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Models and Software for Urban and Regional Transportation Planning : The Contributions of the Center for Research on Transportation

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Models and Software for Urban Regional Transportation Planning: The Contributions of the Center for Research on Transportation

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Abstract. The aim of this article is to give a semi-technical and somewhat journalistic account of the contributions to the methods used for quantitative transportation planning by professors, researchers and graduate students who have been active at the Centre for Research on Transportation (CRT) of the University of Montreal since its inception.

Keywords. Transportation planning, network optimization models, transit assignment, network equilibrium models.

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1. Introduction

The aim of this article is to give a semi-technical and somewhat journalistic account of the contributions to the methods used for quantitative transportation planning by professors, researchers and graduate students who have been active at the Centre for Research on Transportation (CRT) of the University of Montreal since its inception.

This paper is organized as follows. The first section gives a historical background to the developments of models, algorithms and software at the CRT. The next sections present the framework for the models used for transportation planning and then review the CRT contributions to the state-of-the-art of network equilibrium models, transit route choice models, combined mode and class equilibrium models and large-scale transportation planning models. The presentation is a technical overview of the main models and algorithms developed during this period. Next, the software development activities in this area are described in chronological order. The model formulations are presented but the solution algorithms are only referred to. The great variety of algorithms that were developed for the models addressed in this article make it difficult to describe them all in detail. A sample of typical applications is presented to illustrate the applications of these models. The final section addresses software developments that made it possible to transfer these research findings into practice.

2. Background

In 1970, Transport Canada initiated a program that aimed at establishing transportation research centers at major universities across the country. Even though the Centre for Research on Transportation (CRT) at the University of Montreal was established in 1970, it did not become active until two years later. In 1972 the Ford Motor Company of Canada made an unrestricted grant to the University of Montreal, which was used to initiate a basic research program in transportation. Pierre Robillard, Marc Gaudry and the author were the first active professors to participate in this project. Most of the students who were integrated into this project were from the Department of Computer Science and Operations Research (Informatique et recherche operationnelle). The author was appointed as Director of the CRT in the fall of 1973.

The aim of this paper is to present the modeling and software developments that were carried out at the CRT, from its inception, on urban transportation planning models. In addition, the model innovations that resulted in the introduction of the EMME/2-Emme 3 software package are documented.

The research direction adopted in the Ford sponsored project was to provide a critical evaluation of the methodology used in the transportation planning models of that time and to explore new avenues made possible by the mathematical programming and computer science competence of the team. We started by studying the work of Dafermos (1968, 1971, 1972) and Dial (1971). This was the starting point of the research in this area at the CRT. This project produced several doctoral theses: Michael Trahan (1974), Renee Dionne (1974), Sang Nguyen (1974), Claude Chriqui (1974) and Robert Chapleau (1974), which addressed topics related to network equilibrium models and algorithms, transit route choice models, stochastic route choice models and network design models and algorithms.

3. Demand and Network Models for Transportation Planning

In order to place the modeling contributions made by the CRT in the context of transportation planning and travel demand forecasting, an overview of the models used is presented next.

3.1 The Four-Step Transportation Planning Paradigm

When considering the choices that a traveller makes in his trip form an origin to a destination, it is customary to explicitly identify the following:

- destination choice
- mode choice
- route choice on the road or transit network

It has become common to refer to a four-step travel demand forecasting sequence of models:

- The generation and attraction of trips; these are econometric models that determine the number of trips departing from origins, and the number of trips arriving at destinations. There may be more than one model used for the generation and attraction of trips, for instance when the traveller population is subdivided by trip purpose (work, study, service, etc.).

- Destination choice; one or more trip distribution models.

-Mode choice; one or more mode choice models.

- Traffic assignment models are designed to describe the traffic patterns formed by users of a transportation network such as an urban street system. They also adapt to serve as models for travel on rail or airline networks. It is assumed that the performance characteristics of the network are known and that the travel demand is defined by an origin-destination demand matrix, as described in the preceding section, or defined by demand functions.

The diagram below shows a schematic diagram of this four-stage process.

