stable iff the Jury stability test [26] is met. In particular, the characteristic polynomial of (17) is given by:

$$z^2 + (bf(1+hK) - a(2-cK))z + a(a-bf)(1-cK). \tag{28}$$

Thus, by applying Lemma 1 with

$$a_0 = 1, \tag{29}$$

$$a_1 = bf(1+hK) - a(2-cK), \tag{30}$$

$$a_2 = a(a-bf)(1-cK), \tag{31}$$

the theorem is proved. □

The next theorem gives individual state mean square stability conditions.

Theorem 3 *Individual agent systems described by (11) are MSS iff:*

$$K(ac(1-a) + bf(ac+h)) > bf(a-1) - (1-a)^2, \tag{32}$$

$$K(-ac(1+a) + bf(ac-h)) > bf(a+1) - (1+a)^2, \tag{33}$$

$$|a - bf| < 1, \tag{34}$$

$$|a(1-cK)| < 1. \tag{35}$$

Proof. We first note that, in view of the measurement structure of agent i, MSS of the mean population dynamics is necessary for the MSS of individual agents. Thus, inequalities (25)–(27) form necessary conditions. In addition since the pair $(A, D\sqrt{\Sigma_w})$ is controllable, by Theorem 3.13, p. 31 of [20], the complete state (including individual agent state and agent state estimate) is MSS iff matrix A is stable. However, given the block triangular structure of matrix A, its eigenvalues are the union of those of the diagonal blocks. Therefore, in addition to inequalities (25)–(27), one must also satisfy (34)–(35) obtained from the Jury stability criterion for the upper block. The concatenation of all these inequalities leads to (32)–(35) as necessary and sufficient conditions for MSS of individual agent state and state estimate dynamics. □

Remark 2 *For given* a*, the* (K, f) *stability region is independent of* N *because the stability conditions (25)–(27) and (32)–(35) are independent of* N*.*

The next lemma is about the stability region for f.

Lemma 3 *For given* a*,* $K^*(a)$*, the stability region for* f*, if non empty, is an interval* $(f_{\inf}(a), f_{\sup}(a))$*.*

Proof. This lemma is proved using the stability conditions of Theorem 3. In particular, under stability condition $|a(1-cK^*)| < 1$ and noting that $c, h > 0$, $K^* \geq 0$ we have:

$$-1 + a(1-cK^*) - hK^* < 0, \tag{36}$$

$$1 + a(1-cK^*) + hK^* > 0. \tag{37}$$

Moreover, condition (32) can be written as:

$$ac(1-a)K^* + (1-a)^2 > bf(-1 + a(1-cK^*) - hK^*). \tag{38}$$

Thus, combining (36) and (38) yields

$$f > \frac{1-a(1-cK^*)}{1-a(1-cK^*)+hK^*}\left(\frac{a-1}{b}\right). \tag{39}$$

Similarly, from condition (33) and (37) we have:

$$f < \frac{1+a(1-cK^*)}{1+a(1-cK^*)+hK^*}\left(\frac{a+1}{b}\right). \tag{40}$$

Furthermore, condition (34) can be written as:

$$\frac{a-1}{b} < f < \frac{a+1}{b} \tag{41}$$

Note that (39), (40), and (41) have to be all satisfied simultaneously. Therefore, noting that

$$0 < \frac{1-a(1-cK^*)}{1-a(1-cK^*)+hK^*} < 1, \tag{42}$$

$$0 < \frac{1+a(1-cK^*)}{1+a(1-cK^*)+hK^*} < 1, \tag{43}$$

we have:

$$f_{\inf}(a) = \frac{(a-1)(1-a(1-cK^*))}{b(1-a(1-cK^*)+hK^*)}, \; if \quad a < 1; \tag{44}$$

$$f_{\inf}(a) = \frac{a-1}{b}, \quad if \quad a \geq 1; \tag{45}$$

$$f_{\sup}(a) = \frac{a+1}{b}, \quad if \quad a \leq -1; \tag{46}$$

$$f_{\sup}(a) = \frac{(a+1)(1+a(1-cK^*))}{b(1+a(1-cK^*)+hK^*)}, \; if \; a > -1. \tag{47}$$

□

4.2.1 Numerical results II

Figure 3 is a representation of the stability regions defined by Theorem 3, and associated with the parameter set in Section 4.1.1 when a varies from $a = 0.2$ to $a = 5.5$. It is observed that the stability region gradually shrinks as a increases until it all but vanishes at $a = 5.5$.

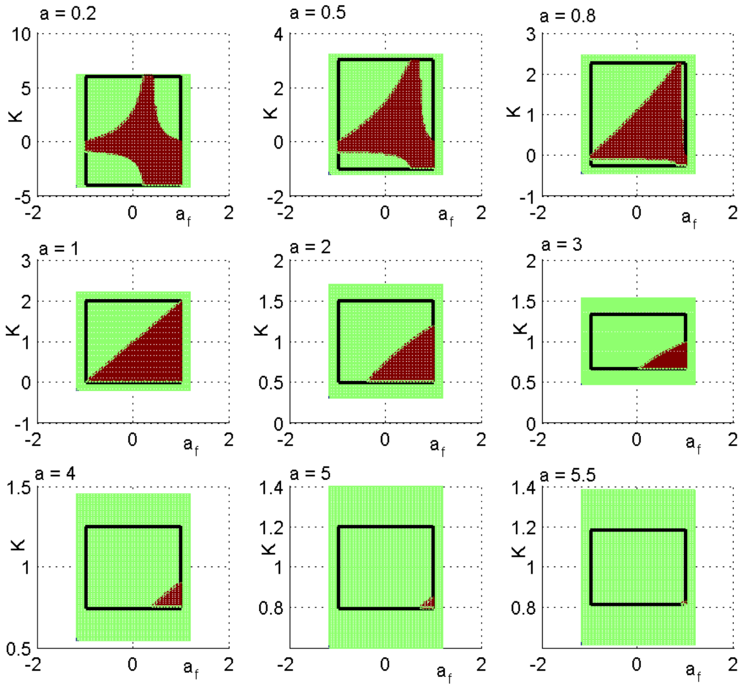

Figure 3: Stability regions: the box inside the black frame defines the region where (34) and (35) are met. This bounded area has been numerically explored to determine the stability regions (brown areas).

4.3 Reverse engineering agent cost functions for stability

One of the main goals of the paper is to identify parameter sets for which asymptotically, as the number of agents increases without bound, separated optimal control policies (i.e., based on $K^*(a)$ and $f^*(a,r)$) become optimal for the measurements coupled agent system itself. Since stability is clearly a necessary condition for this to happen, to this end, we shall be concerned with the following question: Given parameter a, the optimal isolated Kalman gain $K^*(a)$, and a pair $(K^*(a), f)$ within the stability region, is it always possible to reverse engineer the choice of parameter $r > 0$ in the individual agent cost function (5) so that $f = f^*(a,r)$? In order to answer that question, we develop the following steps called Algorithm 1.

Algorithm 1

- Apply the naive Kalman filter

$$\hat{x}_{k+1,i} = (a - bf)\hat{x}_{k,i} + K(y_{k+1,i} - c(a - bf)\hat{x}_{k,i}),$$

with the steady-state scalar gain

$$K^*(a) = \frac{cP(a)}{c^2 P(a) + \sigma_v^2},$$

where P is the unique positive solution of

$$c^2 P^2(a) + ((1 - a^2)\sigma_v^2 - c^2\sigma_w^2)P(a) - \sigma_w^2\sigma_v^2 = 0.$$

- For some fixed a, let

$$u_{k,i} = -f\hat{x}_{k,i},$$

and find the possible values of stabilizing f using Theorem 3.

- For each one such stabilizing f, reverse engineer if possible the cost structure

  $$J_i \triangleq \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(x_{k,i}^2 + ru_{k,i}^2),$$

  for parameter r by verifying if $0 < \frac{-b(a-bf)}{f[a(a-bf)-1]} < \infty$. If so, set

  $$r = \frac{-b(a-bf)}{f[a(a-bf)-1]}\,,$$

  that is the unique value of r for which f is optimal.

