Panchromatic Chains and Paths

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Abstract

A generalization of the Roy-Gallai theorem on the chromatic number of a graph is derived which is also an extension of several other results of Berge and of Li. A simple inductive proof is given which provides a direct way of deriving the theorem of Li.

We also show that any k-chromatic graph contains at least k! chains meeting every color exactly once.

Keywords:
Graph coloring, chromatic number, longest path, panchromatic chains, stable sets.

Résumé

On présente une généralisation du théorème de Roy-Gallai sur le nombre chromatique d'un graphe qui étend également d'autres résultats de Berge et de Li. On donne une preuve inductive simple qui permet d'obtenir d'une manière directe le théorème de Li. On montre également que tout graphe k-chromatique contient au moins k! chaînes rencontrant chaque couleur exactement une fois.

Mots-clés :
Coloration de graphes, nombre chromatique, plus long chemin, chaîne panchromatique, ensembles stables.

1. Introduction

Bounds on the chromatic number of a graph have been obtained by using various coloring techniques like sequential coloring of nodes or construction of circuit-free orientations of edges.

In this note we exploit further the relation between orientations and colorings and we generalize the results in (Li 1998) and in (de Werra, Hansen, 2001); in addition we provide a simple inductive proof of a theorem of (Li 1998). From these results, the Roy-Gallai theorem (Roy 1967, Gallai 1968) as well as some generalizations of (Berge 1982) can be derived.

For all graph theoretical terms not defined here the reader is referred to (Berge 1973).

2. Preliminary results

Recall that a graph G = (V, E) is simple if it has neither multiple edges nor loops. V (or V(G)) is a finite set of nodes and E (or E(G)) is a family of edges [v, w] (unordered pairs). A graph G = (V, U) is oriented if U consists of a family of ordered pairs (v, w), called arcs.

A set S of nodes of G is stable if no two nodes in S are linked by an edge (or an arc).

A k-coloring of G is a partition of its node set V into k stable sets $S_1, S_2, \ldots, S_k$; this amounts to assigning to each node v a color c(v) in {1,…,k} in such a way that no two adjacent nodes have the same color.

It is not difficult to see that a simple graph G admits a k-coloring iff its edges can be oriented in such a way that the resulting graph G’ has no (oriented) circuit and no path in G’ has more than k nodes.

Exploiting this connection between colorings and orientations, B. Roy and T. Gallai obtained independently the following result for oriented graphs where circuits may be present:

Theorem 1 - (Roy 1967, Gallai 1968): *If in an oriented graph G, every elementary path has at most k nodes, then G has a k-coloring.*

Let us recall the definition of the chromatic number of G denoted by $\chi(G)$): it is the smallest k for which G has a k-coloring. If s(P) is the number of nodes on a path P, theorem 1 can be stated as follows:

$$\chi(G) \leq \max \left\{ s(P) \mid P \text{ is an elementary path in } G \right\}.$$

In (Li 1998) a direct extension of theorem 1 was given by considering the root of a graph G; the root is a node r such that for any other node w of G there is a path from r to w. Calling R(G) the set of roots of G (which may be empty) and defining

p(r) = max $\{s(P) \mid P$ is an elementary path in G starting at node r $\}$, one can state:

Theorem 2 - (Li 1998): *For any oriented graph G = (V, U)*

$$\chi(G) \leq min \left(|V|, \min\left(p(v) \mid v \in R(G)\right)\right).$$

It is obvious that theorem 2 implies theorem 1 for all graphs such that R(G) ≠ Ø. In the same paper, Li obtains another result which can be formulated as follows:

Theorem 3 - (Li 1998): *Let G be a connected graph and let k = χ (G); then for any k-coloring of G and for any node v there exists an elementary chain starting at v which meets all k colors.*

This result generalizes a result for undirected graphs conjectured by the automated system Graffiti of (Fajtlowicz 1988) and proved by that autor (Fajtlowicz 1999).

However, as mentioned in (Li 1998), theorem 3 cannot be applied to oriented graphs (a cycle C_5 on 5 nodes with an adequate orientation of its edges provides a counter-example).

In (de Werra, Hansen 2001) a generalization of theorem 2 was derived by using the concept of basis of an oriented graph; it is a subset B of nodes of G such that:

a) for every node w not in B, there is some node b in B from which an elementary path reaches w;

b) no two nodes in B are linked by a path.

Observe that a basis B is a stable set; unlike roots, bases always exist.

Let us define for a stable set S the number p(S) by

$$p(S) = \max \left\{ p(b) : b \text{ is a node of } S \right\}.$$

Then the following result can be stated where a stable set is called basic if it contains some basis of G.