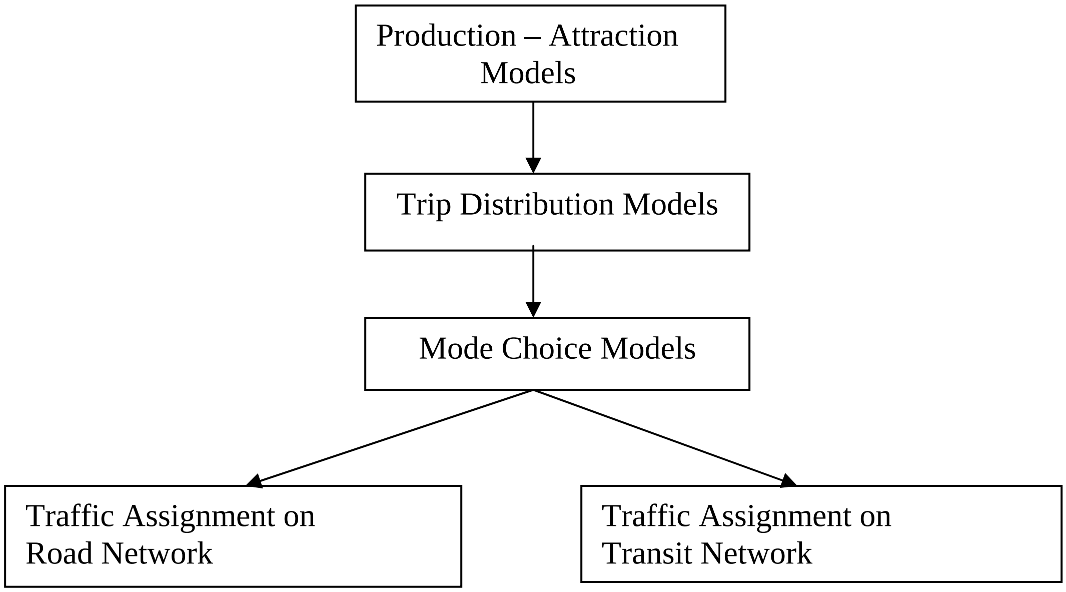

Figure 1. The four-step transportation planning paradigm

One of the main contributions of the research done at the CRT was the development of rigorous models and of algorithmic solutions for the traffic assignment models on the road and transit networks. New integrated models that would render this process simultaneous rather than sequential were also significant contributions.

In the following, the classical destination choice model and very general formulations of the mode choice models used in practice are presented. They are a necessary component for the presentation of multi-modal network equilibrium models that are introduced in later sections of this article.

3.2 Destination choice models

Destination choice modelling uses trip distribution or spatial interaction models. These models assume that the total trips from an origin node and the total trips to a destination node are known. The travel times (costs) are also known, and the result of the model is an origin-destination matrix that contains the trips from origins to destinations in its cells. The spatial interaction models were developed prior to the 70s (see Wilson, 1970). Many variants have been implemented in practice. The model described below is known as the entropy spatial interaction model and is perhaps one of the most common in practice.

Consider a transportation network that permits the flow of one type of traffic (vehicles or passengers) on its links. The nodes $n \in N$ represent origins, destinations and intersections on links. The links $a \in A \subset N \times N$ represent the transportation infrastructure.

If the number of trips that start from origins $p \in P \subset N$ is O_p and the number of trips destined to destinations $q \in Q \subset N$ is D_q, then the issue of interest is to determine g_{pq} (or g_i where $i = (p,q)$) given the time or cost of travel u_{pq}.

The classical model that is used to determine the origin-destination matrix g_{pq} is known as the entropy model. Conservation of flow at origins and destinations implies that

$$\sum_{q \in Q} g_{pq} = O_p,\ p \in P \tag{1}$$

and

$$\sum_{p \in P} g_{pq} = D_q,\ q \in Q \tag{2}$$

and, evidently the demand for travel is nonnegative

$$g_{pq} \geq 0,\ \ p \in P,\ \ q \in Q. \tag{3}$$

In the absence of other information, it is postulated that the origin-destination matrix is the “most likely to occur”, which leads to the objective function

$$\max - \sum_p \sum_q g_{pq} \ln g_{pq} \tag{4}$$

subject to (1)-(3). The objective has the interpretation of entropy maximization; the formalism originates from information theory (see Jaynes (1957,1957b)). The model was introduced to transportation and regional analysis by Wilson (1967, 1970). When some *a priori* information is known about the matrix, say $g_{pq}^0, \forall (p,q)$, Kullback (1959) and Snickars and Weibull (1977) have suggested the use of the objective

$$\max - \sum_p \sum_q g_{pq} \ln \left(\frac{g_{pq}}{g_{pq}^0} \right) \tag{5}$$

In order to characterize the dispersion of trips, a constraint is added to the total travel time, where C is an observed total travel time

$$\sum_p \sum_q g_{pq} u_{pq} = C \tag{6}$$

which leads to the objective function

$$\min \sum_p \sum_q g_{pq} \left(\ln g_{pq} + \theta u_{pq} \right). \tag{7}$$

θ has the interpretation of the dual variable associated with the constraint (6). It is trivial to verify that, by applying the Karush-Kuhn-Tucker conditions, any solution of (7) subject to (1)-(3) has the general form

$$\begin{aligned} g_{pg} &= \exp\left(\alpha_p + \beta_q - 1\right) \exp\left(-\theta u_{pq}\right), \forall (p,q) \\ &= A_p B_q \exp\left(-\theta u_{pq}\right), \end{aligned} \tag{8}$$

where α_p and β_q are the dual variables associated with the conservation of flow constraints. With the convention $g_{pq} \ln g_{pq} = 0$ when $g_{pq} = 0$, it is possible to obtain solutions to this class of problems by using any primal convex programming algorithm.

