Consensus Decision Process: Models, Theory and Experimental Verification

(Revised and Extended)

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Abstract

A management decision based on voting by a team of experts has been commonly used and has been examined by the social scientists. Various approaches have been considered for optimizing the probability of obtaining a correct consensus decision while minimizing that of making an erroneous one. We have derived a mathematical model which independently proves and extends the results of Condorcet to a group of individuals with different abilities in making correct decisions and avoiding risks in making wrong decisions. Our derivations include some quantitative conditions based on which we would be able to choose optimally a team of experts from a pool of those with diverse capabilities. One would also tend to attach different weights to different individuals so as to attain the best possible results. However, we have found that if the difference in abilities among those experts is not so diverse, the assignment of equal weighting to all the individuals seems to be locally maximal. Thus we have to assign a significantly uneven weight-distribution to such a group for a substantial improvement. As a consequence, a thorough investigation in some basic procedures in augmenting weights has been carried out. More crucial results so obtained together with extensive experimental verifications lead to a much better understanding of the consensus decision process as well as the possibility of constructing a combinatorial algorithm which would give the optimal enhancement in those cases where this method is useful and acceptable.

Résumé

La décision de gestion basée sur le vote par une équipe d'experts a été utilisée couramment et étudiée par des chercheurs en sciences sociales. Différentes approches ont été considérées afin d'optimiser la probabilité d'obtenir une décision correcte par consensus tout en minimisant le risque d'une décision erronée. Nous avons créé un modèle mathématique indépendant qui vérifie et étend les résultats de Condorcet à un groupe d'individus montrant diverses aptitudes à décider correctement et à éviter les risques de faire de mauvaises décisions. Nos résultats incluent quelques conditions quantitatives sur lesquelles nous nous basons pour choisir optimalement une équipe d'experts parmi un large groupe possédant diverses aptitudes. On aurait tendance à attacher un poids différent à chaque individu afin d'obtenir un meilleur résultat. Toutefois, nous avons trouvé que les différences d'aptitudes des experts ne sont pas si grandes et que l'assignation de poids égal à tous semble être localement maximale. Nous devons donc assigner une distribution de poids très inégale à ce groupe afin d'obtenir une amélioration significative. Conséquemment, une analyse en profondeur des procédures de base de l'augmentation des poids a été entreprise. Les résultats obtenus avec les vérifications extensives par expérimentation mènent à une meilleure compréhension du processus de décision consensus et donnent la possibilité de construire un algorithme combinatoire qui apporterait une amélioration maximale dans les cas où cette méthode est applicable.

1 Introduction

The consensus decision process by voting was studied by de Condorcet (1785) based on binomial probabilistic concept. Since then, it has been considered by many social scientists and mathematicians (Grofman, and Owen (1986), Pincus (1990), Tataru and Merlin (1997), Lam and Suen (1997), Lepelley and Gehrlerin (2000)). Normally, they simplified the model by assuming that all the persons in the group has approximately equal ability for making correct decisions, or working on a normally distributed mean of the individual abilities within a group. In this paper, an entirely different approach has been used. A complete set of results based on Condorcet's model is first obtained. This includes a formula which gives an estimate of the improvement for adding more persons to a group. In Section 3, a group of $2n + 1$ experts having different probabilities $p_1, p_2, \ldots, p_{2n+1}$ of making correct decision is considered. Using a combinatorial–probabilistic argument, the above results have been successfully generalized with the mild assumption that if for any two new experts added with their respective probabilities $\widetilde{p}_1$ and $\widetilde{p}_2$ for making correct decision satisfying

$$\left(\frac{\widetilde{p}_1}{1-\widetilde{p}_1}\right)\left(\frac{\widetilde{p}_2}{1-\widetilde{p}_2}\right) > \frac{p_i}{1-p_i} \quad \text{for all } i = 1, 2, \ldots, 2n+1,$$

then the probability for making correct consensus decision will definitely increase. Furthermore, an upper estimate of the improvement for continuously adding experts to this group can be obtained readily from the results in section one.

Finally, a less democratic but more practical approach is considered. That is, we enhance the probability of consensus correct decision making by attaching different weights to different individuals. This certainly makes sense, if those experts are not human.

However, it has been found that the attachment of equal weights to all individuals behaves like a local maximum. (This is not too surprising as the voting process is discrete.) On further investigation into the effectiveness of adding unit weights to a few individuals and some special cases of large redistribution of weights, one would conclude that a simple procedure or a combination of some simple basic procedures is all that is required to obtain a close to optimal solution in most cases.

Several experiments are designed and conducted with a large number of randomly generated examples. The results obtained are to be compared with all our theoretical conclusions for verifications.

2 Consensus Decision for a Group of Individual with Equal Ability of Making Correct Decisions

Let $P(n, p)$ be the probability of making a correct consensus decision for a group of n individuals each with probability p for making correct decision. Since the process is binomial, we have

$$P(n,p) = \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

where

$$m = \begin{cases} \frac{n}{2}+1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We shall prove the following proposition.

Proposition 1 *If n is a positive integer ≥ 1, then $P(2n+1,p) - P(2n,p) = p^{n+1}$ $(1-p)^n \binom{2n}{n}$, and $P(2n,p) - P(2n-1,p) = -p^n(1-p)^n \binom{2n-1}{n}$.*

Proof

$$\begin{aligned} P(2n+1,p) &= \sum_{k=n+1}^{2n+1} p^k (1-p)^{2n+1-k} \binom{2n+1}{k} \\ &= \sum_{k=n+1}^{2n+1} p^k (1-p)^{2n+1-k} \left[\binom{2n}{k} + \binom{2n}{k-1} \right] \\ &= (1-p) \sum_{k=n+1}^{2n+1} p^k (1-p)^{2n-k} \binom{2n}{k} \\ &\qquad + p \sum_{k=n+1}^{2n+1} p^{k-1} (1-p)^{2n-(k-1)} \binom{2n}{k-1} \\ &= (1-p) \sum_{k=n+1}^{2n} p^k (1-p)^{2n-k} \binom{2n}{k} + p \sum_{k=n}^{2n} p^k (1-p)^{2n-k} \binom{2n}{k-1} \\ &\qquad \left(\text{since} \binom{2n}{2n+1} = 0 \right) \\ &= \sum_{k=n+1}^{2n} p^k (1-p)^{2n-k} \binom{2n}{k} + p^{n+1} (1-p)^n \binom{2n}{n} \\ &= P(2n,p) + p^{n+1} (1-p)^{2n-k} \binom{2n}{n} \qquad (1) \end{aligned}$$

Similarly, we obtain,

$$P(2n,p) = P(2n-1,p) + p^n (1-p)^n \binom{2n-1}{n} \qquad (2)$$

Hence we have the following:

Corollary 1

$$P(2n+1,p) - P(2n-1,p) = p^n(1-p)^n \binom{2n-1}{n}(2p-1) \tag{3}$$

Proof From Proposition 1,

$$\begin{aligned}
P(2n+1,p) - P(2n-1,p) &= p^{n+1}(1-p)^n\binom{2n}{n} + P(2n,p) \\
&\quad -\left[P(2n,p) + p^n(1-p)^n\binom{2n-1}{n}\right] \\
&= p^n(1-p)^n\left\{p\binom{2n}{n} - \binom{2n-1}{n}\right\} \\
&= p^n(1-p)^n\left\{p\left[\binom{2n-1}{n} + \binom{2n-1}{n-1}\right] - \binom{2n-1}{n}\right\} \\
&= p^n(1-p)^n\left\{2p\binom{2n-1}{n} - \binom{2n-1}{n}\right\} \\
&\qquad \left(\text{since } \binom{2n-1}{n} = \binom{2n-1}{n-1}\right) \\
&= p^n(1-p)^n(2p-1)\binom{2n-1}{n}
\end{aligned}$$

□

Corollary 2 $P(2n+2,p) - P(2n,p) = p^{n+1}(1-p)^n\binom{2n}{n}\left(\frac{2np+p-n}{n+1}\right)$.

Proof of this corollary is similar to that of Corollary 1.