Theorem 4 - (de Werra, Hansen 2001): *For any oriented graph G*

$$\chi(G) \leq min \left\{ p(S) \mid S \text{ is a basic stable set} \right\}.$$

Clearly theorem 4 implies theorem 2 (and theorem 1). It also implies a result of (Berge 1982) as shown in (de Werra, Hansen 2001), i.e., if k is the largest number of nodes in a path of an oriented graph G and if P is a path with k nodes, there is a k-coloring of G such that P meets every color (exactly once).

For the unoriented case, we may define q(S) as the maximum number of nodes in an elementary chain starting at some node in S and having no other node in S. Then one gets:

Corollary 5 - (de Werra, Hansen 2001): *For any simple graph G*

$$\chi(G) \leq \min \left\{ q(S) \mid S \text{ is a stable set} \right\}.$$

In the next section we shall derive a result which is a generalization of all unoriented versions of the above theorems.

3. Panchromatic chains

Let us now state a direct extension of theorem 3 which is similar to theorem 4 and corollary 5 in the sense that it involves stable sets as well. Recall that a k-critical graph G is such that $\chi(G) = k$, but removal of any node v gives a graph $G' = G - v$ with $\chi(G') \leq k - 1$.

In a graph G with $\chi(G) = k$ and with a k-coloring $(S_1,\ldots,S_k)$ a set C of nodes (or a partial graph defined on C) will be called panchromatic if $C \cap S_i \neq \emptyset$ for i = 1,…,k.

Theorem 6: *Let G be a connected graph and let k be an integer with $k \geq \chi(G)$; let furthermore H be a $\chi(G)$-critical subgraph of G and $(S_1,S_2,\ldots,S_k)$ a k-coloring of G.*

Then for any subset of nodes $S \subseteq S_1$, there is an elementary panchromatic chain C which starts at some node of S, has no other node in S and meets at least $\chi(G)$ colors.

In addition after entering into H, chain C remains inside H.

Corollary 7: *Let G be a connected graph with $\chi(G) = k$ and $(S_1,\ldots S_k)$ a k-coloring of G. Then for any $S \subseteq S_1$, there is an elementary panchromatic chain C starting in S and such that $|C \cap S| = 1$.*

It is clear that theorem 6 implies theorem 3: take a subset S with |S| = 1 and $k = \chi(G)$; it also implies corollary 5 since the chain C is allowed to have only its starting node in the stable set S, in the same way as in corollary 5.

Proof of theorem 6: We shall give a proof by induction on the number n of nodes and the chromatic number $\chi(G)$ of G. The result is trivial for a connected G with 2 nodes and with $\chi(G) = 2$.

Let us assume that the result is true for graphs with at most n-1 nodes and chromatic number at most p-1.

Let now G = (V, E) be a graph with |V| = n nodes and chromatic number $\chi(G) = p \le k$; $(S_1,\dots,S_k)$ is a k-coloring of G. Consider now a $\chi(G)$-critical subgraph H of G; notice that $\chi(H) = p$ and H is connected.

Let $S \subseteq S_1$ and let $S^H = S \cap V(H)$. Let V* be a set of nodes such that the subgraph H' = H – $(S_1 \cup V^*)$ obtained by removing the nodes of $S_1 \cup V^*$ from H, has $\chi(H') = p-1$. Then H' (which is possibly not connected) has some connected component, say H'', with $\chi(H'') = p-1$ (otherwise we would have $\chi(H') < p-1$). Since $S_1 \neq \emptyset$, we have $|V(H'')| \le |V(H')| \le n-1$; so the property holds for H'', i.e., by choosing a subset S = $\{v''\}$ there exists from any node v'' in $V(H'')$ an elementary chain C'' which meets at least p-1 of the colors 2, 3,…,k and which remains inside H'' and hence inside H.

A) Assume now that $S^H \neq \emptyset$. Let us choose some node v in S^H; since H is connected there exists an elementary chain between node v and any node in H''. Let us consider such a chain $\hat{C}$; we follow its edges from v in S^H and interrupt it as soon as we reach a node v_o'' in H''. All intermediate nodes of such a $\hat{C}$ are in $V(H) - V(H'')$ ($\hat{C}$ remains in H); now following the chain $\hat{C}$ in reverse order from v_o'' we follow its edges until we meet a first node in S^H; let w be this node. Such a w exists since v is in S^H; we remove all edges of $\hat{C}$ between v and w. The remaining chain $\overline{C}$ from w to v_o'' can be extended by a chain from v_o'' in H'' which meets at least p-1 of the colors 2, 3,…k. The resulting chain C meets at least p colors (since its first node w is in $S^H \subseteq S_1$); it remains inside H and it is elementary. Finally $V(C) \cap S^H = \{w\}$.