However, one of the properties of this problem is that the primal variables may be expressed in terms of the dual variables, as will be shown next. The Lagrangean dual problem of (7) subject to (1)-(2) is

$$D(\alpha,\beta,g)=\max_{\alpha,\beta}\left\{\min_{g}\sum_{p}\sum_{q}g_{pq}\left(\ln g_{pq}+\theta u_{pq}\right)+\sum_{p}\alpha_{p}\left(O_{p}-\sum_{q}g_{pq}\right)\right.$$
$$\left.+\sum_{q}\beta_{q}\left(D_{q}-\sum_{p}g_{pq}\right)\right\} \tag{9}$$

By using (8) to replace the primal variables g_{pq} one obtains, after some simplifications

$$D(\alpha,\beta)=\max_{\alpha,\beta}\left\{\sum_{r}\alpha_{p}O_{p}+\sum_{s}\beta_{q}D_{q}-\sum_{p}\sum_{q}\exp\left(\alpha_{p}+\beta q-1-\theta u_{pq}\right)\right\} \tag{10}$$

This property led to the development of an efficient solution procedure known as the “balancing method”, which is a dual ascent method for one variable at a time. The balancing method algorithm may be referenced in standard texts. This method dates back to at least 1937 when Kruithof used it for the prediction of telephone traffic distribution. Deming and Stephan (1940) independently rediscovered this method and applied it to a cross-classification problem in statistics as a simplification of least squares fitting. Evans and Kirby (1974) and Andersson (1981) also made important contributions to the analysis of the mathematical structure of the model.

The CRT made several contributions to the study of entropy trip distribution models. Variants of the balancing method were studies by Robillard and Stewart (1974). Erlander, Nguyen and Stewart (1979) considered the calibration of the entropy model by using survey data. An algorithm for this model was also studied by Jefferson and Scott (1979). Lamond and Stewart (1981) showed that the balancing method and its variants may be viewed as a special case of Bregman’s (1967) non-orthogonal projection method to solve certain problems of convex programming. An accomplishment of the collaboration between the CRT and the University of Linkoping is the book by Erlander and Stewart (1990). This text is an excellent synthesis of the theory and application of trip distribution models based on the entropy and gravity principles.

3.3 Mode choice models

The formulation and calibration of mode choice models represents a very large body of work. The text by Ben Akiva and Lerman (1980) is a very good reference for the great variety of econometric models that are used in representing choices among competing alternatives. One of the early contributions to choice theory, by Domencich and McFadden (1975), was recognized by awarding Dan McFadden the 2000 Nobel Prize in Economics.

The simplest model can be stated generically as the probability of using a particular mode

$m^{'}$ among a set of modes $m \in M$ as a function of the socio-economic characteristics of the traveller, the travel times and the travel costs:

$$p\,(\text{using mode } m^{'}) = f\,(\text{utility of mode } m^{'} / \sum_{m} \text{ utility of mode } m\,)$$

A common functional form is that of the logistic or logit function. An example of a simple mode choice function among two modes of traffic is

$$p(\text{using } m^{1}\,) = \frac{1}{1+\exp(\sum \gamma_i x_i + \alpha\ \Delta\text{cost} + \beta\ \Delta\text{time})},$$

where the $\sum \gamma_i x_i$ are the socio-economic variables that characterize the traveller. Δcost and Δtime are the differences in travel cost and travel times between the two modes. The parameters γ_i, α and β are calibrated by using survey data.

Mode choice functions play a central role in the formulation of mathematical models that are used in transportation planning.

4. Traffic assignment models and methods

4.1 The Network Equilibrium Model

Traffic assignment models are designed to describe the traffic patterns formed by users of a transportation network such as an urban street system. They also adapt to serve as models for travel on rail or airline networks. It is assumed that the performance characteristics of the network are known and that the travel demand is defined by an origin-destination demand matrix, as described in the preceding section, or it is defined by demand functions.

In order to highlight the contributions made by research staff at the CRT in the area of network equilibrium problems, it is necessary to introduce the basic network equilibrium model as first formulated by Beckmann et al (1976).

For notational simplicity, assume a transportation network model with one type of vehicular flow on the directed links of the network. The nodes $i,\ i \in N$, represent origins, destinations, and intersections and the arcs $a,\ a \in A$ model the transportation links (streets, highways, …). Origin to destination (O-D) demands give rise to link flows $v_a,\ a \in A$ and the cost of traveling on a link is given by a user cost (travel time) function $s_a(v)$, where v is the vector $(v_a)_{a \in A}$ of link flows over the entire network. Cost functions model time delay on a link or more general costs such as tolls or fuel consumption, and are assumed to be nonnegative. Let I be the set of O-D pairs, $K_i, i \in I$, be a set of directed paths connecting O-D pair i, and K be the set of all paths. The demand between

O-D pair $g_i, i \in I$ uses directed paths and the path flows h_k obey conservation of flow and non negativity

$$\sum_{k \in K_i} h_k = g_i, \quad \forall i \in I \tag{11}$$

$$h_k \geq 0 \quad \forall k \in K. \tag{12}$$

Link flows are given by

$$v_a = \sum_{i \in I} \sum_{k \in K_p} \delta_{ak} h_k \quad \forall a \in A \tag{13}$$

where $\delta_{ak} = 1$ if link a belongs to path k and is zero otherwise.