From above, we have the following:

1. If $p > 0.5$, then $P(2n+1,p) > P(2n-1,p)$ and $P(2n+2,p) > P(2n,p)$ for all positive integer n. That is, both $P(2n+1,p)$ and $P(2n,p)$ are monotonically increasing with n, (since $2p-1 > 0$). Furthermore, $P(2n+1,p) \to 1$ as $n \to \infty$. On the other hand, if $p < 0.5$, than both $P(2n+1,p)$ and $P(2n,p)$ are monotonically decreasing with n, and $P(2n+1,p) \to 0$ as $n \to \infty$.
2. Observe that $P(2n+1,p) > P(2n,p)$ and $P(2n-1,p) > P(2n,p)$ for all n and p, this means that a group of odd number of individuals is always better in ability in making correct consensus decision than those group with even number of individuals.

Now, for all integer $n \geq 2$

$$\begin{aligned}
\frac{P(2n+1,p)-P(2n-1,p)}{P(2n-1,p)-P(2n-3,p)} &= \frac{P^n(1-p)^n\binom{2n-1}{n}(2p-1)}{p^{n-1}(1-p)^{n-1}\binom{2n-3}{n-1}(2p-1)} \\
&= p(1-p)\frac{(2n-1)!}{n!(n-1)!} * \frac{(n-1)!(n-2)!}{(2n-3)!} \\
&= 2p(1-p)\left(\frac{2n-1}{n}\right)\ldots \qquad (4)
\end{aligned}$$

Hence,

$$\left[\frac{P(2n+1,p)-p(2n-1,p)}{P(2n-1,p)-P(2n-3,p)}\right]\left[\frac{P(2n-1,p)-P(2n-3,p)}{P(2n-3,p)-P(2n-5,p)}\right]\ldots\left[\frac{P(5,p)-P(3,p)}{P(3,p)-P(1,p)}\right]$$

$$= 2^{n-1}p^{n-1}\frac{(2n-1)(2n-3)\ldots 3}{n(n-1)\ldots 1}.$$

Thus

$$\frac{P(2n+1,p)-P(2n-1,p)}{P(3,p)-P(1,p)} = 2^{n-1}p^{n-1}(1-p)^{n-1}\frac{(2n-1)(2n-3)\ldots 5.3}{n!}\ldots \qquad (5)$$

This gives an estimate of the improvement for adding more individuals to a group.

3 Consensus Decision for a Group with Different Abilities for Correct Decision

In reality, not everybody in committee has equal ability in making correct decision. We shall see if one can obtain similar results for these more general and realistic cases.

Let us consider the case of adding two additional individuals to a group (or committee) consists of $2n+1$ individuals. The following two types marginal cases will be affected:

(i) Close to win cases: n individuals say "yes" to a correct decision, while the other $n+1$ individuals say "no" (or otherwise). The total number of possible cases is ${}^{2n+1}C_n$.

(ii) Close to loss cases: $n+1$ individual say "yes", while the other say "no" (or otherwise). The total number of possible cases is ${}^{2n+1}C_{n+1} = {}^{2n+1}C_n$.

It is clear that if one of these 2 additional experts says "yes", while the other say "no", then there is no effect what so ever on all cases in both (i) and (ii). However, if these two experts both say "yes", then those cases in (i) become winning cases, whereas if both of them say "no", the all those cases in (ii) become losing cases. Let us consider a matching of marginal cases in (i) with those in (ii) induced by identifying a case in (i) with the case in (ii) where the grouping for "yes" and "no" is the same, except for one expert (say i) which is swapped from "no" to "yes" in (ii). This matching is one to one and onto (as the total numbers of possible cases in (i) and (ii) are equal).

Let $x_i, i = 1,\ 2,\ 3,\ \dots\ ,\ 2n+1$ be the random variable defined by

$x_i = \begin{cases} 1 & \text{if } i \text{ says "yes"} \\ 0 & \text{if } i \text{ says "no"} \end{cases}$, and let $A = \{i : x_i = 1\}$ and $B = \{j : x_j = 0\}$.

Now, it is easily seen that the total change in probability ΔP of making correct decision is given by

$$\begin{aligned} \Delta P &= \widetilde{p}_1\widetilde{p}_2 \sum \Big(\prod_{k\in A} p_k\Big)\Big(\prod_{j\in B}(1-p_j)\Big) \\ &\quad -(1-\widetilde{p}_1)(1-\widetilde{p}_2)\sum\Big(\prod_{i\in A} p_i\Big)\Big(\prod_{j\in B}(1-p_j)\Big) \end{aligned} \tag{6}$$

where, the first term of the right hand side of the above expression, the product Π is taken over all k such that $x_k = 1$ and $\sum x_k = n+1$ and the summation taken over all such possible cases. Similarly, in the second term, the product Π is taken over all k such that $x_k = 1$ and $\sum x_k = n$, and the second product Π is taken over all j such that $x_j = 0$ and $\sum x_j = 0$. The summation $\sum$ is taken all such possible cases.

In view of the matching between cases in (i) and those in (ii).

$$\Delta P = \sum_i \Big\{ \Big[\widetilde{p}_1\widetilde{p}_2(1-p_i) - (1-\widetilde{p}_1)(1-\widetilde{p}_2)p_i\Big] \sum\Big(\prod_{k\neq i} p_k\Big)\Big(\prod_{j\neq i}(1-p_j)\Big)\Big\} \tag{7}$$

Hence, the change ΔP is positive, i.e. there is increase in probability of making correct consensus decision if

$$\widetilde{p}_1\widetilde{p}_2(1-p_i) - (1-\widetilde{p}_1)(1-\widetilde{p}_2)p_i > 0 \text{ for all } i = 1,2,\dots,2n+1$$

or equivalently

$$\Big(\frac{\widetilde{p}_1}{1-\widetilde{p}_1}\Big)\Big(\frac{\widetilde{p}_2}{1-\widetilde{p}_2}\Big) > \frac{p_i}{1-p_i} \text{ for all } i = 1,2,\dots,2n+1. \tag{8}$$

Notice that the condition (8) is easily satisfied, except when the values of $\widetilde{p}_1$ and $\widetilde{p}_2$ are low in comparison with those of the original group.

e.g. suppose $\widetilde{p}_1 = \widetilde{p}_2 = 0.6$ and $p_i \geq 0.75$ for all i, then

$$\Big(\frac{\widetilde{p}_1}{1-\widetilde{p}_1}\Big)\Big(\frac{\widetilde{p}_2}{1-\widetilde{p}_2}\Big) = \Big(\frac{0.6}{0.4}\Big)^2 = 2.25 < 3 = \frac{0.75}{0.25} \leq \frac{p_i}{1-p_i}.$$

Let there be a sequence of individuals $\{a_k\}$ whose probabilities for making correct decision are given by $\{p_k\}$. Define $\widetilde{p}_n = \max\{p_k : 1 \leq k \leq 2n+1\}$, the expression $B_n = \sum_{k=n}^{2n} {}^{2n+1}C_{k+1}\widetilde{p}_n^{\,n+1}(1-\widetilde{p}_n)^{2n-k}$ is readily seen to be an upper bound for the probability of making correct consensus decision for a group of $2n+1$ individuals whose probabilities for correct decision are given by $p_1, p_2, \ldots, p_{2n+1}$. Clearly a lower bound L_n can be constructed in the similar manner, i.e. $L_n = \sum_{k=n}^{2n} {}^{2n+1}C_{n+1}\varrho_n^{k+1}(1-\varrho_n)^{2n-k}$ where $\varrho_n = \min\{p_k : 1 \leq k \leq 2n+1\}$.

Now, we have the following generalization of Proposition 1.

Proposition 2 *If for all positive integer k, such that $1 < k \leq n$*

$$\left(\frac{p_{2k+2}}{1-p_{2k+2}}\right)\left(\frac{p_{2k+3}}{1-p_{2k+3}}\right) > \frac{p_i}{1-p_i} \quad \textit{for all } i, \quad 1 \leq i \leq 2k+1 \tag{9}$$

then the probability P_n for correct consensus decision for this group is monotonic increasing with n and P_n tends to 1, as n tends to ∞ provided $\varrho = \lim_{n\to\infty} \sup \varrho_i > 0.5$.