B) Assume that $S^H = \emptyset$, i.e., $S \subseteq S_1 - V(H)$. Since G is connected there is an elementary chain from an arbitrary node v in S to some arbitrary node u in H; we interrupt such a chain at the first node which is in H. As before we may destroy from the beginning of the chain all edges which are traversed before we leave S for the last time. Let C_1 be the resulting chain from a node $\overline{v}$ in S to a node $\overline{u}$ in H.

Now if $\overline{u}$ is in $V(H'')$, we are done: there is by the induction assumption an elementary chain C_2 in H'' from $\overline{u}$ which meets at least p-1 of the colors 2, 3,…,k. Concatenation of C_1 and C_2 gives the required chain of G.

Finally assume $\bar{u}$ is in V(H) – V(H''); now since H is connected, there is an elementary chain (inside H) from $\bar{u}$ to some arbitrary node v'' in V(H''); this chain from $\bar{u}$ can be interrupted at the first node $\bar{v}''$ which is in V(H''); then, as before, it can be extended by a chain from $\bar{v}''$ which remains inside H'' and meets at least p-1 of the colors 2, 3,…,k. Combining this chain C_2 with the chain C_1 from $\bar{v}$ to $\bar{u}$, we get the required chain C in G; it satisfies $V(C) \cap S = \{u\}$: since $S_H = Ø$, when the chain enters into H, it remains inside and it does not meet any node of S other than v. ■

Remark 1: The above inductive proof (where k = $\chi(G)$, H is omitted and S chosen with |S| = 1) provides a simple way of proving theorem 3 directly; the resulting derivation appears to be simpler than the original proof of (Li 1998).

Remark 2: The statement of theorem 6 is best possible in the following sense: if S is not monochromatic, the statement is not true:
consider a cycle C_7 on 7 nodes a, b, c, d, e, f, g; take S_1 = {a, c, e}, S_2 = {b, d, f}, S_3 = {g} and S = {b, e, g}. Then there is no elementary chain out of S which meets all colors and has only its first node in S.

In addition, one cannot choose arbitrarily the node v in S from which the chain C is constructed: take the same coloring of the graph C_7 and S = S_1, then there is no chain out of node c in S which meets all colors. ■

Finally we mention another direct consequence of the above result:

Corollary 8 - (de Werra, Hansen 2001): *Let G = (V, E) be a connected simple graph and let k = $\chi(G)$. Let H be a k critical subgraph of G. Then for any k-coloring of G and for any subset $S \subseteq V$, there is a collection $\mathcal{C}$ of |S| node disjoint chains starting at nodes in S, having no other nodes in S and meeting together all k colors in H. Furthermore if a chain of $\mathcal{C}$ enters into H it remains inside.*

4. Stars and flutes

The above results may also be viewed in a different perspective related to the concept of diameter of a graph; it is by definition the maximum number of edges in a shortest chain between two nodes of the graph.

We have shown in the previous sections that for any k-coloring (S_1,…,S_k) with $k = \chi(G)$ there exists a chain C with $|C \cap S_i| \geq 1$ for i = 1,…,k. In G a chain may be viewed as a partial connected subgraph H with the largest possible diameter (for a given number of |C| - 1 edges).

On the other hand we may ask whether there exists a connected partial subgraph H with a small diameter such that $\left|V(H) \cap S_i\right| \geq 1$ for i = 1,…,k where V(H) is the node set of H.

The following statement can be formulated in this direction.

Proposition 9: *Let (S_1,…,S_k) be any k-coloring of a graph G with* $\chi(G) = k \geq 2$. *Then there is a connected subgraph H of G with diameter at most 2 which meets every color class exactly once.*

Proof: Consider the nodes v in S_k; there is at least one node, say $\bar{v}$, such that its set N(v) of neighbors satisfies $N(v) \cap S_i \neq \emptyset$ for i = 1,…,k-1. Indeed, assume that for each such v there is a color i(v) $\leq$ k-1 with $N(v) \cap S_{i(v)} = \emptyset$, then v can be given color i(v) and we get a (k-1)-coloring of G, which is impossible.

Hence node $\bar{v}$ in S_k has one neighbor v(i) in S_i for i = 1,…,k-1.