Define Δ as the $|A| \times |K|$ arc-path incidence matrix (δ_{ak}) so that $v = \Delta h$, where h is the vector $(h_k)_{k \in K}$ of path flows for all O-D pairs. The cost $s_k\left(= s_k(h)\right)$ for each path k is then defined by

$$s_k = \sum_{a \in A} \delta_{ak} s_a(v) = \sum_{a \in A} \delta_{ak} s_a(\Delta h) \quad \forall k \in K_i, i \in I, \tag{14}$$

and $u_i\left(= u_i(h)\right)$ is by definition the cost of the least cost path for O-D pair i:

$$u_i = \min_{k \in K_i} s_k \quad \forall i \in I. \tag{15}$$

For each $i \in I$, the travel demand t_p may be obtained from a fixed O-D demand matrix, in which case we write $g_i = \bar{g}_i$, or it is given by a specific demand function $g_i(u)$, where u is the vector of least cost travel values, $(u_i)_{i \in I}$ for all the O-D pairs of the network:

$$g_i = g_i(u) \quad \forall i \in I. \tag{16}$$

System–optimal traffic assignment models assume that travel on the network follows paths such that network utilization is for the common good. If the demands g_i are fixed, then the objective is to satisfy a normative principle that states the average travel cost (or time) is to be minimized. Since total demand is a constant, it is equivalent to minimize total system cost, and the fixed demand system optimization model is

$$\min \sum_{a \in A} s_a(v) v_a \tag{17}$$

subject to (11), (12), and (13)

with $g_i = \bar{g}_i$.

If, however, travel demand is *elastic*, that is, dictated by demand functions (6), the system-optimization model aims to maximize the net economic benefit to the network users. From standard economic principles, the benefit to travellers between any O-D pair $i \in I$ is measured by the area under their demand curve $g_i(u)$. It is assumed that this function has an inverse $w_i\left(g_i\right)=u_i$. Hence the economic benefit can be expressed as $\int_0^{t_p} w_i\left(y\right)dy$. Therefore, in this case, the system optimization model becomes

$$\max \sum_i \int_0^{t_p} w_i\left(y\right)dy - \sum_{a\in A} s_a\left(v\right)v_a \tag{18}$$

subject to (11), (12), (13) and (16).

These system problems have counterparts in user-optimization models that aim to more accurately describe the situation where travellers on the network distribute themselves so that no single user can unilaterally improve travel costs. The descriptive models of traffic flow, therefore, assume the users are in a Wardrop equilibrium (Wardrop, 1952), a special case of Nash equilibrium. The mathematical statement is:

Determine h^* and u^* such that the following conditions are satisfied:

$$\left(s_k\left(h^*\right)-u_i^*\right)h_k^*=0 \quad \forall k\in K_i, i\in I \tag{19}$$

$$s_k\left(h^*\right)-u_i^*\geq 0 \quad \forall k\in K_i, i\in I \tag{20}$$

$$\sum_{k\in K_i} h_k^*-g_i=0 \quad \forall i\in I \tag{21}$$

$$h^*\geq 0,\ u^*\geq 0, \tag{22}$$

where $g_i=\overline{g}_i$ for fixed demand and $g=g\left(u^*\right)$ when demand is elastic. The equilibrium link flows, v^* are calculated from the path flows h^* using (13).

Another equivalent way to state the equilibrium conditions (19),(20) is

$$\left.\begin{array}{ll} s_k\left(h^*\right)=u_i^* & \text{if } h_k\geq 0, \\ s_k\left(h^*\right)\geq u_i^* & \text{if } h_k=0, \end{array}\right. \qquad \forall k\in K_i, i\in I$$

subject to (21), (22), which is a direct statement of Wardrop's user optimal principle.

The first two conditions ensure that, for all $i\in I$, only minimum cost paths are used, and the third equates the total path flows to the total demand, given the minimum path costs. This general version of the problem is known as the network equilibrium model (NEM) which has applications in many areas, including electrical networks, water pipe networks and spatial price equilibrium problems. Florian and Hearn (1995) provide numerical examples of these applications.

The basic NEM reformulations used in transportation planning, however, are optimization problems. The primary assumption is that the cost and demand functions are *separable*, that is, they have the form $s_a\left(v\right)=s_a\left(v_a\right)$ and $g_i\left(u\right)=g_i\left(u_i\right)$, respectively. In other words, the cost on a link depends only on the flow on that link, and the demand for O-D pair p depends only on the minimum travel time for that O-D pair.

It is further assumed that the cost functions are convex and the demand functions are strictly monotone. Under these conditions, the elastic demand user-optimization problem may be stated as the convex program:

$$\min \sum_{a \in A} \int_0^{v_a} s_a(x)\,dx - \sum_{p \in P} \int_0^{g_i} w_i(y)\,dy \tag{23}$$

subject to (11), (12), (13) and (16).

In the fixed demand case, the user problem becomes

$$\min \sum_{a \in A} \int_0^{v_a} s_a(x)dx \tag{24}$$

subject to (11), (12) and (13).
with $g_i = \bar{g}_i$.

That the solutions of these problems are equivalent to (23) – (26), in the elastic and fixed demand cases, respectively, follows directly from the Karush-Kuhn-Tucker conditions for the two problems. The connection between the NEM conditions and the system optimal models is also revealed by their Karush-Kuhn-Tucker conditions. It is straight-forward to verify that they have the same form, with the terms $s_k(h^*)$ calculated from the *marginal* link costs, $s_a(v_a^*) + s'_a(v_a^*)v_a^*$. Thus, the interesting and important connection is that the solution to a system optimal model is in equilibrium with respect to marginal costs, while the solution of a user optimal model is in equilibrium with respect to average costs.