Lemma 4 *For $a \neq 0$, $[K^*(a), f]$ is an optimum isolated agent vector of control and estimation gains for some r iff*

$$0 < r = \frac{-b(a-bf)}{f[a(a-bf)-1]} < \infty. \tag{48}$$

Proof. In order to reverse engineer the cost structure in Algorithm 1, we use the expression of the optimal feedback gain assumed to be f to express Σ,

$$\Sigma = \frac{rf}{b(a-bf)}\,, \tag{49}$$

the positive solution of the algebraic Riccati equation in

$$b^2\Sigma^2 + (r - a^2r - b^2)\Sigma - r = 0. \tag{50}$$

Then solving the resultant equation for the candidate r yields:

$$r = \frac{-b(a-bf)}{f[a(a-bf)-1]}\,. \tag{51}$$

Thus for $a \neq 0$, whenever the expression in the right-hand side of (51) is strictly positive and finite, the pair $(K^*(a), f)$ will be isolated agent optimal policies for the corresponding value of r. Note that for $a = 0$, the optimal feedback gain is zero, and any $r \in (0,\infty)$ will correspond to a potential reverse engineering cost solution. Except for the indetermination at $a = 0$, either the reverse cost solution does not exist, or it exists uniquely. □

4.3.1 Numerical results III

The reverse engineering regions are depicted in Figure 4 and Figure 5. It is observed that the stability limit on a is obtained as $|a| \leq 3.68$. Moreover, for all $a \in (-2.52, 2.52)$ and (a, a_f) in the stability region, where a_f is defined as $a_f = a - bf$, it is possible to reverse engineer a cost parameter r for isolated optimality. In particular, Figure 5 confirms Lemma 3; in fact, for each a and $K^*(a)$, the stabilizing f belongs to an interval $(f_{inf}(a), f_{sup}(a))$. Lemma 4 has been used to evaluate if each stabilizing f corresponds to a positive value of r (i.e., if reverse engineering holds for that f).

4.4 Asymptotic optimality and ϵ-Nash equilibrium properties

In this section, we first establish conditions for the asymptotic optimality of isolated agent optimal gains.

Theorem 4 *Let $S(a)$ designate the (K, f) stability region associated with parameter a, and let $K^*(a)$ and $f^*(a,r)$, respectively, be the optimal estimation and control gains associated with the isolated agent optimal control problem (when $h = 0$, i.e., with zero average coupling term) for some $0 < r < \infty$. If $(K^*(a), f^*(a,r)) \in$*

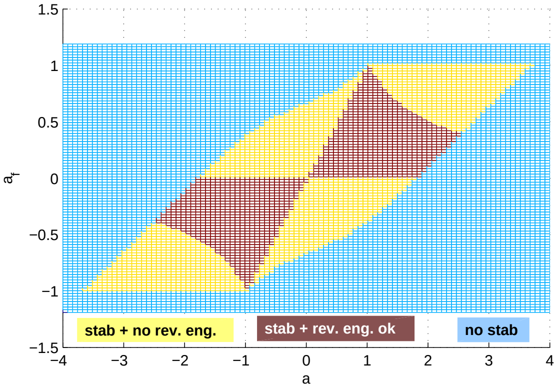

Figure 4: Stability and reverse engineering regions in (a, a_f)

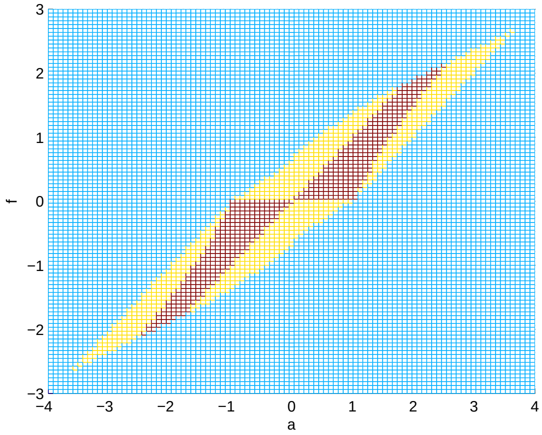

Figure 5: Stability and reverse engineering regions in (a, f) (same color meaning as Figure. 4)

$S(a)$, then the couple $(K^*(a), f^*(a,r))$ defines an asymptotically optimal policy for the coupled agents problem as $N \to \infty$. Furthermore, for N finite, it is ϵ-optimal with ϵ of order $1/N$ over any closed subset of $S(a)$ containing $(K^*(a), f^*(a,r))$.

Proof. We first note that for any $(K, f) \in S(a)$, from conditions (34) and (35), respectively, f and K will be bounded. Then using (3) and under the assumption that $(K^*(a), f^*(a,r)) \in S(a)$, a direct calculation of the actual cost functions yields

$$\begin{aligned} J_i &= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(x_{k,i}^2 + ru_{k,i}^2), \\ &= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(X_{k,i}^T\bar{Q}X_{k,i}), \\ &= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}Tr(\bar{Q}X_{k,i}X_{k,i}^T), \\ &= Tr(\bar{Q}\bar{P}_\infty), \end{aligned} \tag{52}$$

where

$$X_{k,i} = \begin{bmatrix} x_{k,i} \\ \hat{x}_{k,i} \\ m_k \\ \tilde{m}_k \end{bmatrix}, \bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & rf^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{53}$$

$\bar{P}_k = \mathbb{E}[X_{k,i}X_{k,i}^T]$, and $\bar{P}_\infty = \lim_{k\to\infty} \bar{P}_k$. Note that $\bar{P}_\infty$ can be directly calculated from the covariance equation of the closed-loop system (11) and is given as the unique positive definite solution of the Lyapunov equation:

$$A\bar{P}_\infty A^T - \bar{P}_\infty + D\Sigma_w D^T = 0, \tag{54}$$

where Σ_w denotes the covariance matrice of $W_{k,i}$. Now Σ_w can be written as:

$$\Sigma_w = \Sigma_{w_1} + \frac{1}{N}\Sigma_{w_2}, \tag{55}$$

where

$$\Sigma_{w_1} = \begin{bmatrix} \sigma_w^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_v^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Sigma_{w_2} = \begin{bmatrix} 0 & \sigma_w^2 & 0 & 0 \\ \sigma_w^2 & \sigma_w^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_v^2 \\ 0 & 0 & \sigma_v^2 & \sigma_v^2 \end{bmatrix}. \tag{56}$$

Since (54) is a linear equation, superposition holds and $\bar{P}_\infty$ can be split into two components respectively, $\bar{P}_{\infty_1}$ and $\bar{P}_{\infty_2}$ corresponding to the respective contributions of Σ_{w_1} and $\frac{1}{N}\Sigma_{w_2}$. In addition, let the corresponding contributions to the total cost be respectively denoted $\alpha(K,f)$ and $\frac{1}{N}\beta(K,f)$. Thus

$$J_i(K,f,\frac{1}{N}) = \alpha(K,f) + \frac{1}{N}\beta(K,f). \tag{57}$$

Since (K,f) belongs to $S(a)$ the nature of which is independent of N, then both $\alpha(K,f)$ and $\beta(K,f)$ will be bounded. As a result, for every $(K,f) \in S(a)$, the total cost converges pointwise to $\alpha(K,f)$ as N goes to infinity. Furthermore, $\alpha(K,f)$ can be computed from (52) as:

$$\begin{aligned}
\alpha(K,f) \equiv J_i(K,f,\frac{1}{N}) \mid_{(\frac{1}{N}=0)} &= (-K^3a^4c^3\sigma_w^2 \\
&- rK^3a^4cf^2\sigma_v^2 + 3K^3a^3bc^3f\sigma_w^2 + rK^3a^3bcf^3\sigma_v^2 \\
&- 3K^3a^2b^2c^3f^2\sigma_w^2 - K^3a^2b^2cf^2\sigma_v^2 - rK^3a^2c^3f^2\sigma_w^2 \\
&+ rK^3a^2cf^2\sigma_v^2 + K^3ab^3c^3f^3\sigma_w^2 + K^3ab^3cf^3\sigma_v^2 \\
&+ rK^3abc^3f^3\sigma_w^2 + rK^3abcf^3\sigma_v^2 + 3K^2a^4c^2\sigma_w^2 \\
&+ rK^2a^4f^2\sigma_v^2 - 9K^2a^3bc^2f\sigma_w^2 - rK^2a^3bf^3\sigma_v^2 \\
&+ 9K^2a^2b^2c^2f^2\sigma_w^2 + K^2a^2b^2f^2\sigma_v^2 + rK^2a^2c^2f^2\sigma_w^2 \\
&- K^2a^2c^2\sigma_w^2 - 2rK^2a^2f^2\sigma_v^2 - 3K^2ab^3c^2f^3\sigma_w^2 \\
&- K^2ab^3f^3\sigma_v^2 - rK^2abc^2f^3\sigma_w^2 + rK^2abf^3\sigma_v^2 \\
&+ K^2b^2c^2f^2\sigma_w^2 + K^2b^2f^2\sigma_v^2 + rK^2c^2f^2\sigma_w^2 \\
&+ rK^2f^2\sigma_v^2 - 3Ka^4c\sigma_w^2 + 9Ka^3bcf\sigma_w^2 + 3Ka^2c\sigma_w^2 \\
&- 9Ka^2b^2cf^2\sigma_w^2 + 3Kab^3cf^3\sigma_w^2 - 3Kabcf\sigma_w^2 \\
&+ a^4\sigma_w^2 - 3a^3bf\sigma_w^2 + 3a^2b^2f^2\sigma_w^2 - 2a^2\sigma_w^2 - ab^3f^3\sigma_w^2 \\
&+ 3abf\sigma_w^2 - b^2f^2\sigma_w^2 + \sigma_w^2)/((a^2(1-cK)^2 - 1) \\
&((a-bf)^2 - 1)(1 - a(a-bf)(1-cK))).
\end{aligned} \tag{58}$$