Then $\bar{v}$, v(1),…,v(k-1) induce a subgraph H of G with diameter at most 2 (a star) and such that $\left|V(H) \cap S_i\right| = 1$ for i = 1,…,k. ■

Remark 3: Observe that Proposition 9 is not true if k > χ(G): a chain P_4 on 4 nodes colored with 1, 2, 3, 4 is a counter example. ■

At this stage, we would like to mention a consequence of the equivalence between colorings and circuit-free orientations.

In a graph G with $\chi(G) \leq k$, given a k-coloring (S_1,…,S_k), a <u>flute</u> F will be a chain such that $\left|F \cap S_i\right| \leq 1$ for i = 1,…,k. So a panchromatic flute meets all k colors exactly once. The <u>size</u> $\left|F\right|$ of a flute F will be the number of its nodes.

Let us now consider a k-coloring of a graph G with $\chi(G) = \omega(G) = k$ where $\omega(G)$ is the maximum size of a clique in G. Then all k colors occur in a maximum clique of G, so we have the following property:

for any k-coloring of G with $\chi(G) = k$ and for any permutation $a_1,a_2,\ldots,a_k$ of the colors 1,2,…,k there exists a panchromatic flute which meets colors $a_1,a_2,\ldots,a_k$ in this order.

It turns out that this property holds also for arbitrary graphs G since there exists a k-coloring if and only if there exists a circuit-free orientation of the edges such that the longest path has at most k nodes: when $k = \chi(G)$ a longest oriented path in G gives a panchromatic flute meeting consecutively colors 1,2,…,k.

More generally, we can state

Proposition 10: *Let G be a graph and* $\chi(G)$ *its chromatic number. Let k be an integer with* $k \geq \chi(G)$. *For any k-coloring of G there is at least one flute F with* $|F| \geq \chi(G)$ *meeting its colors in increasing order.*

Proof: We construct the orientation associated to the k-coloring (edge [u,v] becomes arc (u,v) if the colors c(u), c(v) of its end nodes satisfy c(u) < c(v)).

In the oriented graph G, there is no circuit and there is at least one path P with at least $\chi(G)$ nodes; indeed if there would be no such paths, G could be colored with less than $\chi(G)$ colors, a contradiction.

Now path P is a flute which meets at least $\chi(G)$ colors in increasing order. ■

For the case k = $\chi(G)$, we have the following:

Corollary 11 *For every graph G with* $\chi(G) = k$, *any k-coloring and any permutation* $a_1, a_2, \ldots a_k$ *of the colors, there exists a panchromatic flute meeting colors* $a_1, a_2, \ldots a_k$ *in that order. Therefore G contains at least k! panchromatic flutes.*

Corollary 11 is indeed somewhat surprising, it implies for instance the following: for any optimal k-coloring (k = $\chi(G)$) we may choose an ordered subset of colors, say $a_1, a_2, \ldots a_r$, and we know that there will be a flute F with $|F|$ = r meeting colors $a_1, a_2, \ldots a_r$ consecutively.

This property does of course not hold when k > $\chi(G)$: for a chain on three nodes colored with 1, 2, 3 there is no flute F with $|F|$ = 2 meeting consecutively colors 1 and 3.

A panchromatic flute in such a graph G is a partial graph H with diameter k-1 which meets all colors exactly once. Proposition 9 showed the existence of an H with diameter 2. More generally we have the following.

Proposition 12: *Given a graph G with* $\chi(G) = k$, *an integer d* $(2 \leq d \leq k-1)$ *and a k-coloring* $(S_1, \ldots, S_k)$, *there exists a partial graph H with diameter d such that* $|H \cap S_i| = 1$ *for i = 1,…,k.*

Proof: Consider the nodes $v \in S_k$; some of them are the end nodes of panchromatic flutes.

At least one of these endnodes v is the center of a panchromatic set $N[v] = \{v\} \cup N(v)$.

Indeed, if all endnodes of panchromatic flutes in S_k had some color missing in their neighbor set, they could be recolored with a smaller color.

The associated oriented graph has then no more paths with k nodes. So it can be recolored with fewer thank k colors, a contradiction.

Now the panchromatic flute F ending at v gives us a partial H with diameter k-1.

By removing consecutively the edges of F from the initial nodes and introducing edges $[v, w]$ of $N[v]$ with colors 1,2,…, consecutively, we get partial graphs with diameter decreasing from k-1 to 2. ■

For claw free graphs we can give a more specific statement on the existence of a panchromatic flute.

A graph is <u>claw-free</u> if it does not contain a complete bipartite graph $K_{1,3}$ as an induced subgraph. $N[v]$ will again denote the set $\{v\} \cup N(v)$ where N(v) is the set of neighbors of v.