It is important to mention that the total link flows of the NEM presented above are unique under the assumptions made on the link cost functions. Any decomposition of the total link flow by origins or by paths is not necessarily unique.

4.2 Contributions to the study and solution of single class network equilibrium models

The first contribution made for the solution of the NEM was the doctoral thesis of Nguyen (1974). He explored algorithms for the network equilibrium model with fixed and variable demand in the space of link flows, origin flows and origin-destination path flows (see also Nguyen, 1974, 1976). These algorithms were the adaptation of the classical methods of nonlinear programming such as the linear approximation method, the convex simplex method and the reduced gradient method. At the time, the computer memory (RAM) available was quite limited and the most useful algorithm was the adaptation of the linear approximation method of Frank and Wolfe (1956) since it required relatively little RAM and was quite simple to implement. The heuristic methods used at the time, such as the incremental assignment method or capacity restrained method were to be replaced eventually by rigorous methods; however the validation of the method had to be demonstrated. A calibration and validation of the method was carried out by using the network of the City of Winnipeg, Canada (Florian and Nguyen, 1976). This was one of the first studies that showed that a rigorous solution of the NEM could be calibrated and validated successfully.

The CRT took the initiative to organize conferences on the topic of network equilibrium models and their algorithmic solution. The first such meeting, which took place in 1974, brought together all the researchers who were interested in this emerging area of research. It is worth mentioning the participation of Martin Beckmann, Harold Kuhn, Stella Dafermos, Marvin Manheim, Dirck Van Vliet, Suzanne Evans and Bob Dial who also provided some support from UMTA. The proceedings were published by Springer Verlag (see Florian, ed., 1976). Two other conferences on traffic network equilibria and network modeling topics took place in 1977 and 1981 that helped to establish the CRT internationally as a leading academic research centre in the field. Several papers that were presented at the 1977 conference were published in a special issue of *Transportation Research* B (see Florian and Gaudry, 1980). In 1982, at the invitation of the National Research Council of Italy, the author organized a week long course in Amalfi, Italy that brought together some of the best researchers in this field at that time and was attended by many young Italian researchers who will go on and make significant contributions to this field of endeavor. The lectures were published in a book edited by the author (see Florian, ed., 1984).

The interest in solution algorithms for the NEM continued with developments that addressed variants of the linear approximation method such as the “away step” (see Florian, 1977, Guelat and Marcotte, 1986), and the PARTAN method (see Florian, Guelat and Spiess, 1987). During the same period, other results that could be computed after carrying out a traffic assignment on the road network, such as fuel consumption, were studied (see Le-Van Nguyen, 1982). A collaboration with Italian researchers resulted in several contributions. A dual shortest path algorithm (see Florian, Nguyen and Pallottino, 1981) was one of the first such common works. A survey paper (Florian, 1986) summarized some of the developments carried out at the C.R.T. and elsewhere. However, one other main area of research was the development of multi-modal network equilibrium models that could consider, in an integrated way, the choices made by travelers as to mode and routes on both the road and the transit networks. This is described in more detail below. The texts by Sheffi (1985) and the monograph by Patrickson (1994) provide more detail on network equilibrium models and solution algorithms.

4.3 Contributions to the study and solution of multi-class network equilibrium models

Travellers are not homogeneous**.** They may be distinguished by the vehicle they drive (car, truck, etc..) or by their socio-economic characteristics. The extension of the single class NEM to multiple classes is relatively straightforward. Some more complex models arise when tolls are introduced into the model since the willingness to pay tolls depends on the socio-economic characteristics of the population. The modelling of the response to tolls has been considered in the context of discrete multiple classes of traffic by Florian (2006). There is a value of time associated with each class and the resulting model is a rather complex multi-class network equilibrium model with variable demand associated with the choice between tolled links and free links. The current trend of developing new highway facilities by using tolls as a means of financing the project has led to very common use of such models. If one assumes that the value of time is given by a continuous distribution in the population then the model is different and was solved by Marcotte et al (1996).

Multi-class network equilibrium models involving cars and trucks have been studied by Wu et al (2006), who consider the contribution of trucks to congestion to depend on the mix of traffic and Noriega and Florian (2007), who consider different volume/delay functions for each class of traffic.

4.4 Contributions to the study and solution of network design models

The network design problem has attracted the attention of many researchers. Dionne (1974) studied the optimal design of a network when the network is not subject to congestion. Marcotte (1982) has considered versions of this problem when congestion prevails in his doctoral thesis. He considered both single level (see Marcotte, 1983) and bi-level (see Marcotte,1986) versions of this problem.

5. Transit route choice models

5.1 Contributions to the study of un-congested transit route choice models

Transit route choice models or transit assignment models aim to describe the traffic flows on a network of transit lines that operate at known frequencies. The main difference with the traffic assignment models for road networks is the waiting phenomenon: transit travellers experience a waiting time for the first vehicle (bus) of the line which they have chosen. In addition, the access to the transit line stop implies an access time (which is usually a walk time), transfers between lines if more then one line is taken and the in-vehicle time. The contributions of the CRT researchers to the formulation and solution of this problem are numerous and significant.

One of the first contributions to the study of transit route choice models was the work of Chriqui and Robillard (1975) who considered a simple network of one origin and one destination pair connected by several non-overlapping transit lines, or common lines. Their seminal paper introduced the notion that, on a simple network of one origin and one destination, passengers can select a subset of attractive lines and board the first one of these that arrives at a stop in order to minimize the expected sum of waiting plus travel times.