On the other hand, the isolated (separated) agents cost with $h = 0$, denoted $J_i^{(s)}$, can be similarly calculated. In particular, we have:

$$
\begin{aligned}
J_i^{(s)}(K,f) &= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(x_{k,i}^2 + ru_{k,i}^2),\\
&= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}(\begin{bmatrix} x_{k,i} \\ \hat{x}_{k,i}\end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & rf^2\end{bmatrix} \begin{bmatrix} x_{k,i} \\ \hat{x}_{k,i}\end{bmatrix}),\\
&= \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\sum_{k=0}^{T-1}Tr(\begin{bmatrix} 1 & 0 \\ 0 & rf^2\end{bmatrix} \begin{bmatrix} x_{k,i} \\ \hat{x}_{k,i}\end{bmatrix} \begin{bmatrix} x_{k,i} \\ \hat{x}_{k,i}\end{bmatrix}^T),\\
&= \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=0}^{T-1}Tr(\begin{bmatrix} 1 & 0 \\ 0 & rf^2\end{bmatrix} \bar{P}_s),\\
&= Tr(\begin{bmatrix} 1 & 0 \\ 0 & rf^2\end{bmatrix} \bar{P}_s),
\end{aligned} \tag{59}
$$

where $\bar{P}_s$ is directly calculated from the covariance equation given by

$$A_s\bar{P}_sA_s^T - \bar{P}_s + D_s\Sigma_{w,s}D_s^T = 0, \tag{60}$$

with

$$A_s = \begin{bmatrix} a & -bf \\ acK & a(1-cK)-bf\end{bmatrix}, D_s = \begin{bmatrix} 1 & 0 \\ cK & K\end{bmatrix}, \tag{61}$$

$$\Sigma_{w,s} = \begin{bmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_v^2\end{bmatrix}. \tag{62}$$

Solving (60) and replacing $\bar{P}_s$ in (59) yields $J_i^{(s)}(K,f) = J_i(K,f,0)$. In particular, denoting (K^*,f^*) the (K,f) couple minimizing $J_i^{(s)}(K,f)$, we have: $J_i^{(s)}(K^*,f^*) = J_i(K^*,f^*,0)$. Thus the isolated agent optimal policy is asymptotically optimal for the coupled agents provided that $(K^*(a),f^*(a,r)) \in S(a)$. Moreover, if (K,f) is constrained to an arbitrary closed subset M of $S(a)$ containing (K^*,f^*), then invoking the continuity of $\beta(K,f)$ over a closed and bounded set, $\beta(K,f)$ is uniformly bounded on M, and in view of equation (57), the convergence of $Argmin_{(K,f)\in M}(J_i(K,f,\frac{1}{N}))$ to (K^*,f^*) will be uniform in M, as N goes to infinity. As a result, over an arbitrary such M, and for N finite, (K^*,f^*) is epsilon optimal where epsilon is $O(\frac{1}{N})$. $\square$

We proceed to study the ϵ-Nash equilibrium property by deriving the dynamics model of a so-called *deviant agent*, say the i_0^{th} agent that tries to improve its cost by choosing a different set of control and estimation gains denoted by $(\tilde{K},\tilde{f})$. In particular, define:

$$m_k^- = \frac{1}{N}\sum_{j\neq i_0}^{N} x_{k,j}, \quad \tilde{m}_k^- = \frac{1}{N}\sum_{j\neq i_0}^{N} \hat{x}_{k,j}, \tag{63}$$

$$\tilde{w}_k^- = \frac{1}{N}\sum_{j\neq i_0}^{N} w_{k,j}, \quad \tilde{v}_k^- = \frac{1}{N}\sum_{j\neq i_0}^{N} v_{k,j}. \tag{64}$$

Then following a procedure similar to that in Section 3.1, the 4^{th} order model of the closed-loop deviant agent dynamics can be expressed as:

$$X_{k+1,i_0} = A_dX_{k,i_0} + D_dW_{k,i_0}, \tag{65}$$

where the augmented state is

$$X_{k,i_0} = [x_{k,i_0}, \hat{x}_{k,i_0}, m_k^-, \tilde{m}_k^-]^T, \tag{66}$$

and matrix A_d is given by

$$A_d = \begin{bmatrix} a & -b\tilde{f} & 0 & 0 \\ a(c+\frac{h}{N})\tilde{K} & a'_{2,2} & ah\tilde{K} & -bhf^*\tilde{K} \\ 0 & 0 & a & -bf^* \\ \frac{(N-1)}{N^2}ahK^* & -\frac{(N-1)}{N^2}bh\tilde{f}K^* & a'_{4,3} & a'_{4,4}\end{bmatrix}, \tag{67}$$

with

$$a'_{2,2} = a(1 - c\tilde{K}) - b\tilde{f}(1 + \frac{h}{N}\tilde{K}), \tag{68}$$

$$a'_{4,3} = a(c + \frac{(N-1)}{N}h)K^*, \tag{69}$$

$$a'_{4,4} = a(1 - cK^*) - bf^*(1 + \frac{(N-1)}{N}hK^*), \tag{70}$$

and also we have:

$$D_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (c + \frac{h}{N})\tilde{K} & h\tilde{K} & \tilde{K} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{(N-1)}{N^2}hK^* & (c + \frac{(N-1)}{N}h)K^* & 0 & K^* \end{bmatrix}. \tag{71}$$

Furthermore, the noise vector W_{k,i_0} and its covariance matrix $\Sigma_{w,d}$ are given by:

$$W_{k,i_0} = \begin{bmatrix} w_{k,i_0} \\ \tilde{w}^{-}_{k} \\ v_{k+1,i_0} \\ \tilde{v}^{-}_{k+1} \end{bmatrix}, \Sigma_{w,d} = \begin{bmatrix} \sigma_w^2 & 0 & 0 & 0 \\ 0 & \frac{(N-1)}{N^2}\sigma_w^2 & 0 & 0 \\ 0 & 0 & \sigma_v^2 & 0 \\ 0 & 0 & 0 & \frac{(N-1)}{N^2}\sigma_v^2 \end{bmatrix}. \tag{72}$$

The next lemma gives a necessary condition for mean square stability of the deviant agent system.