Proof The assertion (9) follows from (8). For the last assertion, since $1 \geq P_n \geq L_n = \sum_{k=n}^{2n} {}^{2n+1}C_{k+1}\varrho_n^{k+1}(1-\varrho_n)^{2n-k}$ which tends to 1 as n tends to infinity, this implies that P_n tends to 1 as n tends to ∞.

Following the same line of argument as the previous section, one should see that the following assertions are true.

1. Adding a single individual to a group (or committee) of $2n+1$ individuals brings no improvement in probability of making correct consensus decisions. As the close to winning cases will not be affected.

2. The addition of one expert to a group of $2n$ expert will bring about definite improvement. If we look at the marginal cases; the close to win cases are those with n of the group say "yes" while the other n experts say "no" which incidentally are also all the close to loss cases. The addition of the one who says "yes" will turn each of these cases into a winning case, while the one added who says "no" will turn each of these cases into a losing one. Since the P value of the expert is suppose to be greater than 0.5, an increase in probability in making correct consensus decision is always guaranteed.

3. Suppose $\widetilde{p}_1', \widetilde{p}_2'$ and $\widetilde{p}_i'$ are the respective probabilities for making wrong decision for those individuals concerned, using similar argument, for minimizing consensus error.

The condition is described by the same inequality in (5) but with the sign reversed. That is

$$\left(\frac{\widetilde{p}'_1}{1-\widetilde{p}'_1}\right)\left(\frac{\widetilde{p}'_2}{1-\widetilde{p}'_2}\right) < \frac{p'_i}{1-p'_i} \qquad \text{for all } i = 1, 2, \ldots, 2n+1. \tag{10}$$

One might have to consider (5) and (6) separately, since for each i, $p_i + p'_i + p''_i = 1$, where p''_i is the probability for indecision.

4 Consensus Decision a Group of Persons with Diverse Abilities

In order to increase the probability of making correct consensus decision, an attempt was made to put some weighting on each individual in accordance with their abilities (i.e. their probabilities p_i's of making correct decision). This is by no means a democratic process. It applies to a decision group where there are leaders or a group of individuals which are not human (e.g. some methods or devices). A natural model is given below:

$$\text{(M)} \quad \begin{array}{ll} \text{Maximize} \displaystyle\sum_{i,j} \prod_{i\in A} p_i \prod_{j\in B} (1-p_j) & \begin{array}{l} \text{where } A = \{i : x_i = 1\} \\ \qquad B = \{j : x_j = 0\} \text{ and} \\ x_i\text{'s are the random variables} \\ \text{described in the previous section} \end{array} \\ \text{Subject to} \displaystyle\sum_{i=1}^{2n+1} w_i x_i \geq n+1 & \\ \displaystyle\sum_{i=1}^{2n+1} w_i = 2n+1 & \end{array}$$

We observe that the even distribution with $w_1 = w_2 = \ldots w_{2n+1} = 1$ is a local maximum. For if we wish to change all the marginal winning cases in which n individuals say "yes", while the other $n+1$ say "no" or otherwise to a truly winning case, we must increase the total amount of weighting of some individuals who say "yes" by 1 unit. Since $\sum_{i=1}^{2n+1} w_i = 2n+1$, this requires that the total reduction of weight from some other individuals who say "no" must also be equal to one unit. Hence we must have

$$|w_1 - 1| + |w_2 - 1| + \cdots + |w_n - 1| + \cdots + |w_{2n+1} - 1| \geq 2 \tag{11}$$

Thus, in the neighborhood $\sum_{i=1}^{2n+1} |w_i - 1| < 2$, no increase in probability in making correct consensus decision is possible. However, this is not surprising, as the voting process is discrete in nature.

Further, investigation shows that even in the boundary case, where a redistribution of weight such that a subgroup of less than n individuals has a total net gain of one unit weight, the improvement is not likely.

Let us consider a group consisting of $2n+1$ individuals. Suppose one additional unit is distributed to k individuals while a total of one unit weight is reduced from r individuals. Clearly, we must have $k \leq n$ and $r \leq n+1$.

The marginal (close to win) case is having k individuals whose weights are to be augmented, all included in the "yes" group while those r persons whose weights are to be reduced, all included in the group of $n+1$ individuals who say "no". The total number of cases is given by

$$^{2n+1-r-k}C_{n-k} = \frac{(2n+1-r-k)!}{(n-k)!(n+1-r)!} \tag{12}$$

On the other hand, the total number of marginal (close to loss) cases are given by

$$\sum_{j+k' \leq n+1} \frac{(2n+1-r-k)!}{(n+1-j-k')![n-(r-j)-(k-k')]!} + \frac{(2n+1-r-k)!}{(n+1-r)!(n-k)!} \tag{13}$$

Here, the summation in the first term of the above expression is taken over all subgroups of j individuals belonging to the group of r individuals whose weighting are to be reduced and all subgroups of k' individuals belonging to the group whose weighting are to be augmented such that the total weight gain by those k' individuals is less than the total weight loss by those j individuals. The last term refers to the cases where all those k individuals with weight gain are in the "no" group and all those r individuals with weight reduced are in the group of $n+1$ individuals who say "yes".

Observe that the number of terms under this summation turns out to be large, this means that the total number of the (close to loss) cases is much larger than that of the (close to win) cases, irrespective of the distribution of weighting is even or uneven. Extensive calculation of change in probabilities of making correct consensus decision in a number of cases has been performed. It is found that the improvement in all cases is unlikely. Detail working shown in the following extreme cases illustrates the general methodology and the validity of our argument.

Case 1a. $k = 1$, $3 \leq r \leq n$

The marginal (close to win) cases are those cases with the i^{th} individual whose weight to be increased in the "yes" group of n individuals while these r individuals whose weights are to be reduced all in the "no" group.

The total number of such cases is given by $^{2n-r}C_{n-1}$.

One group of those marginal (close to loss) cases are those with i in the "no" group while any two (say j and k) of those r individuals swapped to the "yes" side as illustrated below.

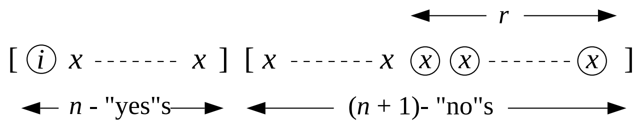


Figure 1

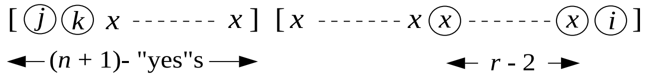


Figure 2

The total number of cases is given by ${}^{r}C_2 \cdot {}^{2n-r}C_{(n+1)-2} = {}^{r}C_2 \cdot {}^{2n-r}C_{n-1}$.

by computing the change in probability as in the previous section, $\Big($cf (7) & (8)$\Big)$, we must have

$$\left(\frac{p_i}{1-p_i}\right) > \sum_{1\le j,k\le r} \left(\frac{p_j}{1-p_j}\right)\left(\frac{p_k}{1-p_k}\right) \tag{14}$$

in order that the total increase in probability by those (close to win) marginal cases as a result of weight increase is not being off set by the decrease in probability in those (close to loss) cases mentioned above. This condition is very difficult to fulfill. Besides, there are still some more marginal (close to loss) cases including the one where i says ”no” while all those in the r group say ”yes” not yet being considered.

Case 1b. $k = 1$ and $r = 2$ (say i and j, k)

The marginal (close to win) cases are those in 1a) with the total number of cases equal to ${}^{2n-2}C_{n-1}$ while the (close to loss) cases are those with i swapped to the “no” side, while either both k and j being swapped to the “yes” side or either one of them (j or k) together with one (say λ) from the remaining individuals. Hence, by using the similar argument as before. We see that the probability of correct consensus decision will be increased if

$$\left(\frac{p_i}{1-p_i}\right) > \left(\frac{p_j}{1-p_j}\right)\left(\frac{p_k}{1-p_k}\right) + \sum_{\lambda\neq j,k}\left[\left(\frac{p_j}{1-p_j}\right)\left(\frac{p_\lambda}{1-p_\lambda}\right) + \left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\lambda}{1-p_\lambda}\right)\right] \tag{15}$$

is satisfied. However, this is by no means easy, as all the p_n's are greater than 0.5.