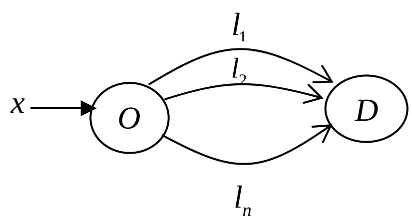

Figure 2: The common-line problem

The ideas of Chriqui and Robillard were extended to general transit networks in two ways. Spiess (1984) and Spiess and Florian (1989) introduced the notion of strategy, which is the choice of an attractive set of lines at each decision point; that is, at each node where boarding occurs. The resulting model and algorithm achieve the minimization of the expected value of the total travel time, which includes access, wait and in-vehicle time. Nguyen and Pallottino (1988) provided a graph theoretic interpretation of a strategy as an acyclic directed graph, and denoted it as a hyperpath. These models considered congestion aboard the vehicles by associating discomfort functions with each segment of a transit line, so that the resulting equilibrium models could be solved by standard algorithms for convex minimization. However, the waiting times are underestimated since they do not consider the fact that, in a period of heavy congestion, passengers may not be able to board the first vehicle to arrive at a stop.

The results of Chriqui and Robillard were used in a different way by Chapleau (1974) and DeCea and Fernandez (1989) in a transit assignment model based on a restricted notion of strategy, which allows choices among multiple lines at a given stop only if they all share the next stop to be served (for a comparison with the strategy approach, see DeCea *et al*. (1988)).

A more formal study of congestion at bus stops based on results from queuing theory was initiated by Gendreau (1984) in his doctoral thesis. He was the first to formulate a general transit assignment model with congestion. For more recent results on the waiting processes at bus stops, see Bouzaïene-Ayari *et al*. (2001) and Cominetti and Correa (2001), as well as the recent thesis by Cepeda (2002).

Consider a transit network that consists of a set of nodes, a set of transit lines, each defined by an ordered list of nodes at which boarding and alighting are permitted, and a set of walk links, each defined by two nodes. The times associated with each walk link and each transit line segment are constant. At each node that is on the itinerary of a transit line, the distribution of the interarrival times of the vehicles is known for each line that serves the node. As a consequence, one can compute the combined expected time for the arrival of the first vehicle, for any subset of lines incident at a node, as well as the probability that each line arrives first.

In order to state the mathematical model that corresponds to the transit route choice selection, it is noted that each walk link may be replaced (conceptually) by a transit line of one link with a zero waiting time (infinite frequency). Also, it is assumed that the underlying network is strongly connected. The objective is taken to be the minimization of *expected* waiting and travel time, or the *expected* total generalized cost if waiting times and travel times have different weights (e.g. waiting is more onerous that in-vehicle time).

The network is composed of four types of arcs: wait arcs (no travel time), in-vehicle (no waiting), alighting (no travel time, no waiting) and walk arcs (travel time, no waiting). Thus, the segment of a transit line is an arc that is *served* by a vehicle at given intervals and the transit traveller waits for the link to be *served* by a vehicle.

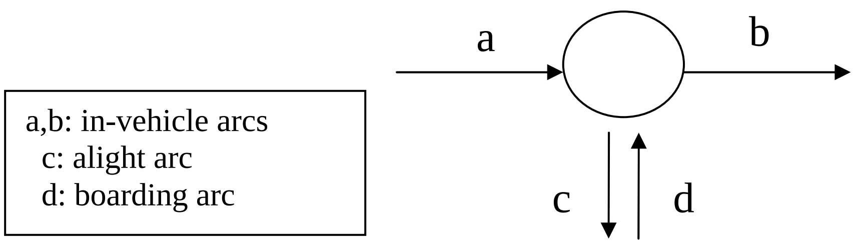

Figure 3. The link representation of a transit network

The arcs that will be included in a solution of the model are denoted by $\bar{A} \in A$, where A is the set of arcs and N is the set of nodes. Thus the solution for a single destination s is a sub-graph $b_s = \left(N, \bar{A}\right)$. The demand for travel from nodes $i, i \in N$ to destination q is denoted $\bar{g}_i$. Among the links included in a solution $\bar{A}$, at each node $i, i \in N$, a traveller boards the first vehicle that serves any of the lines in the $\bar{A}_i^+ \left(\bar{A} = \cup_i A_i^+\right)$. The set $\bar{A}_i^+$ corresponds to the lines that will be chosen by the traveller to yield one or more routes from i to s in a solution of the model. At each stop i, it is convenient to refer to the set $\bar{A}_i^+$ as the *set of attractive lines*.