Lemma 5 *A necessary condition for the closed-loop deviant agent dynamics (65) to be MSS is that*

$$|a^2(1 - cK^*)(1 - c\tilde{K})(a - bf^*)(a - b\tilde{f})| < 1. \tag{73}$$

Proof. Since $\forall N$, $D_d\sqrt{\Sigma_{w,d}}$ is invertible, then the pair $(A_d, D_d\sqrt{\Sigma_{w,d}})$ is controllable, and by virtue of Theorem 3.13, p. 31 in [20], the deviant agent system (65) will be MSS iff A_d is a stable matrix. Now A_d will be stable iff the Jury stability test is met. In particular, the characteristic polynomial of (65) is given by:

$$z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0, \tag{74}$$

where

$$\alpha_0 = a^2(1 - cK^*)(1 - c\tilde{K})(a - bf^*)(a - b\tilde{f}), \tag{75}$$

and the expressions of α_1, α_2, α_3 are complex and are dropped for brevity. Now one of the five stability conditions from the Jury test [26] is that $|\alpha_0| < 1$, which yields (73). □

Theorem 5 *The set of gains $(K^*(a), f^*(a,r)) \in S(a)$ defines an ϵ-Nash equilibrium.*

Proof. The deviant agent must always opt for a $(\tilde{K}, \tilde{f})$ couple which would stabilize matrix A_d (Hurwitz matrix), no matter what N is. Note that the set of such stabilizing couples is non empty since it includes (K^*, f^*) irrespective of N. Also, note that in view of (73), $\tilde{K}$ can become unbounded only if $a = b\tilde{f}$. Additionally, $\tilde{f}$ can be unbounded only if $c\tilde{K} = 1$. Let us refer to these cases as (i) and (ii), respectively. We now establish that both (i) and (ii) are excluded if the deviant agent optimizes its choices. Indeed case (i) would imply from (4) that $\overline{\lim}_{k\to\infty}\hat{x}_{k,i_0}$ can only remain of bounded variance, if y_{k,i_0} goes to zero almost everywhere as $k \to \infty$. However, this is impossible in view of the independent measurement noise that enters the i_0^{th} agent observations. Case (ii) on the other hand, would imply from (3) that $\overline{\lim}_{k\to\infty}\hat{x}_{k,i_0} = 0$ a.e., otherwise $\overline{\lim}_{k\to\infty}\mathbb{E}[u_{k,i_0}^2] \to \infty$; which would be clearly suboptimal since f^*, K^* achieve a finite limiting cost. However, this cannot happen since from (4) and $\tilde{K} \neq 0$, this would mean $\overline{\lim}_{k\to\infty}y_{k,i_0} = 0$ a.e., and we know the latter to be impossible. As a result for any $a \in S^{opt}(a,r)$, where $S^{opt}(a,r)$ denotes the set of couples $\{(a,r)|(K^*(a), f^*(a,r)) \in S(a)\}$, $\tilde{K}(N)$ and $\tilde{f}(N)$ will be bounded. Thus, if the deviant agent maintains its choice fixed of $\tilde{K}(N_0)$ and $\tilde{f}(N_0)$, for any $N \geq N_0$, then this choice becomes eventually suboptimal for N

large enough. Indeed, as N increases to infinity, the dynamics of $[x_{k,i_0}, \hat{x}_{k,i_0}]^T$ becomes entirely decoupled from that of $[m_k^-, \tilde{m}_k^-]^T$, as $\frac{(N-1)}{N^2} x_{k,i_0}$, $\frac{(N-1)}{N^2} \tilde{f}(N_0)\hat{x}_{k,i_0}$, $\tilde{K}(N_0) m_k^-$, $\tilde{K}(N_0)\tilde{m}_k^-$ go to zero almost surely. In that case, the deviant agent optimal choice for $(\tilde{K}(N), \tilde{f}(N))$ becomes precisely (K^*, f^*). As a result, any cost improvement attributed to some optimal decision $(\tilde{K}^*(N_0), \tilde{f}^*(N_0))$ by the deviant agent can only improve its cost by $\epsilon(N)$, with $\epsilon(N)$ going to zero as $N \to \infty$. This establishes that a finite N_0-based policy, is such that the (K^*, f^*) policy is an ϵ-Nash equilibrium. Finally, note that if the deviant agent starts at the outset with the $N = \infty$ policy, from Theorem 1, it would have to pick the (K^*, f^*) policy as an optimal response. ☐

4.4.1 Numerical results IV

In this section, we present a numerical example on the deviant agent (namely, the 50^{th} agent) in a population of $N = 100$ agents via simulations of $T = 100000$ steps. In particular, we consider $a = 2.44$, where the optimal Kalman-Riccati couple $(K^* = 0.8595, a_{f^*} = 0.3428)$ belongs to the stability region and all the agents except the deviant agent, apply this couple. We let the deviant agent apply another stabilizing couple $(\tilde{K} = 0.8307, a_{\tilde{f}} = 0.3929)$ such that all eigenvalues of A_d lie inside the unit circle. The improvement on the cost achieved by the deviant agent $i_0 = 50$ is shown in Figure 6. However, performing the same simulation with $N = 1000$ agents, we observed that the cost difference has disappeared.

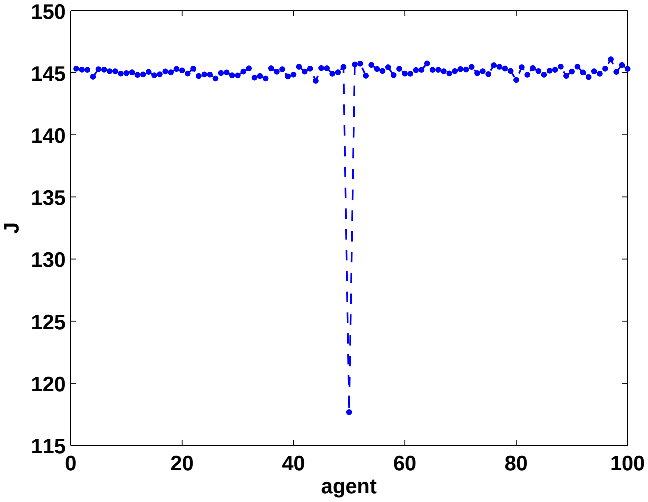

Figure 6: Cost comparison between the deviant agent and each of the other agents when $a = 2.44$, under $r = 1$, $N = 100$, $i_0 = 50$.

4.4.2 Boundaries of stability regions of separated optimal gains

Assuming $a > 0$ for the sake of analysis in the whole forthcoming discussion, we first define a threshold on a.

Definition 1 *a_{Nash} is the (unique) value of a if it exists, such that:*

$$a_{Nash} - b f_{\sup}(a_{Nash}) = 0. \tag{76}$$

Also, we have the following Lemma.

Lemma 6 *Assume Eq. (76) admits a solution a_{Nash}, and let $f_{\sup}(a)$ be given by (47). Then we have:*

(i) a_{Nash} exists uniquely;

(ii) for $a > a_{Nash}$, $a/b > f_{\sup}(a)$;

(iii) for $a < a_{Nash}$*,* $a/b < f_{\sup}(a)$*;*

(iv) a_{Nash} *can assume any positive value by properly modifying* h*.*

Proof. First of all, from the expression of $K^*(a)$ (see Algorithm 1), it is straightforward to see that $K^*(a) > 0$ for all a and that $cK^*(a) \in (0, 1)$ for all a. A direct computation of the derivative of $K^*(a)$ allows one to also verify that $K^*(a)$ is a strictly increasing function of $a > 0$ which ranges from $\frac{1}{c(1+\delta)}$ when $a = 0$ (where $\delta \triangleq \sigma_v^2/(c^2\sigma_w^2)$ is a positive quantity) and goes toward $1/c$ as $a \to \infty$.

Consider now the difference $f_{\sup}(a) - a/b$, which can be written as follows:

$$f_{\sup}(a) - a/b = \frac{ag(a,h)}{b[1 + a(1 - cK^*(a)) + hK^*(a)]}\,, \tag{77}$$

where

$$g(a,h) = 1 - (c+h)K^*(a) + 1/a. \tag{78}$$

Notice that from (37) the denominator of (77) is positive for all $a > 0$, so the sign of $f_{\sup}(a) - a/b$ coincides with the sign of $g(a,h)$ and $f_{\sup}(a) = a/b$ if and only if $g(a,h) = 0$. Given the aforementioned properties on $K^*(a)$, we have that $g(a,h)$ is strictly decreasing for $a > 0$ and ranges from $+\infty$ (when $a \to 0$) to $-h/c$ (when $a \to \infty$). Therefore, by continuity, $g(a,h)$ will cross zero at some $a = a_{Nash}$, and by the strict decreasing character of $g(a,h)$, the intersection will be unique. This concludes the proof of (i) and also of (ii) and (iii) since, as mentioned, the sign of $f_{\sup}(a) - a/b$ coincides with the sign of $g(a,h)$.