Case 2. $2 \le k \le n$ and $r = 2$.

Clearly, the marginal (close to win) cases are those with all k individuals on the “yes” side, while those two individual (i and j) whose weights to be reduced are on the “no” side as shown in the following:

The total number of cases is ${}^{2n-k-1}C_{n-k}$.

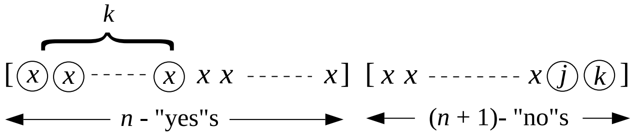


Figure 3

Now, one sort of the marginal (close to loss) cases are those with one of those k individuals (say i) swapped to the “no” side, while both j and k being swapped to the “yes” side as shown below:

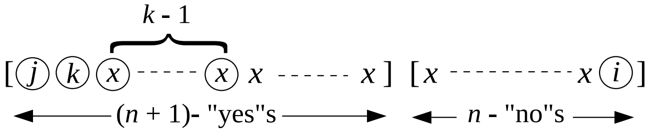


Figure 4

Thus, the total number of those cases is $k \cdot {}^{2n-k-1}C_{(n+1)-2-(k-1)} = k \, {}^{2n-k-1}C_{n-k}$. Similarly, we obtain, the condition for increase in probability of correct consensus decision is possible only if

$$\left(\frac{1-p_j}{p_j}\right)\left(\frac{1-p_k}{p_k}\right) > \sum_{i=1}^{k}\left(\frac{1-p_i}{p_i}\right) \tag{16}$$

which is very difficult to fulfill especially for large k, since $\left(\frac{1-p_m}{p_m}\right) < 1$ for all m. Besides, there are other marginal (close to loss) cases not yet being considered.

Case 3. $k = n$ and $r = n + 1$.

There is one and only one marginal (close to win) case, which is the one with n-individual whose weights are to be augmented all say “yes” while the $n+1$ individuals whose weights are to be reduced all say “no”.

The obvious marginal (close to loss) case is that with $k(= n)$ of those all swapped to the “no” side, while all the $n + 1$ individuals mentioned above all swapped to the “yes” side. Besides this there are numerous marginal (close to loss) cases including those with $(n - 1)$ individuals whose weights are to be augmented swapped to the “no” side while $(n - 1)$ of those whose weights are to be reduced swapped to the “yes” side.

Unless the weight redistribution is very uneven on both group, there are quite a number of such marginal (close to loss) cases. Moreover, for an uneven redistribution, one can keep swapping those individuals with greatest weight reduction with those with greatest weight gain, one will soon arrive at a situation where the net gain in weight on the “yes” side will become negative. Continuing the swapping will produce more marginal (close to loss) cases. Thus, in all cases, there will be many marginal (close to loss) cases which may off-set the increase in probability of making correct consensus decision by a single marginal (close to win) case.

From above, one concludes that the allocation of equal weighting to all individuals (i.e., $p_1 = p_2 = \cdots = p_{2n+1}$) behaves like a local maximum. Any small redistribution of weight will not improve the probability of making correct consensus decision of a group especially when the individual abilities for making correct decision do not vary too much among the members of this group.

5 Analysis of Some Basic Steps of Attaching Different Weights

From the previous arguments, it seems that a sensible and economical way of enhancing the probability of making correct consensus decision can be derived by considering the following simple cases.

Case A. Adding 2 unit weights to a single individual (say i), while reducing a total weights of 2 units from three others. For simplicity, we shall assume that equal weights are reduced from each of them. Let us call those individuals whose weight are to be reduced the r-group.

I. The Marginal (close to win) cases are:

(i) i says "yes"
all in r-gp say "no"

(ii) i says "yes"
2 in r-gp say "no"

Total number of such cases are respectively given by:

$$^{(2n+1)-4}C_{n-1} = \frac{(2n-3)!}{(n-1)!(n-2)!} \qquad {}^3C_2 \cdot {}^{2n-3}C_{n-2} = 3 \cdot \frac{(2n-3)!}{(n-2)!(n-1)!}$$

II. The Marginal (close to loss) cases are:

(i) i says "no"
all in r-gp say "yes"

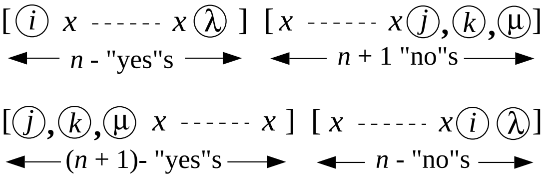
(ii) i says "no"
2 in r-gp say "yes"

(iii) i says "no"
1 in r-gp says "yes"

Total number of such cases are respectively given by:

$$^{(2n+1)-4}C_{(n+1)-3} = \frac{(2n-3)!}{(n-2)!(n-1)!} \qquad {}^3C_2 \cdot {}^{2n-3}C_{n-1} = 3 \cdot \frac{(2n-3)!}{(n-1)!(n-2)!} \qquad {}^3C_1\, {}^{2n-3}C_n = 3 \cdot \frac{(2n-3)!}{n!(n-3)!}$$

The matching between cases in I(i) and II(i) can be defined by swapping i together with one other individuals (say λ) to the "no" side, while swapping all three in the r-group from the "no" side to the "yes" side as illustrated below:



By considering the difference in probabilities between these two marginal cases as before, we see that there will be net gain in probability if

$$\left(\frac{p_i}{1-p_i}\right)\left(\frac{p_\lambda}{1-p_\lambda}\right) > \left(\frac{p_j}{1-p_j}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\mu}{1-p_\mu}\right) \tag{17}$$

for all $\lambda \neq i$, $\lambda \notin$ the r-group and for all j, k, μ in the r-group .

Again the matching between I(ii) and II(ii) is induced by swapping i to the "no" side, while k in r-group and another individual (say α) are being swapped back to the "yes" side. Thus net gain in probability will result, if

$$\left(\frac{p_i}{1-p_i}\right) > \left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\alpha}{1-p_\alpha}\right), \tag{18}$$

for all k in the r-group and for all $\alpha \neq i$.

Observe that there is also an injective mapping from II(iii) to I(ii) induced by swapping i from the "no" side to the "yes" side, while 2 other individuals (say β, γ) not belonging to the r-group from the "yes" side to the "no" side. Adding this to the above cases, we have, the sufficient condition for the net gain in probability for making correct consensus decision are (17) together with the following:

$$\left(\frac{p_i}{1-p_i}\right) > \left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\alpha}{1-p_\alpha}\right) + \left(\frac{p_\beta}{1-p_\beta}\right)\left(\frac{p_\gamma}{1-p_\gamma}\right) \tag{19}$$

for all k in the r-group and for all α, β, $\gamma \neq i$.

Normally, we reduce weights from individuals with the lowest p_k values, while adding them on the individuals with the highest p_i value. We must have $\frac{p_i}{1-p_i} > \frac{p_\alpha}{1-p_\alpha} > \frac{p_k}{1-p_k}$ for all $\alpha \neq i$ and for all k belongs to the r-group. Hence the condition

$$\left(\frac{p_i}{1-p_i}\right) > 2\left(\frac{p_\alpha}{1-p_\alpha}\right)\left(\frac{p_\beta}{1-p_\beta}\right) \tag{20}$$

clearly implies both condition (17) and (19).

The addition of 2 units of weights to a single individual, while reducing 2 units of weights from 4 others, has also been considered, similar results are obtained. It can be seen that the general case where these 2 units of weight are to be reduced from r individuals could be verified to have similar sufficient condition for improvement.

Case B. Adding 1 unit of weight to each of individuals (say i and j), while a total of 2 units of weight are to be reduced from 3 other individuals (μ, k, ℓ).