Let $W\left(\bar{A}_i^+\right)$ be the expected waiting time for the arrival of the first vehicle serving any of the links $a \in \bar{A}_i^+$, which is denoted as the *combined waiting time* of links $a \in \bar{A}_i^+$. Let

$P_a\left(\bar{A}_i^+\right)$ be the probability that link a is the first line to be served among the links $\bar{A}_i^+$. If an exponential distribution of interarrival times is assumed then

$$W\left(\bar{A}_i^+\right) = \frac{1}{\sum_{a \in \bar{A}_i^+} f_a} \tag{25}$$

and

$$P\left(A_i^+\right) = \frac{f_a}{\sum_{a' \in \bar{A}_i^+} f_a}, \ a \in \bar{A}_i^+ \tag{26}$$

where f_a is the frequency of link (line) a

Since $\bar{A}$ is not known *a priori*, the single destination model is formulated by using binary variables x_a

$$x_a = \begin{cases} 0 & \text{if } a \notin \bar{A} \\ 1 & \text{if } a \in \bar{A} \end{cases}, \ a \in A$$

The optimization model for a single destination may be stated now as follows:

$$\min \sum_{a \in A} s_a v_a + \sum_{i \in I} \frac{V_i}{\sum_{a \in A_I^+} f_a x_a} \tag{27}$$

subject to

$$v_a = \frac{x_a f_a}{\sum_{a' \in A_i^+} f_{a'} x_{a'}}, \ a \in A_i^+, \ i \in N \tag{28}$$

$$V_i = \sum_{a \in A_i^-} v_a + \bar{g}_i, \ i \in N \tag{29}$$

$$V_i \geq 0, \ i \in N \tag{30}$$

$$x_a = 0 \text{ or } 1, \ a \in A, \tag{31}$$

where s_a is the travel cost on link a and V_i is the total volume at node i. At first sight, the problem (15)-(19) is a mixed integer nonlinear optimization problem. Fortunately, the problem may be reduced to a much simpler linear programming problem by making the following observations. (16) may be replaced by the non-negativity constraints of the link volumes $v_a \geq 0, \ a \in A$ since $\sum_{a \in A_i^+} v_a = V_i, \ i \in N$. Then, by introducing new variables w_i, which denote the total waiting time of all trips at node i, $w_i = \frac{V_i}{\sum_{a \in A_i^+} f_a x_a}, \ i \in N$, one obtains the equivalent problem

$$\min \sum_{a \in A} s_a v_a + \sum_{i \in N} w_i \tag{32}$$

subject to

$$v_a = x_a f_a w_i,\ a \in A_i^+,\ i \in I \tag{33}$$

$$\sum_{a \in A_i^+} v_a - \sum_{a \in A_i^-} v_a = \bar{t}_i,\ i \in N \tag{34}$$

$$v_a \geq 0,\ a \in A. \tag{35}$$

The objective function (20) is now linear and the 0-1 variables are only used in constraints (21), which are the only nonlinear constraints. These constraints may be relaxed by replacing (21) with

$$v_a \leq f_a w_i,\ a \in A_i^+,\ i \in N \tag{36}$$

which yields the linear programming problem (32), (34), (35), (36). It may be shown, by using the extreme point properties of the solutions of a linear programming model, that this problem is equivalent to (32)-(35).

5.2 Contributions to the study of congested transit route choice models

It is clear that congestion at but stops does not only increase the waiting times but it also affects the flow share of each attractive line. In the case of independent arrivals with exponential distribution, the flow split is proportional to the so-called *effective frequency*, that is to say, the inverse of the waiting time of each line. The stop models are called *semi-congested* if they consider only the increase in waiting times, and *full-congested* if they also include the effects on the flow split. Wu, Florian and Marcotte (1994) considered a semi-congested transit network model in which the time required to board a vehicle increases with flow, but the distribution of flows among attractive lines is done in proportion to the nominal frequencies. It is worth mentioning the work of Wu and Florian (1993) in solving the *semi-congested* problem by using a simplicial decomposition approach.

Bouzaïene-Ayari (1996) and Bouzaïene-Ayari *et al*. (1995) extended the latter to a full-congested model that combines a fixed-point problem in the space of arc flows with a variational inequality in the space of hyperpath flows. An algorithm reminiscent of the method of successive averages was also proposed in Bouzaïene-Ayari (1996) and Bouzaïene-Ayari *et al*. (1995), but the combinatorial character of hyperpaths seems to limit its applicability to small networks. Bouzaïene-Ayari *et al* (2001) provided a survey of the models used to represent the behaviour of transit passengers at stops.

More recently, Cominetti and Correa (2001) analyzed a full-congested version of the common lines problem of Chriqui and Robillard and used it to develop a transit network model that can deal with general arc travel times as well as more realistic waiting time functions with asymptotes at bus capacity. The latter are introduced by considering effective frequency functions that vanish when the flows exceed the capacity of the line. Although Cominetti and Correa (2001) established the existence of a network equilibrium, they fail to propose an algorithm to compute it. Nevertheless, since the model is stated as a fixed point in the space of arc flows only, it opens the way to dealing with large-scale networks. Building on their work, Cepeda (2002) devised a non-differentiable formulation of the strategy transit assignment with flow-dependent travel times and perceived frequencies. This model is described in detail in Cepeda et al (2005). In the following, only the basic models of Chriqui and Robillard (1975), Spiess (1984) and the extension to the consideration of capacities by Cepeda et al (2005) are described in detail. The strategy model and its extensions for congested networks are described in detail in the following.