Property (iv) can be obtained by considering, for $h_2 > h_1$, the difference:

$$g(a,h_2) - g(a,h_1) = (h_1 - h_2)K^*(a) < 0. \tag{79}$$

In particular:

$$g(a_{Nash_1}, h_2) - g(a_{Nash_1}, h_1) = g(a_{Nash_1}, h_2) < 0, \tag{80}$$

which indicates: $a_{Nash_1} > a_{Nash_2}$, that is, the solution $a_{Nash}(h)$ of (76) is strictly decreasing with h. Moreover, as $h \to 0$, we have $a_{Nash} \to +\infty$ and, as $h \to +\infty$, since $f_{\sup}(a) \to 0$ then $a_{Nash} \to 0$. Thus, by moving h from zero to infinity, one moves a_{Nash} over its complete possible range. □

Remark 3 *Noting the expression of* $g(a,h)$ *given by (78), it is possible to evaluate the threshold* h_t *such that* a_{Nash} *crosses* 1 *(i.e.* $a_{Nash} > 1$ *for all* $h < h_t$ *and* $a_{Nash} < 1$ *for all* $h > h_t$*) which is* $h_t = (2 - cK^*(1))/K^*(1)$*.*

The threshold a_{Nash} represents the maximum value of a past which it is not always possible to apply so-called isolated optimal control policies without causing potential instability problems. More specifically, if $a < a_{Nash}$, we have that $(K^*(a), f^*(a,r))$ stabilizes the population for all r while, for $a > a_{Nash}$, the range of r progressively decreases. The following proposition provides a formal characterization of this behavior.

Proposition 1 *Assume* $a > 0$*, let* $\bar{f}_{sup}(a) = \min\{a/b,$ $f_{\sup}(a)\}$ *and define*

$$r_{\inf}(a) = \frac{-b(a - b\bar{f}_{sup}(a))}{\bar{f}_{sup}(a)[a(a - b\bar{f}_{sup}(a)) - 1]}\,. \tag{81}$$

Then there exists a threshold $a_{\sup} > \max\{a_{Nash}, 1\}$*, which satisfies*

$$a_{\sup}[a_{\sup} - bf_{\sup}(a_{\sup})] - 1 = 0, \tag{82}$$

such that, for all $a < a_{\sup}$*,* $(K^*(a), f^*(a,r)) \in S(a)$ *for all* $r > r_{\inf}(a) \geq 0$ *(with* $r_{\inf}(a) = 0$ *for all* $a \leq a_{Nash}$*). Moreover,* $r_{\inf}(a) \to +\infty$ *as* $a \to a_{\sup}$*, and, if* $a > a_{\sup}$*,* $(K^*(a), f^*(a,r)) \notin S(a)$ *for all* r*.*

Proof. See the Appendix. □

Remark 4 *According to Proposition 1 and Lemma 6, for all* $0 < a < a_{Nash}$*,* $(K^*(a), f^*(a,r)) \in S(a)$ *for all positive* r *while, for* $a_{Nash} < a < a_{\sup}$*,* $(K^*(a), f^*(a,r)) \in S(a)$ *only if* r *is larger than a positive threshold* $r_{\inf}(a)$*. Moreover, since* $r(\bar{f}_{inf}(a)) = +\infty$*, for* $a = a_{\sup}$*,* $r_{\inf}(a)$ *reaches* $+\infty$*. So, at that point, the range of admissible* r*'s shrinks to zero. Past* $a_{\sup}$*, isolated interference indifferent optimal controls can no longer stabilize the system for any value of* r*, and one has to resort to cooperative control.*

4.4.3 Numerical results V

Figure 7 shows a numerical investigation on $r_{\text{inf}}(a)$. More specifically, considering values of a ranging on a grid with step 0.01 one can observe that:

- For $0 < a \leq 1.79$, we have $r_{\text{inf}}(a) = 0$;
- For $1.8 \leq a \leq 2.53$, $r_{\text{inf}} > 0$ and rapidly goes toward infinity;
- For $a = 0$ and $a \geq 2.54$, reverse engineering does not apply for any r, so r_{inf} is not defined.

Thus, we have: $a_{Nash} = 1.79$, and $a_{\text{sup}} = 2.53$. We note that the analytical values of a_{Nash} and a_{sup} obtained through (76) and (82), i.e., respectively 1.7963 and 2.5369, confirm the numerical findings.

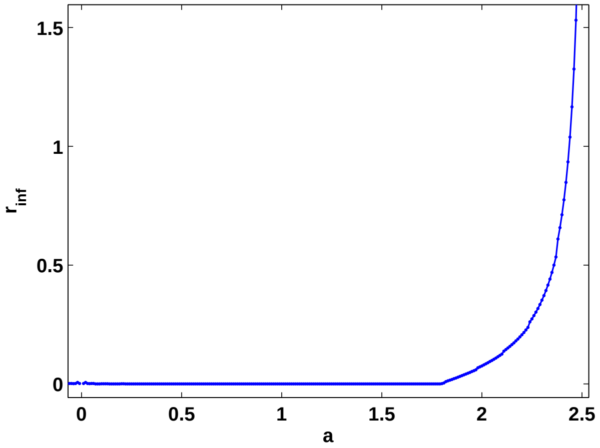

Figure 7: $r_{\text{inf}}(a)$ as calculated using increments of 0.01 by evaluating for values of a ranging on a grid of step 0.01.

5 Cooperative decentralized separated policies

Definition 2 *Cooperative decentralized separated policies are defined as decentralized separated policies (see Section 2) with common gains K, f such that the resulting social cost*

$$J_{soc}^{(N)} = \frac{1}{N}\sum_{j=1}^{N} J_j \tag{83}$$

is minimized.

If $a > a_{\text{sup}}$, then agents must cooperate for otherwise, they risk having to pay an infinite cost. This is a situation where the optimal Kalman-Riccati couple (K^*, f^*) is outside of the stability region. On the other hand, even when $a \leq a_{\text{sup}}$, agents may still be interested in achieving optimal *cooperative* decentralized separated policies. We have the following lemma and proposition for the cooperative cost.

Lemma 7 *(K, f) based local control policies, $\forall (K, f) \in S(a)$, when uniformly applied by all agents, for a given N lead to a steady-state cost given by:*

$$J_i = Tr(\bar{Q}\bar{P}_\infty) \tag{84}$$

where $\bar{Q} = diag([1, rf^2, 0, 0])$, and $\bar{P}_\infty$ is the steady-state solution of the covariance equation given by

$$\bar{P}_{k+1} = A\bar{P}_k A^T + D\Sigma_w D^T \tag{85}$$

Proof. See the cost calculations in Theorem 4. □

Proposition 2 *If the optimal Kalman-Riccati couple* (K^*, f^*) *belongs to stability region* $S(a)$*, then it characterizes an* ϵ*-optimal cooperative decentralized separated control policy for system (1), (2), (5), with* ϵ *of order* $1/N$*, over any closed subset of* $S(a)$ *including* (K^*, f^*)*.*

Proof. The proof follows readily by recalling individual cost expression (57), and recognizing that (i) $\beta(K, f)$ is continuous in (K, f) over a closed subset of $S(a)$, and therefore is uniformly bounded over that subset, (ii) the minimum of $\alpha(K, f)$ is $\alpha(K^*, f^*)$. □

In the following, we numerically explore the situation when (K^*, f^*) does not belong to the stability region $S(a)$.

5.1 Numerical results VI

In this section, we present some numerical results on cooperative control. Figures 8–12 show the simulation results for $a = 2.44$, where $(K^* = 0.8595, a_{f^*} = 0.3428)$ belongs to the stability region. It is observed that the optimal cooperative $(K = 0.8307, a_f = 0.3929)$ is near the Kalman-Riccati couple $(K^* = 0.8595, a_{f^*} = 0.3428)$ and approaches this point as N goes to infinity.

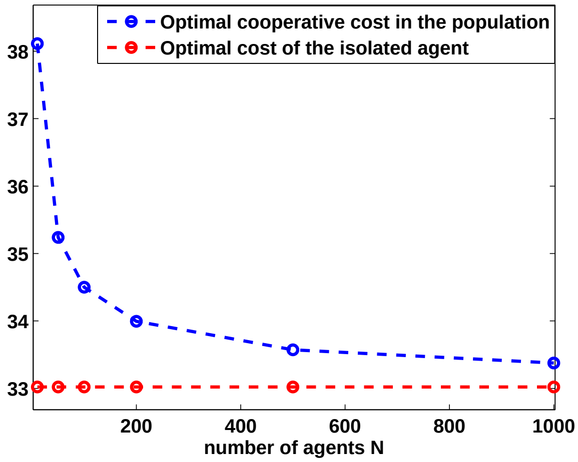

Figure 8: Optimal cost J_i^* when $a = 2.44$, under $r = 1$, $b = c = h = 1$, $\bar{x}_0 = 0$, $\sigma_0 = \sigma_w = \sigma_v = 1$.