I. The Marginal (close to win) cases are:

(i) i and j say "yes"
all in r-gp say "no"

(ii) i and j say "yes"
1 in r-gp say "yes"

(iii) i or j says "yes"
all in r-gp say "no"

The respective total numbers of cases are:

${}^{(2n+1)-5}C_{(n-2)} = \frac{(2n-4)!}{(n-2)!(n-2)!}$

${}^{3}C_2 \cdot {}^{2n-4}C_{n-3} = 3 \cdot \frac{(2n-4)!}{(n-3)!(n-1)!}$

$2 \cdot {}^{2n-4}C_{(n+1)-4} = 2\frac{(2n-4)!}{(n-3)!(n-1)!}$

II. The Marginal (close to loss) cases are and their respective numbers of cases are:

(i) i and j say "no"
all in r-gp say "yes"

(ii) i and j say "no"
2 in r-gp say "yes"

(iii) i and j say "no"
1 in r-gp says "yes"

${}^{(2n+1)-5}C_{(n+1)-3} = \frac{(2n-4)!}{(n-2)!(n-2)!}$

${}^{3}C_2 \cdot {}^{2n-4}C_{(n+1)-2} = 3 \cdot \frac{(2n-4)!}{(n-1)!(n-3)!}$

${}^{3}C_1 \, {}^{2n-4}C_{(n+1)-1} = 3 \cdot \frac{(2n-4)!}{n!(n-4)!}$

(iv) i or j says "no"
all in r-gp say "yes"

(v) i or j says "no"
2 in r-gp say "yes"

$2 \cdot {}^{2n-4}C_{(n+1)-4} = 2 \cdot \frac{(2n-4)!}{(n-3)!(n-1)!}$

$2\,{}^{3}C_2 \cdot {}^{2n-4}C_{n-2} = 6 \cdot \frac{(2n-4)!}{(n-2)!(n-2)!}$.

Again, one can match the cases of I(i) with that of II(i) by swapping i, j together over the "no" side, while swapping all those in r-group over to the "yes" side. Hence one obtains the net gain in probability will result, if

$$\left(\frac{p_i}{1-p_i}\right)\left(\frac{p_j}{1-p_j}\right) > \left(\frac{p_\mu}{1-p_\mu}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\lambda}{1-p_\lambda}\right) \tag{21}$$

where μ, k and λ are all in the r-group .

Similarly, for cases I(ii) and II(ii), we obtain the condition for increase in probability for correct consensus decision to be

$$\left(\frac{p_i}{1-p_i}\right)\left(\frac{p_j}{1-p_j}\right) > \left(\frac{p_\mu}{1-p_\mu}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\alpha}{1-p_\alpha}\right) \tag{22}$$

for all μ, k in the r-group and for all $\alpha \neq i, j$ and α not in r-group.

Also, for cases I(iii) and II(iv), the sufficient condition is found to be

$$\left(\frac{p_\alpha}{1-p_\alpha}\right)\left(\frac{p_\beta}{1-p_\beta}\right) > \left(\frac{p_\mu}{1-p_\mu}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_i}{1-p_i}\right) \tag{23}$$

for all α, $\beta \neq i, j$ and not in r-gp, and μ, k, i all belong to the r-group.

This means that the p_i value of individuals belonging to the r-group have to be relatively small. One sees that these conditions are not difficult to fulfill. However, there are still two groups of marginal (close to loss) cases having no corresponding marginal (close to win) cases to counterbalance them. The number of these cases are fairly large $\left(\frac{3(2n-4)!}{n!(n-4)!} + \frac{6\cdot(2n-4)!}{(n-2)!(n-2)!}\right)$. Thus the enhancement in probability for making correct consensus decision in Case B needs not be easier. A similar conclusion is obtained with the case where 2 units that are added to i and j are reduced from 4 other individuals. Computation for the case of adding two units of weight to three individuals has been carried out and the results indicate there are less number of marginal (close to win) cases relative to the number of marginal (close to loss) cases which has increased significantly.

Summing up, the weight increase should be large (say at least more than 1 unit) and concentrated to a few individuals who are really capable for optimal effect. Of course it is understood that the corresponding reduction should only be taken from those individuals with weak ability for making correct decision.

Case C. Simply adding weights to a few particular individuals.

1. Adding 2 units of weights to an individual (say i). Clearly the number of marginal (close to win) cases is ${}^{(2n+1)-1}C_{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$, where i says "yes", and the number of (close to loss) cases is ${}^{(2n+1)-1}C_{(n+1)-2} = \frac{(2n)!}{(n+1)!(n-1)!}$ where i says "no". A natural matching between these two groups is induced by swapping i to the "no" side, while any 2 individuals (say j and k) are being swapped back to the "yes" as shown in the following:

   [ⓘ x ------ x] [x ------ x, ⓙ, ⓚ]
   ◄— n - "yes"s —► ◄—(n + 1)- "no"s —►

   [ⓙ ⓚ x ------ x] [x ------ x, ⓘ]
   ◄—(n + 1)- "yes"s —► ◄— n - "no"s —►

   Clearly, this matching is a bijection and the probability for making correct consensus decision will be increased if

$$\left(\frac{p_i}{1-p_i}\right) > \left(\frac{p_j}{1-p_j}\right)\left(\frac{p_k}{1-p_k}\right) \quad \text{for all } j,k \neq i\ . \tag{24}$$

2. Now, if we increase the weighting to i further (say to 4 units), the additional marginal (close to win) cases will be those with $(n-1)$-"yes" and $(n+2)$-"no"s and with i on the "yes" side, while the additional marginal (close to loss) cases will be those with $(n-1)$-"no's" and $(n+2)$-"yes" and with i saying "no". Their respective numbers of cases are both equal to ${}^{2n}C_{n-2}$. The mapping which induced by swapping i to

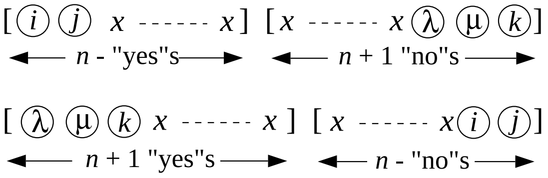
the "no" side, while swapping 4 other individuals (say j, k, ℓ, m) to the "yes" side is bijective. Hence, there will be a definite gain in probability if

$$\frac{p_i}{1-p_i} > \left(\frac{p_j}{1-p_j}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\ell}{1-p_\ell}\right)\left(\frac{p_m}{1-p_m}\right) \quad \text{for all } j,k,\ell,m \neq i \qquad (25)$$

Thus, it requires an individual with exceptional ability in making correct decision to ensure this additional gain. For example, if the other members of the group have an average value of $p = 0.75$, it requires that this particular individual must have $p = 0.9878$ or greater in order to achieve this gain. Since $\frac{p}{1-p} > \left(\frac{0.75}{1-0.75}\right)^4 = 3^4 = 81$ implies $p > \frac{81}{82}$. One might say, "A dictator's rule is never a good rule for making correct decision."

3. Adding 1 unit of weights to each of i and j (2 leaders!). No effect will be on the marginal cases if one says "yes" while the other says "no". The only marginal (close to win) cases are those with both i and j belonging to the group of n individuals who say "yes", while the marginal (close to loss) cases are those with i and j both on the "no" side. Clearly the total numbers of such cases are both equal to ${}^{2n+1-2}C_{n-2} = \frac{(2n-1)!}{(n-2)!(n+1)!}$.

   From the bijective correspondence illustrated below,



   We obtained as before the winning condition is

$$\left(\frac{p_i}{1-p_i}\right)\left(\frac{p_j}{1-p_j}\right) > \left(\frac{p_\lambda}{1-p_\lambda}\right)\left(\frac{p_\mu}{1-p_\mu}\right)\left(\frac{p_k}{1-p_k}\right) \quad \text{for all } \lambda,\mu,k \neq i,j \quad (26)$$

   which is less demanding than that given in (24). However, if we compare the number of marginal cases with that described in 1, their relative ratio is ${}^{2n-1}C_{n-2} \,/\, {}^{2n}C_{n-1} = \frac{n-1}{2n} < \frac{1}{2}$. This means that the number of cases is less than half of the previous. The net gain may even be less, although the condition for definite gain is less demanding.