For such graphs we can state

Proposition 13: *Let G be a claw-free graph with chromatic number* $\chi(G)$ *; then for any k-coloring (*$S_1, \ldots, S_k$*) with* $k \geq \chi(G)$ *and for any node v, the set N[v] contains a flute meeting all* colors present in N[v].

Proof: Consider a node v and assume $\bigcup \{ i \mid S_i \cap N(v) \neq \emptyset \} = \{1,\ldots,r\}$; Let v_i be a node of $N(v) \cap S_i$ for i = 1,…,r.

Claim: Let Q be a connected component of N(v); then there is a chain meeting all nodes of Q.

Proof of the claim: The result is trivial if Q has at most 3 nodes. Let Q have at least 4 nodes and assume the result is not true. Let R be a chain in Q meeting the largest possible number of nodes of Q. There is at least one node, say $\bar{v}$, which is not met by R. Since Q is connected, we may even assume that $\bar{v}$ is at distance 1 from R, i.e., there is an edge $[\bar{v}, u]$ where u is in the chain. Observe that u cannot be the first nor the last node of R since otherwise $R \cup [\bar{v}, u]$ would form a chain meeting more nodes of Q than R. So let a and b be the first and the last nodes of R; clearly a and $\bar{v}$ (resp. b and $\bar{v}$) are not linked. Hence a and b must be linked, otherwise v, a, b, $\bar{v}$ induce a claw in G. But then starting at $\bar{v}$ and traversing edge $[\bar{v}, u]$ and the subchain R(u, b) of R between u and b, followed by [b, a] and the subchain $R(a, u^*)$ of R between a and u^* (the predecessor of u in R), we get a chain meeting more nodes of Q than R. This is not possible. Hence there exists in Q a chain which visits all nodes of Q. This ends the proof of the claim.

Now since G is claw-free, the set $\{v_1,...,v_r\} \subset N(v)$ has at most two connected components.

If there is one, say Q_1, there exists from the claim a chain R_o meeting all nodes of Q_1; then C is obtained from R_o by going from its last node b to node v by following [b, v].
If there are two components Q_1, Q_2, let b_1 (resp. a_2) be the last (resp. first) node of a chain R_1 (resp. R_2) meeting all nodes of Q_1 (resp. Q_2). Then C is obtained by following consecutively R_1, $[b_1, v]$, $[v, a_2]$, R_2. ■

Corollary 14: *If G is a claw-free graph with chromatic number $\chi(G)$ and $(S_1,\ldots,S_k)$ a k-coloring with $k = \chi(G)$, then there exists a node v such that N[v] contains a panchromatic flute.*

Proof: This follows from Proposition 13 and Proposition 9.

As a conclusion, we mention a conjecture which would generalize previous results.

Conjecture: *Given a connected graph with a k-coloring $(S_1,\ldots,S_k)$ for some $k \geq \chi(G)$ and a node v of G, there exists an elementary chain C = $(v_o, v_1,..., v_q)$ such that v_o = v and the last $\chi(G)$ nodes form a flute.*

The conjecture is true for $\chi(G)$ = 3 as shown below.

Proposition 15: *Let G be a connected graph with $\chi(G) = 3$, $(S_1,\ldots,S_k)$ a k-coloring for some $k \geq 3$ and v a node of G. There exists an elementary chain starting at v and such that the last three nodes have different colors.*

Proof: G contains a 3-critical subgraph H (which is an odd cycle). Assume first $v \notin H$ then there exists an elementary chain C from v to some node v' of H containing no other node of H.

If $v \in H$, we set v = v'. If H is a triangle, we are done since its nodes have 3 different colors and for any choice of v' we can extend the chain as required to meet all three colors.

Let H be an odd cycle with at least 5 nodes; then for any k-coloring with $k \geq 3$ of H there are at least two triples (no necessarily disjoint) of consecutive nodes with different colors. A node v' of H cannot be the middle point of two triples, so we will always be able to extend the chain around H from v' and meet a triple of consecutive nodes with all different colors. ■

We also ask the following question:

Does there exist for any k-coloring of a graph G with chromatic number $\chi(G) \leq k$, a node v for which $N[v] = \{v\} \cup N(v)$ meets at least $\chi(G)$ colors?
The answer is positive for graphs G with $\chi(G) \leq 3$ according to Proposition 14 and also for k = $\chi(G)$ as shown in Proposition 9.

However in general it is negative. Ivo Bloechliger at EPFL has constructed a graph G with 30 nodes and with $\chi(G)$ = 4. It has a 5-coloring such that N[v] contains 3 colors for each node v.

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