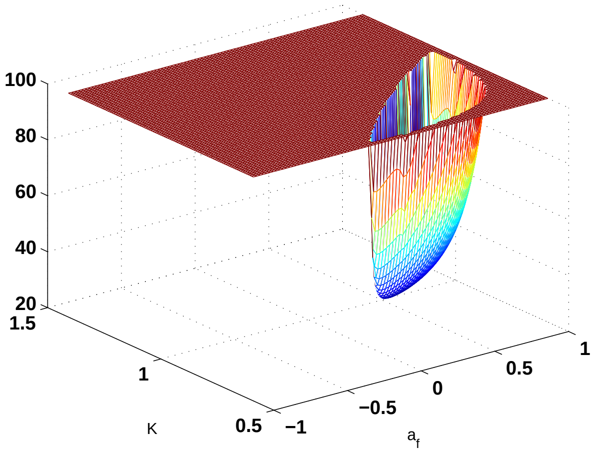

Figure 9: Analytical cost J_i as a function of K and a_f, saturated at 100, where $a_f = a - bf$, $a = 2.44$, $N = 100$.

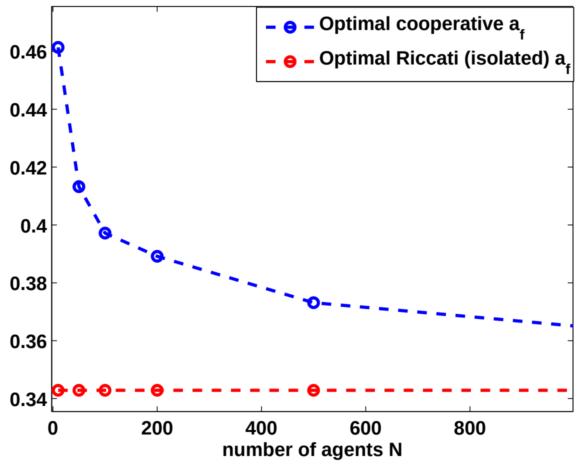

Figure 10: Optimal cooperative a_f as a function of the number of agents, where $K = K^*$ and $a = 2.44$.

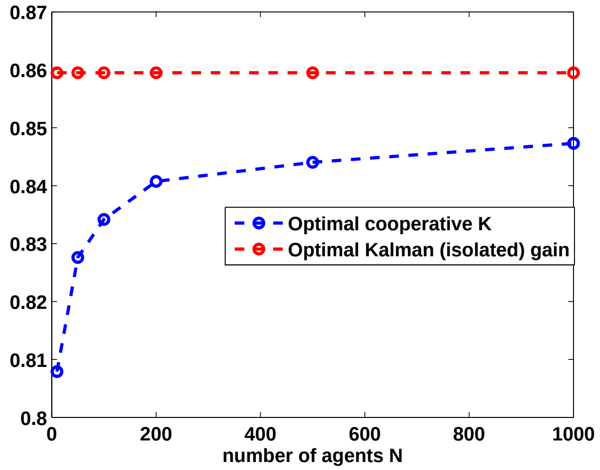

Figure 11: Optimal cooperative K as a function of the number of agents, where $f = f^*$ and $a = 2.44$.

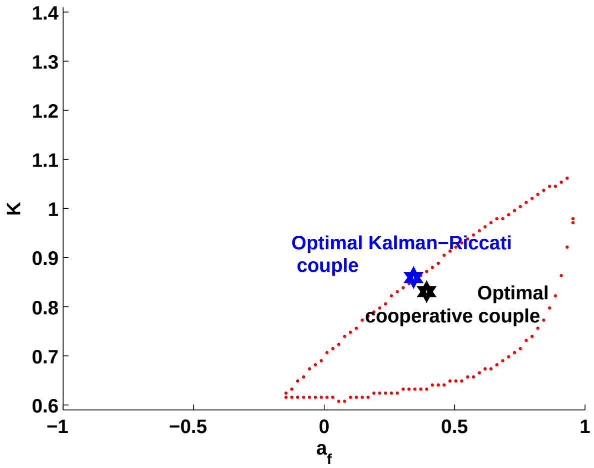

Figure 12: Stability region (inside the red border) when $a = 2.44$, $N = 100$.

Moreover, the simulation results for $a = 4$ are depicted via Figures 13–14. We note that the reverse engineering of r does not apply for $a = 4$, and that K and a_f do not converge toward K^* and a_{f^*} as N increases. Also in this case, $(K^* = 0.9414, a_{f^*} = 0.2344)$ does not belong to the stability region.

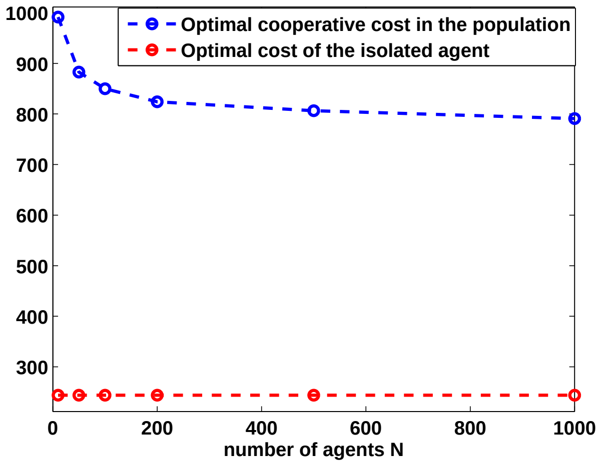

Figure 13: Optimal (analytical) cost J_i^* when $a = 4$.

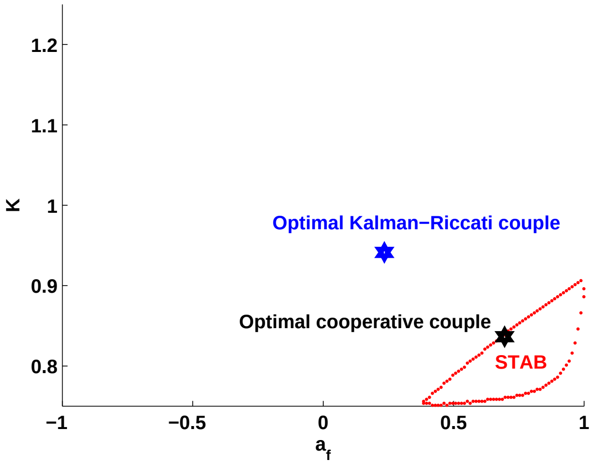

Figure 14: Stability region (inside the red border) when $a = 4$, $N = 100$.

Furthermore, for $a = 4$, we present a numerical example on the deviant agent (i.e., the 50^{th} agent) in a population of $N = 100$ agents via simulations of $T = 100000$ steps. In particular, all the agents except the deviant agent, apply the optimal cooperative couple $(K = 0.8365, a_f = 0.6955)$. We let the deviant agent apply another stabilizing couple $(\tilde{K} = 0.8842, a_{\tilde{f}} = 0.4857)$ such that all eigenvalues of A_d lie inside the unit circle. The improvement on the cost achieved by the deviant agent $i_0 = 50$ is shown in Figure 15.

We note that there does not exist an optimum cooperative control setting in the $a > a_{\sup}$ region, but an infimum of the cost which would be its minimal value on the stability border under the assumption that $J_i(K, f)$ is strictly convex in K and f. For example, when $a = 3$, we have performed long simulation runs to evaluate the isolated cost on a grid of the (K, a_f) plane with a step of about 0.01 to determine the minimum. The population stability region (inside the red dot points) is depicted in Figure 16, where we verify that the minimum of the isolated cost on such stability region falls indeed on its border. It also shows the cost of N agents as N increases, and the position of these minima. It is observed that the optimal $(K(N), a_f(N))$ couple is actually approaching the optimal isolated cost on the border as N increases. Moreover, considering the black segment in this figure (called segment Σ) which comprises 100 points numbered from inside to the border of the stability region, Figure 17 illustrates the isolated cost versus the cost of N agents for different

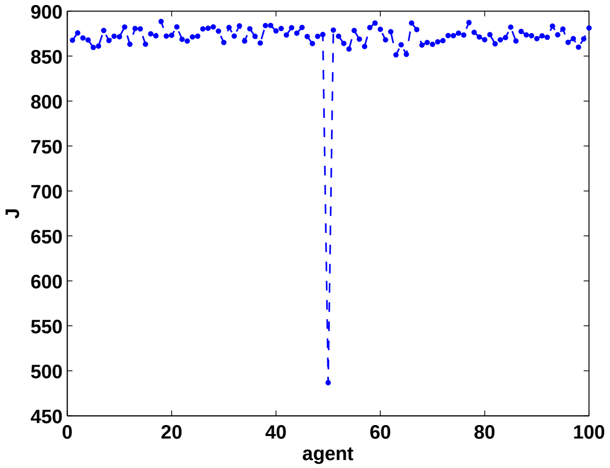

Figure 15: Cost comparison between the deviant agent ($i_0 = 50$) and each of the other agents when $a = 4$, under $r = 1$, $N = 100$.