4. Again, if we instead, add 2 units of weights to each of i and j. Using similar argument as before, the condition for gain in probability is given by:

$$\left(\frac{p_i}{1-p_i}\right)\left(\frac{p_j}{1-p_j}\right) > \left(\frac{p_\lambda}{1-p_\lambda}\right)\left(\frac{p_\mu}{1-p_\mu}\right)\left(\frac{p_k}{1-p_k}\right)\left(\frac{p_\ell}{1-p_\ell}\right) \qquad (27)$$

$$\text{for all } \lambda,\mu,k,\ell \neq i,j$$

which is again weaker than that described in Section 2. But the difference in number of marginal cases between this and that given in 2 is $\left({}^{2n}C_{n-1} + {}^{2n}C_{n-2}\right) - \left({}^{2n-1}C_{n-2} + {}^{2n-1}C_{n-2}\right) = {}^{2n-1}C_{n-1} + \left(\frac{5}{n+3}\right){}^{2n}C_{n-2}$ which is quite large. The net gain in probability for correct consensus decision is not high. This means that two leaders may not necessarily be better than one. However, the risk of making consensus erroneous decision is lower.

6 Experimental Verification

In order to verify those results obtained from our theoretical findings, the following tests on some randomly generated groups of decision makers with their respective values of p_is have been performed.

i). Generate a group of five decision makers (DM) whose ability p_i of making correct decisions (e.g. based on past records) ranging from 0.5 to 0.9. Compute the probability for correct consensus decisions and find enhancement in this probability, if two new members are added. Repeat experiment with various values of p_is for these two new members.

ii). Try small redistribution of weights on individual members to see if there is any improvement in the group decision ability.

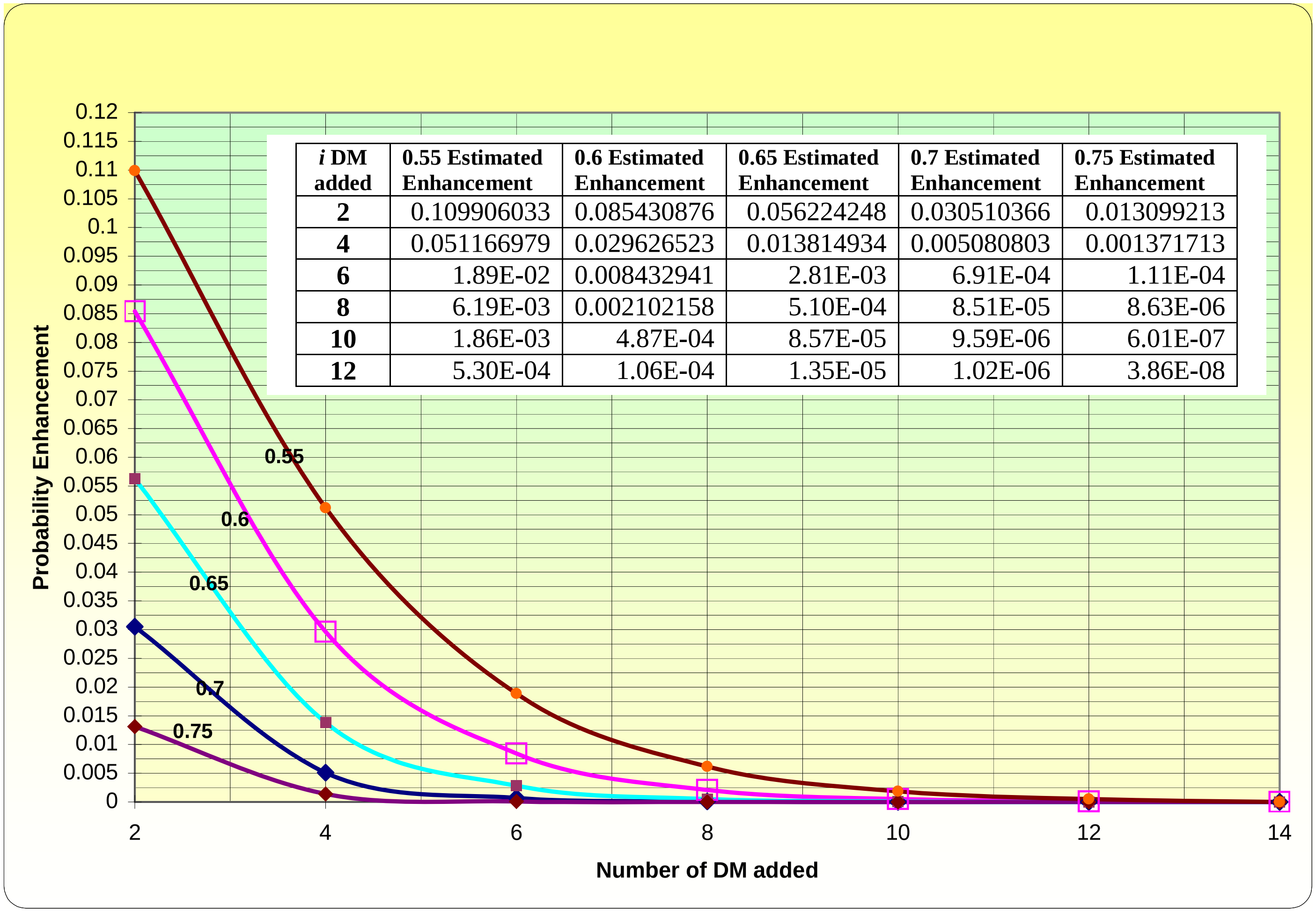
iii). Try larger redistribution of weights as guided by the examples (see case A) in our paper.

iv). Try add weights to a few (e.g. 1 to 2 in a group of 9) exceptional DM to see if there is any substantial improvement for the group.

All results in i) indicate that substantial enhancement in probability of correct decision making by voting as a group, and that the addition of two new members satisfying condition (8) of section 3 enhances the probability for making correct consensus decision. Moreover, the probability for consensus decision increases as the number of decision makers increases if the average value of their pi's is not less than 0.55 and the increase becomes insignificant if the group size reaches 13 (see Table 1). One could determine the optimal group size if he knows the average pi's of the decision makers.

In ii), various weight redistribution in the neighbourhood of $\sum |w_i - 1| \leq 2$ on groups of 7 and 9 with diverse abilities in making correct decision have been tried. It has been found that the one which gives the highest probability value is normally the one with equal weighting, (i.e. $w_1 = w_2 = \ldots = w_{2n+1}$) if the range of the values of p_is is small or if there is no decision maker with very high ability (say, higher than 0.8), this illustrates that any small redistribution of weight tends to decrease rather than enhance the group ability for making correct consensus decision.(See Appendixes Ia and Ib).

Results from iii), reaffirm that large redistribution of weights does not guarantee an enhancement in correct decision making probability. Those uneven weights distributions



i **DM added**	**0.55 Estimated Enhancement**	**0.6 Estimated Enhancement**	**0.65 Estimated Enhancement**	**0.7 Estimated Enhancement**	**0.75 Estimated Enhancement**
2	0.109906033	0.085430876	0.056224248	0.030510366	0.013099213
4	0.051166979	0.029626523	0.013814934	0.005080803	0.001371713
6	1.89E-02	0.008432941	2.81E-03	6.91E-04	1.11E-04
8	6.19E-03	0.002102158	5.10E-04	8.51E-05	8.63E-06
10	1.86E-03	4.87E-04	8.57E-05	9.59E-06	6.01E-07
12	5.30E-04	1.06E-04	1.35E-05	1.02E-06	3.86E-08

Table 1: Probability enhancement against increase in number of DM starting from a group of 5 DM *pi*'s 0.7, 0.75, 0.6, 0.65, 0.5, 0.55

which attain substantial probability enhancement are produced by a reduction of more than one unit of weights from a few weak individuals while adding them to one or two exceptional ones satisfying the required conditions. This further confirms our previous theoretical findings in case A. (See Appendixes IIa and IIb). Finally, results from iv) confirm that adding more units to one or two exceptional individuals enhances the probability of correct consensus decisions. It has been found that only those individuals with exceptional ability will be benefit and significant enhancement will be obtained only if the conditions (26) and (28) are satisfied. Also, there seems to be no advantage at all in doubling the weightings (say from 2 units to 4 units) added to the able individuals. (See Appendixes IIIa and IIIb).

7 Conclusion

Summarizing all our previous findings and analysis, we can conclude that consensus decision process by voting increases the chance of making correct decision, while reduces the risk of making erroneous ones, even if the members of a group may have different abilities in their decision makings. Independent of the size of the group, a group with odd number of individuals is more effective in consensus decision making.

In case the members of a group having diverse abilities in making correct consensus decisions, there is possible enhancement of the probability of making correct consensus decision by attaching different weights to difference individuals in accordance with their abilities. Even if that is the case, excess weighting allocated to those few individuals with exceptional abilities may have an adverse effect. On the other hand, if all the members of decision group do not differ so much in their abilities of correct decision making, equal weighting for all members generally gives an optimal arrangement, as we have demonstrated in Section 3 that $w_1 = w_2 = \cdots = w_{2n+1}$ behaves like a local optimal point.

Our detail study in the last section indicates that a simple procedure of allocating weights to individuals with diverse abilities in making correct decision is probably what is required for optimizing the decision making power of a decision team, whether the members are humans or devices. Most of these deductions have been verified with experiments on a variety of cases as reported in the previous section.

We believe that the results of our findings reported in this paper may help the management of an institution in choosing members of a decision group and in optimizing their functions. An efficient method for these purposes could be developed and implemented in computer as part of the management information system. An Internet version of the client program using the Java programming language is in progress.

Acknowledgments

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Appendix Ia – Change in Probability of Correct Consensus Decision with Small Redistribution of Weightings $\sum |wi-1| = 2$ for 7 D.M.s

P_i	0.6	0.63	0.669	0.71	0.73	0.78	0.8		
	Weight combination							**Probability of Correct Concensus Decision**	**Change in Probability**
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8804	No change
1	0.5	0.5	1.0	1.0	1.0	1.0	2.0	0.8509	-0.0295
2	0.67	0.67	0.67	1.0	1.0	1.0	2.0	0.8310	-0.0494
3	0.75	0.75	0.75	0.75	1.0	1.0	2.0	0.8207	-0.0597
4	0.8	0.8	0.8	0.8	0.8	1.0	2.0	0.8142	-0.0662
5	0.83	0.83	0.83	0.83	0.83	0.83	2.0	0.8142	-0.0662
6	0.5	0.5	1.0	1.0	1.0	1.5	1.5	0.8597	-0.0207
7	0.67	0.67	0.67	1.0	1.0	1.5	1.5	0.8124	-0.0680
8	0.75	0.75	0.75	0.75	1.0	1.5	1.5	0.8376	-0.0428
9	0.8	0.8	0.8	0.8	0.8	1.5	1.5	0.7560	-0.1244
10	0.5	0.5	1.0	1.0	1.33	1.33	1.33	0.8162	-0.0642
11	0.67	0.67	0.67	1.0	1.33	1.33	1.33	0.8311	-0.0493
12	0.75	0.75	0.75	0.75	1.33	1.33	1.33	0.8311	-0.0493
13	0.5	0.5	1.0	1.25	1.25	1.25	1.25	0.8436	-0.0368
14	0.67	0.67	0.67	1.25	1.25	1.25	1.25	0.7784	-0.1020
15	0.5	0.5	1.2	1.2	1.2	1.2	1.2	0.8436	-0.0368

Appendix Ib – Change in Probability of Correct Consensus Decision with Small Redistribution of Weightings $\sum |wi - 1| = 2$ for 9 D.M.s

P_i	0.605	0.625	0.653	0.675	0.7	0.725	0.75	0.775	0.795		
	Weight combination									**Probability of Correct Concensus Decision**	**Change in Probability**
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9039	No Change
1	0.5	0.5	1.0	1.0	1.0	1.0	1.0	1.0	2.0	0.8796	-0.0243
2	0.67	0.67	0.67	1.0	1.0	1.0	1.0	1.0	2.0	0.8637	-0.0402
3	0.75	0.75	0.75	0.75	1.0	1.0	1.0	1.0	2.0	0.8554	-0.0485
4	0.8	0.8	0.8	0.8	0.8	1.0	1.0	1.0	2.0	0.8515	-0.0524
5	0.83	0.83	0.83	0.83	0.83	0.83	1.0	1.0	2.0	0.8488	-0.0551
6	0.86	0.86	0.86	0.86	0.86	0.86	0.86	1.0	2.0	0.8498	-0.0541
7	0.875	0.875	0.875	0.875	0.875	0.875	0.875	0.875	2.0	0.8498	-0.0541
8	0.5	0.5	1.0	1.0	1.0	1.0	1.0	1.5	1.5	0.8858	-0.0181
9	0.67	0.67	0.67	1.0	1.0	1.0	1.0	1.5	1.5	0.8543	-0.0496
10	0.75	0.75	0.75	0.75	1.0	1.0	1.0	1.5	1.5	0.8718	-0.0321
11	0.8	0.8	0.8	0.8	0.8	1.0	1.0	1.5	1.5	0.8398	-0.0642
12	0.83	0.83	0.83	0.83	0.83	0.83	1.0	1.5	1.5	0.8041	-0.0998
13	0.86	0.86	0.86	0.86	0.86	0.86	0.86	1.5	1.5	0.8051	-0.0988
14	0.5	0.5	1.0	1.0	1.0	1.0	1.33	1.33	1.33	0.8527	-0.0512
15	0.67	0.67	0.67	1.0	1.0	1.0	1.33	1.33	1.33	0.8695	-0.0344
16	0.75	0.75	0.75	0.75	1.0	1.0	1.33	1.33	1.33	0.8489	-0.0551
17	0.8	0.8	0.8	0.8	0.8	1.0	1.33	1.33	1.33	0.8615	-0.0424
18	0.83	0.83	0.83	0.83	0.83	0.83	1.33	1.33	1.33	0.9174	0.0135
19	0.5	0.5	1.0	1.0	1.0	1.25	1.25	1.25	1.25	0.8709	-0.0330
20	0.67	0.67	0.67	1.0	1.0	1.25	1.25	1.25	1.25	0.8500	-0.0540
21	0.75	0.75	0.75	0.75	1.0	1.25	1.25	1.25	1.25	0.8698	-0.0341
22	0.8	0.8	0.8	0.8	0.8	1.25	1.25	1.25	1.25	0.8255	-0.0784
23	0.5	0.5	1.0	1.0	1.2	1.2	1.2	1.2	1.2	0.8568	-0.0471
24	0.67	0.67	0.67	1.0	1.2	1.2	1.2	1.2	1.2	0.8301	-0.0738
25	0.75	0.75	0.75	0.75	1.2	1.2	1.2	1.2	1.2	0.8681	-0.0358
26	0.5	0.5	1.0	1.17	1.17	1.17	1.17	1.17	1.17	0.8753	-0.0287
27	0.67	0.67	0.67	1.17	1.17	1.17	1.17	1.17	1.17	0.8301	-0.0738
28	0.5	0.5	1.14	1.14	1.14	1.14	1.14	1.14	1.14	0.8753	-0.0287

Appendix IIa – Change in Probability of Correct Consensus Decision with Large Redistribution of Weightings $\sum |wi - 1| > 2$ for 7 D.M.s

P_i	0.5	0.567	0.6333	0.71	0.772	0.833	0.9		
i	**Weight combination**							**Probability of Correct Concensus Decision**	**Change in Probability**
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8883	No change
1	0.33	0.33	0.33	1.0	1.0	1.0	3.0	0.9000	0.0117
2	0.5	0.5	0.5	0.5	1.0	1.0	3.0	0.9023	0.0140
3	0.6	0.6	0.6	0.6	0.6	1.0	3.0	0.8997	0.0113
4	0.67	0.67	0.67	0.67	0.67	0.67	3.0	0.8957	0.0074
5	0.25	0.25	0.25	0.25	1.0	1.0	4.0	0.9000	0.0117
6	0.4	0.4	0.4	0.4	0.4	1.0	4.0	0.9000	0.0117
7	0.5	0.5	0.5	0.5	0.5	0.5	4.0	0.9000	0.0117
8	0.2	0.2	0.2	0.2	0.2	1.0	5.0	0.9000	0.0117
9	0.33	0.33	0.33	0.33	0.33	0.33	5.0	0.9000	0.0117
10	0.17	0.17	0.17	0.17	0.17	0.17	6.0	0.9000	0.0117
11	0.33	0.33	0.33	1.0	1.0	2.0	2.0	0.8939	0.0057
12	0.5	0.5	0.5	0.5	1.0	2.0	2.0	0.9058	0.0175
13	0.6	0.6	0.6	0.6	0.6	2.0	2.0	0.8427	-0.0456 +
14	0.25	0.25	0.25	0.25	1.0	2.5	2.5	0.8990	0.0107
15	0.4	0.4	0.4	0.4	0.4	2.5	2.5	0.8427	-0.0456 +
16	0.2	0.2	0.2	0.2	0.2	3.0	3.0	0.7727	-0.1156 +
17	0.33	0.33	0.33	1.0	1.67	1.67	1.67	0.8893	0.0010
18	0.5	0.5	0.5	0.5	1.67	1.67	1.67	0.8695	-0.0188
19	0.25	0.25	0.25	0.25	2.0	2.0	2.0	0.9300	0.0417
20	0.33	0.33	0.33	1.5	1.5	1.5	1.5	0.8549	-0.0334

+ Here, the reduction of weights is uniformly over the remaining group of individuals.