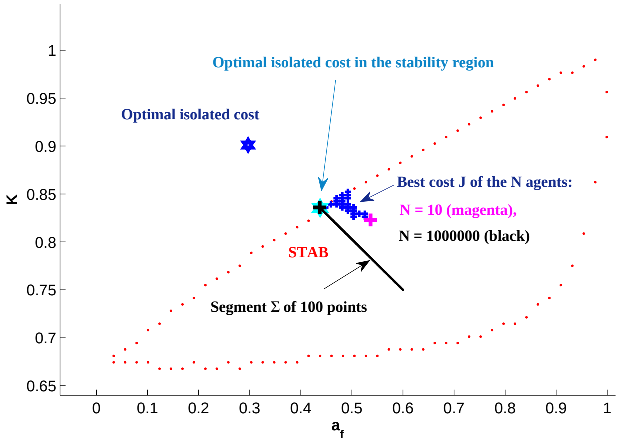

Figure 16: Stability region (inside the red border), and the best cost J of the N agents with increasing N when $a = 3$; the magenta and black colors correspond to $N = 10$ and $N = 1000000$, respectively.

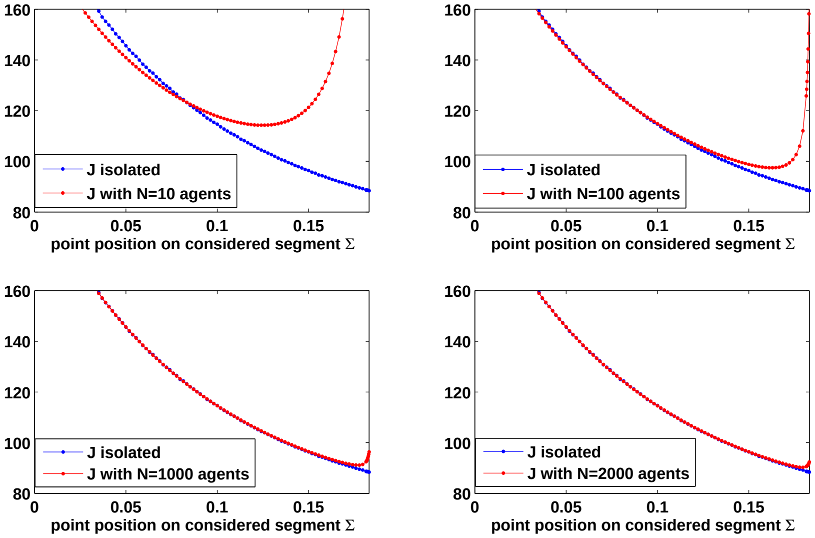

Figure 17: Isolated cost and cost of each agent for different values of N on the points of segment Σ.

values of N. Furthermore, we study the convergence rate with respect to N when approaching the stability border. In particular, consider a couple of gains (K, f); let $\epsilon > 0$ be a small positive number, and also let N_{min} be the number of agents such that $|J^{(S)}(K, f) - J^{(N)}(K, f)| < \epsilon$ for all $N > N_{\text{min}}$, where under (K, f), $J^{(N)}$ denotes the cost of each agent in a population of N elements and $J^{(S)}$ is the isolated cost (see 59). As shown in Figure 18, it is observed that the minimum value of N such that $|J^{(S)}(K, f) - J^{(N)}(K, f)| < \epsilon$ is rapidly increasing as one approaches the stability border. In essence, these numerical results confirm: (i) the infimum cooperative cost is on the stability border; (ii) the slowing down of convergence of actual cost with N agents to the isolated cost within the stability region, when N goes to infinity, as one approaches the edge of the stability region.

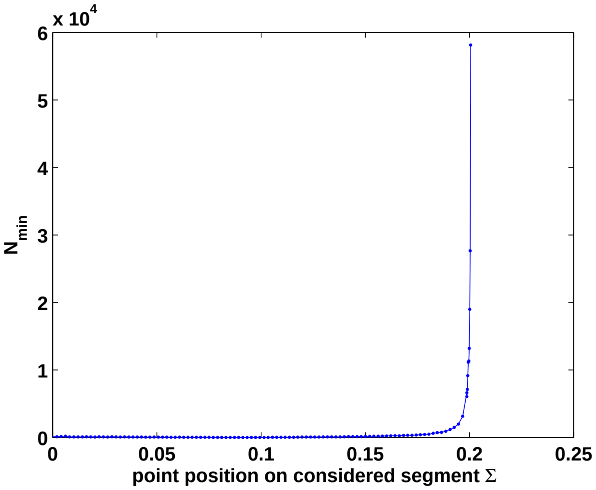

Figure 18: Minimum number of agents N such that $|J^{(S)} - J^{(N)}| < \epsilon$ (with $\epsilon = 1$), along segment Σ.

6 Application to wireless communications

In this section, we present an application example for decentralized power control in code division multiple access (CDMA) cellular telephone systems [3, 12, 19, 28].

Following [34, 35], we consider a mobile system network in the context of a large number of users with a signal processing gain assumed to be proportional to $1/N$. Let $p^{(m)}_{k,i}$ and $\alpha_{k,i}$ denote, respectively, the transmitted power and the mean squared value of the uplink channel gain for the i^{th} mobile user of the network and let $p^{(b)}_{k,i}$ denote the received power at the base station for user i, where $p^{(b)}_{k,i} = \alpha_{k,i} p^{(m)}_{k,i}$. Based on the work in [28], we model the received power dynamics at the base station by

$$p^{(b)}_{k+1,i} = p^{(b)}_{k,i} + u_{k,i} + w_{k,i} \tag{86}$$

with observations given by:

$$y_{k,i} = p^{(b)}_{k,i} + \frac{h}{N} \sum_{j \neq i}^{N} p^{(b)}_{k,j} + \sigma^2_{th} + v_{k,i}, \tag{87}$$

which is the average over slot k of the power of the CDMA signal despread by the spreading sequence of user i, where σ^2_{th} is the variance of the background thermal noise process (modeled as a zero mean Gaussian random variable). Note that the resulting signal processing gain is assumed to be h/N.

The goal is to perform decentralized power control in order to design the control command $d_{dB_{k,i}}$ which updates the transmitted power (on the logarithmic scale, i.e., $d_{dB_{k,i}} = 10 \log_{10}(d_{k,i})$, for its ease of implementation in practical systems) according to

$$p^{(m)}_{dB_{k+1,i}} = p^{(m)}_{dB_{k,i}} + d_{dB_{k,i}}, \tag{88}$$

so that the signal to interference plus noise ratio of each user achieves a target value γ (common to all users), i.e., $SINR_{k,i} = \gamma$. This is feasible at minimum power if the received powers for all users are equal to $\bar{p}^*$ given by [28]

$$\bar{p}^* = \frac{\gamma \sigma_{th}^2}{1 - \gamma h(N-1)/N} \cdot \tag{89}$$

Therefore, we minimize the individual cost function for each user as defined by

$$J_i \triangleq \overline{\lim_{T\to\infty}} \frac{1}{T} \mathbb{E} \sum_{k=0}^{T-1} \{(p_{k,i}^{(b)} - \bar{p}^*)^2 + r u_{k,i}^2\} \cdot \tag{90}$$

By applying change of variables

$$x_{k,i} = p_{k,i}^{(b)} - \bar{p}^*, \quad y'_{k,i} = y_{k,i} - (1 + h - \frac{h}{N})\bar{p}^* \tag{91}$$

to (86), (87) and (90), we obtain a system in the form of (1), (2) and (5) with $a = b = 1$, $c = 1 - \frac{h}{N}$ and $v'_{k,i} = \sigma_{th}^2 + v_{k,i}$. Note that we formulated this problem on the linear scale and that it is trivial to derive a relative change in watts into a change in decibels.