Appendix IIb – Change in Probability of Correct Consensus Decision with Large Redistribution of Weightings $\sum |wi - 1| > 2$ for 9 D.M.s

	0.5	0.52	0.56	0.58	0.6	0.65	0.65	0.856	0.88		
	Weight combination									**Probability of Correct Concensus Decision**	**Change in Probability**
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8270	No change
1	0.6	0.6	0.6	0.6	0.6	1.75	1.75	1.75	1.75	0.8797	0.0527
2	0.49	0.51	0.67	0.68	0.81	1.03	1.49	1.51	1.81	0.8356	0.0086
3	0.51	0.56	0.573	0.67	0.87	0.95	1.302	1.202	2.363	0.8437	0.0167
4	0.5	0.5	0.5	0.5	0.5	0.5	1.0	2.5	2.5	0.8838	0.0568
5	0.46	0.57	0.68	0.78	1.0	1.0	1.0	1.465	2.045	0.8339	0.0069
6	0.56	0.56	0.56	0.56	0.56	1.35	1.54	1.6	1.71	0.8341	0.0071
7	0.25	0.25	0.25	0.25	1.6	1.6	1.6	1.6	1.6	0.8671	0.0401
8	0.44	0.51	0.635	0.778	0.883	0.9	1.16	1.8	1.894	0.8477	0.0207
9	0.33	0.345	0.39	0.4	0.41	1.1	1.7	1.925	2.4	0.8651	0.0381
10	0.6	0.6	0.6	0.6	0.6	1.0	1.0	2.0	2.0	0.8547	0.0277
11	0.25	0.25	0.25	0.25	1.0	1.75	1.75	1.75	1.75	0.8492	0.0222
12	0.2	0.69	0.7	1.0	1.0	1.0	1.09	1.52	1.8	0.8401	0.0131
13	0.2	0.3	0.6	0.65	0.67	0.68	1.76	2.0	2.14	0.8618	0.0348
14	0.44	0.467	0.48	0.49	0.52	1.3	1.5	1.65	2.153	0.8388	0.0117
15	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	7.0	0.8800	0.0530
16	0.5	0.52	0.66	0.67	0.68	0.685	1.23	2.0	2.055	0.8640	0.0369
17	0.678	0.699	0.72	0.741	0.762	0.783	0.804	1.803	2.01	0.8458	0.0188
18	0.5	0.5	0.5	0.5	1.0	1.0	1.0	2.0	2.0	0.8830	0.0559
19	0.2	0.39	0.45	0.555	0.61	0.736	1.65	2.1	2.309	0.8685	0.0415
20	0.25	0.25	0.25	0.25	1.0	1.0	1.0	2.5	2.5	0.8871	0.0600
21	0.4	0.4	0.4	0.4	0.4	1.0	1.0	2.5	2.5	0.8647	0.0377
22	0.62	0.64	0.673	0.756	0.812	0.856	0.9	1.126	2.617	0.8478	0.0208
23	0.31	0.4	0.41	0.5	0.51	1.31	1.5	2.0	2.06	0.8425	0.0155
24	0.6	0.6	0.6	0.6	0.6	1.75	1.75	1.75	1.75	0.8797	0.0527
25	0.456	0.461	0.542	0.567	0.615	0.658	1.69	2.0	2.011	0.8553	0.0283

Appendix IIIa – Probability Enhancement Resulting from Adding 2 or 4 Units of Weights to a Single Individual

For individual p_i	**Winning condition for adding 1 unit weight to** i	**Winning condition for adding2 unit weights to** i	**Initial probability**	**Enhancement from adding 2 units to an individual**	**Enhancement from adding 4 units to an individual**
For the group with set of P(correct) decision {0.55,0.5565, 0.56, 0.562, 0.6, 0.63, p_i***}***					
0.68	Not satisfy	Not satisfy	*0.6937*	0.1409	0.1448
0.78	satisfy	satisfy	*0.7229*	0.1616	0.1661
0.8	satisfy	satisfy	*0.7287*	0.1658	0.1703
0.82	satisfy	satisfy	*0.73457*	0.1700	0.1746
0.838	satisfy	satisfy	*0.7398*	0.1737	0.1784
0.86	satisfy	satisfy	*0.74624*	0.1782	0.1831
For the group with set of P(correct) decision {0.61,0.612, 0.62, 0.623, 0.627, 0.63, p_i***}***					
0.7	Not satisfy	Not satisfy	*0.7693*	0.1045	0.1066
0.8256	satisfy	satisfy	*0.80217*	0.1232	0.1257
0.83	satisfy	satisfy	*0.8033*	0.1239	0.1264
0.845	satisfy	satisfy	*0.80723*	0.1261	0.1287
0.862	satisfy	satisfy	*0.8117*	0.1287	0.1312
0.873	satisfy	satisfy	*0.8145*	0.1303	0.1329
For the group with set of P(correct) decision {0.63,0.633, 0.64, 0.642, 0.644, 0.65, p_i***}***					
0.75	Not satisfy	Not satisfy	*0.81456*	0.0949	0.0966
0.858	satisfy	satisfy	*0.83649*	0.1086	0.1105
0.86	satisfy	satisfy	*0.83698*	0.1089	0.1107
0.862	satisfy	satisfy	*0.83747*	0.1091	0.1110
0.87	satisfy	satisfy	*0.83942*	0.1101	0.1120
0.88	satisfy	satisfy	*0.84187*	0.1114	0.1133

Appendix IIIb – Probability Enhancement Resulting from Adding 2 or 4 Units of Weights to 2 Individuals

For individual		**Winning condition for adding 1 unit weights to each i and j**	**Winning condition for adding2 unit weights to each i and j**	**Probability of adding 2 units to 2 individuals**	**Probability of adding 4 units to 2 individuals**
p_i	p_j				
0.608	0.61	Not satisfy	Not satisfy	0.1436	0.1809
0.63	0.65	Not satisfy	Not satisfy	0.1517	0.1909
0.65	0.675	satisfy	Not satisfy	0.1602	0.2018
0.68	0.7	satisfy	satisfy	0.1654	0.2074
0.7	0.75	satisfy	Not satisfy	0.1684	0.2091
0.716	0.775	satisfy	satisfy	0.1814	0.2250
0.733	0.8	satisfy	satisfy	0.1866	0.2299
0.708	0.71	Not satisfy	Not satisfy	0.0713	0.0833
0.73	0.75	Not satisfy	Not satisfy	0.0752	0.0878
0.75	0.775	Not satisfy	Not satisfy	0.0799	0.0933
0.78	0.8	Not satisfy	Not satisfy	0.0814	0.0946
0.8	0.85	satisfy	Not satisfy	0.0800	0.0920
0.816	0.875	satisfy	satisfy	0.0872	0.1000
0.833	0.9	satisfy	satisfy	0.0883	0.1004
0.808	0.81	Not satisfy	Not satisfy	0.0208	0.0229
0.83	0.85	Not satisfy	Not satisfy	0.0219	0.0240
0.85	0.875	Not satisfy	Not satisfy	0.0235	0.0258
0.88	0.9	Not satisfy	Not satisfy	0.0231	0.0252

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