6.1 Numerical results VII

In this section, we consider a network with $N = 100$ users which are assumed to be equally spaced on a circle around the base station. Choosing $\gamma = 0.95$, $\sigma_{th}^2 = 1$ and $h = 1$ in (89) yields $\bar{p}^* = 15.9664$. We apply Algorithm 1 with $a = b = 1$, $c = 0.99$ and the pair $(K^* = 0.5017, a_f = 0.8)$ within the stability region. Reverse engineering the cost structure (5) for parameter $r > 0$, we get $r = 20$. Figure 19 shows the power tracking error $x_{k,i} = p_{k,i}^{(b)} - \bar{p}^*$ for a representative user (namely, the 50th mobile user). Moreover, Figure 20 illustrates the transmitted power $p_{dB_{k,50}}^{(m)}$ with $\alpha_{50} = 0.7$.

7 Conclusion

We have studied a class of interference induced games in a system of uniform agents coupled via their distinct sets of partial observations, whereby each agent has noisy measurements of its own state. We have shown that interference coupled agents can afford to act non cooperatively provided their individual stability level as characterized by a quantity called a_{Nash}, is sufficient relative to the signal to noise ratio in their observations and the number of agents is sufficiently high. Moreover, there is a lack of stability threshold past which, the only choice left for the majority of agents is to act cooperatively. The apparent role of individual agent

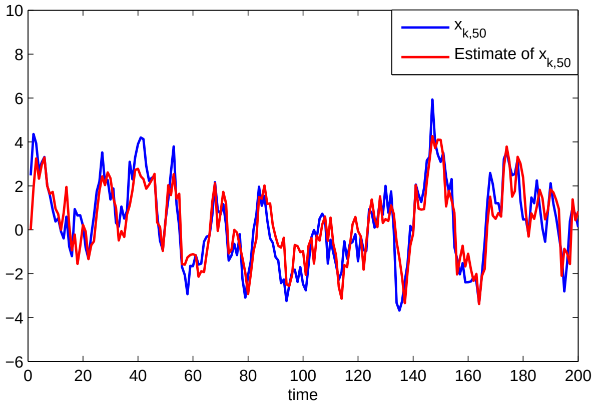

Figure 19: The power tracking error $x_{k,i} = p_{k,i}^{(b)} - \bar{p}^*$ and its estimate when $r = 20$, $\bar{p}^* = 15.9664$, $N = 100$, $i = 50$.

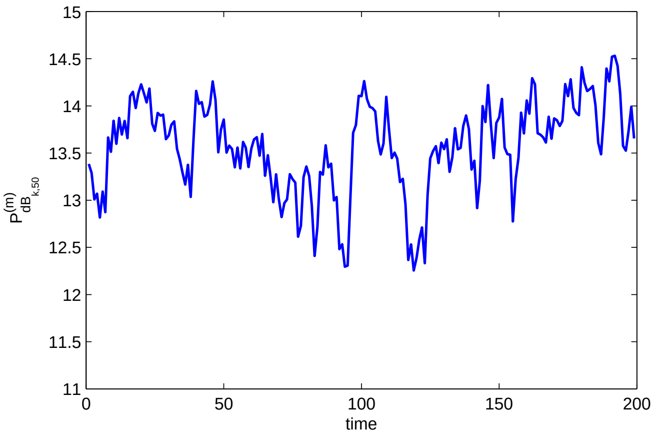

Figure 20: Transmitted power $p^{(m)}_{dB_{k,i}}$ in decibels when $\alpha_i = 0.7$, $i = 50$, in a group of 100 mobile users.

systems lack of stability in interference coupled systems, points at a potential hitherto unsuspected role of instability in more classical mean field games where the mean agent state enters individual dynamics [16].

In future work, we will investigate the use of optimal (growing dimension) estimators to improve the situation. We will also generalize our analysis to the multivariate case. Finally, we note that the finite number of agents version of the game constitutes a challenging problem in signalling and decentralization.

Appendix

Proof of Proposition 1. First of all, $f^*(a,r) = ab\Sigma/(b^2\Sigma + r)$, where Σ is the positive solution of (50). It is straightforward to see that, with $a > 0$, we have $f^*(a,r) > 0$ and $f^*(a,r) < a/b$. This implies also that, if $f^*(a,r)$ is stabilizing, then $0 < a - bf^*(a,r) < 1$.

Based on (48), the fact that $r(f)$ must be positive and taking into account that $a - bf^*(a,r) > 0$, we have that the possible range of $f^*(a,r)$ is $(0, +\infty)$ if $a \in (0,1]$, and $((a^2-1)/(ab), +\infty)$ if $a > 1$.

Finally, if $f^*(a,r)$ is stabilizing the population, then $f^*(a,r) \in (f_{inf}(a), f_{\sup}(a))$, where it is easy to verify that: $f_{inf}(a) \leq 0$ if $a \in (0,1)$ (since $f = 0$ is stabilizing given that with $f = 0$ and $a \in (0,1)$, Eqs. (32)–(35) are satisfied) and that, according to (45), $f_{inf}(a) = (a-1)/b < (a^2-1)/(ab)$ if $a > 1$.

Putting together all the above constraints, we can conclude that $(K^*(a), f^*(a,r)) \in S(a)$ implies that $f^*(a,r) \in (\bar{f}_{inf}(a), \bar{f}_{sup}(a))$, where $\bar{f}_{inf}(a) = 0$ if $a \in (0,1]$ and $\bar{f}_{inf}(a) = (a^2-1)/(ab)$ if $a \geq 1$. Notice that $f_{inf}(a) \leq \bar{f}_{inf}(a)$ for all a.

Now, for all $f \in (\bar{f}_{inf}(a), \bar{f}_{sup}(a))$, the derivative of $r(f)$ with respect to f which is given by:

$$\frac{\partial r}{\partial f} = \frac{ab[(a-bf)^2 - 1]}{f^2[a(a-bf)-1]^2}\,, \tag{92}$$

is negative. This happens as a result of the fact that the interval $(\bar{f}_{inf}(a), \bar{f}_{sup}(a))$ is included in the stability interval $(f_{inf}(a), f_{\sup}(a))$ where $(a-bf)^2 < 1$.

Notice also that $r(\bar{f}_{inf}(a)) = +\infty$. So the values of $r(f)$, for f ranging in $(\bar{f}_{inf}(a), \bar{f}_{sup}(a))$, will decrease from $+\infty$ down to the value $r_{\inf}(a)$ given in (81). Furthermore, there is a one to one map between f and the corresponding r.

Finally, in view of Lemma 6, for $a \in (0, a_{Nash})$, individuals applying optimal isolated policies will stabilize for the whole range of r from 0 to infinity (since in this case $\bar{f}_{sup}(a) = a/b$ and hence $r_{\inf}(a) = 0$). For a past a_{Nash}, $r_{\inf}(a)$ is strictly positive, since $\bar{f}_{sup}(a) = f_{\sup}(a) < a/b$ where r is still positive.

As a keeps on increasing past a_{Nash}, at some point we have $\bar{f}_{sup}(a) \equiv \bar{f}_{inf}(a)$. This can occur only if $a > 1$, since, for $a \in (0, 1]$, $\bar{f}_{sup}(a) > 0$ and $\bar{f}_{inf}(a) \leq 0$. On the other hand, if $a > 1$, $\bar{f}_{sup}(a) = \bar{f}_{inf}(a)$ if and only if a is such that $(c+h)K^*(a) - 1 = \frac{1}{a}(hK^*(a) + 1)$. Following a procedure similar to the one adopted in the proof of Lemma 6, it is possible to show that the value of $a > 1$ such that $(c+h)K^*(a) - 1 = \frac{1}{a}(hK^*(a)+1)$ uniquely exists and that, denoting $a_{\sup}$ this value, we have $a_{\sup} > a_{Nash}$.

Notice also that, for $a > a_{Nash}$, $\bar{f}_{sup}(a) = f_{\sup}(a)$ and that, for $a > 1$, $\bar{f}_{inf}(a) = (a^2 - 1)/(ab)$. For this reason, $a_{\sup}$ is the value of a such that $f_{\sup}(a) = (a^2 - 1)/(ab)$ which yields (82).

For $a > a_{\sup}$, $\bar{f}_{sup}(a) = f_{\sup}(a) < (a^2 - 1)/(ab) = \bar{f}_{inf}(a)$, i.e., $(K^*(a), f^*(a, r)) \notin S(a)$ for all r. This concludes the proof. □

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