It is possible to extend the strategy algorithm to a nonlinear version of this problem, where the link travel times are no longer constants, but are continuous functions $s_a\left(v_a\right), a \in A$ of the arc flows v_a. The resulting model may be solved by an adaptation of the linear approximation algorithm. Further details may be found in Spiess (1984) and Spiess and Florian (1989).

$$\min_{v \in V_o} \sum_{a \in A} \int_0^{v_a} s_a(x)dx + \sum_{\substack{d \in D \\ i \neq d}} w_i^d \tag{37}$$

$$\text{s.t.} \quad \begin{aligned} & v_a^d \leq w_i^d f_a \qquad d \in D, i \neq d, a \in A_i^+ \\ & v_a = \sum\nolimits_{d \in D} v_a^d \quad a \in A. \end{aligned} \tag{38}$$

$$v_a^d \geq 0, a \in A, d \in D \tag{39}$$

A stochastic version of the transit strategy model was developed by Nguyen, Pallottino, and Gendreau (1998). This model considers a logit based choice of strategies for route choice on transit networks.

This model considers congestion aboard the vehicles but does not take into account the fact that waiting times increase as the passenger load on the transit vehicles increases. Passengers may not be able to board the first vehicle that arrives at a stop at which they are waiting. The modelling of waiting times that increase in congested conditions was the topic of the doctoral thesis of Cepeda (2002). The results were also reported in Cepeda, Cominetti and Florian (2005).

In order to extend the strategy model to consider vehicle capacities, some additional notation is required. A transit line is composed of several transit segments. Each line segment $a \in A$ is characterized by an in-vehicle travel time function $s_a(v_a)$ and a saturation flow $\bar{v}_a, \bar{v}_a > 0$. The *effective frequency*, which is perceived by a waiting traveller, is assumed to be decreasing in order to reflect the increment in waiting time induced by an augmentation of flow. For each $d \in D$, the effective frequency $f_a\left(v\right) \to 0$ when $v_a^d \to \infty$ with $f_a\left(v\right)$ strictly decreasing with v_a^d as long as $f_a\left(v\right) > 0$. Note that $f_a\left(v\right)$ may take the value 0, which makes it possible to model waiting times that explode to infinity beyond the line capacity.

The model that considers transit capacities in this way may be restated in the form

$$\underset{v \in V_0}{\text{Minimize}}\ G(v) \tag{40}$$

where

$$G(v) \square \sum_{d \in D} \left[\sum_{a \in A} s_a(v) v_a^d + \sum_{i \neq d} \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} - \sum_{i \neq d} g_i^d u_i^d(v) \right] \tag{41}$$

$$\sum_{a \in A_i^+} v_a^d - \sum_{a \in A_i^-} v_a^d = g_i^d, i \in I, d \in D \tag{42}$$

$$v_a^d \leq \alpha_i^d f_a(v),\ a \in A_i^+,\ i \in N,\ d \in D \tag{43}$$

$$v_a^d \geq 0, \quad a \in A,\ d \in D \tag{44}$$

$G(v)$ is a gap function that has a value of zero when an equilibrium flow is reached.

Now, since the optimal value of is known, a simple algorithm is to use the Method of Successive Averages (MSA), which has been extensively used in transportation applications as a heuristic method, and to evaluate the deviation from optimality of the.

At each iteration, the method computes a transit network equilibrium for the linear network obtained by fixing the travel times and the frequencies at the values determined by the current flows, and then updates these flows by averaging the previous iterate and the newly computed solution. As mentioned earlier, a solution for the linear cost network can be found by solving

$$\min_{v \in V_0} \sum_{d \in D} \left[\sum_{a \in A} t_a v_a^d + \sum_{i \neq d} w_i^d \right]$$
$$\text{s.t.} \quad v_a^d \leq w_i^d f_a \quad d \in D, i \neq d, a \in A_i^+.$$

This method has been used successfully in practice.

5.3 Contributions to the study and solution of network design models

Transit design problems, where the behaviour of transit travellers is governed by a strategy assignment were considered by Constantin (1986), Constantin and Florian (1993), the un-congested case, Noriega (2002) and Noriega and Florian (2003), the semi-congested case. These models optimize the frequency of a set of transit lines given a normative objective of minimizing the cost of the operator. The resulting min/min optimization problems are solved by iterative algorithms that use the projection of the gradients of the upper level objective function.

6. Combined and Multi-Modal models

6.1 Simple combined models

In this section, some of the contributions made to the formulation and solution of multi-mode integrated demand-traffic assignment models are explored in detail.

The transportation choices offered in an urban area include both road and transit facilities. The challenge in the mid 70's was to state an integrated model that would simultaneously consider the choices made by travelers regarding destination, mode and route. The integration of demand models with network models in a single model that would describe all choices made in a city regarding destination, mode and route choice was an innovative idea at the time. An early contribution was the formulation of a combined mode choice-transit and road assignment model (Florian, 1977) and the statement of a solution algorithm (that turned out later to be the adaptation of the Jacobi method) and a model that combined trip distribution, modal split and road traffic assignment (Florian and Nguyen, 1978).

The motivation to formulate and solve combined models comes from the simple fact that the sequential use of travel demand models and network assignment models is inconsistent: the origin to destination travel times that are obtained after the network assignment is carried out are not necessarily consistent with the travel times that were the input to the travel demand model. Hence, some equilibration method was necessary in order to ensure that the models were consistent. The diagram below identifies the need for an equilibration mechanism in order to render the travel demand forecasting process consistent.

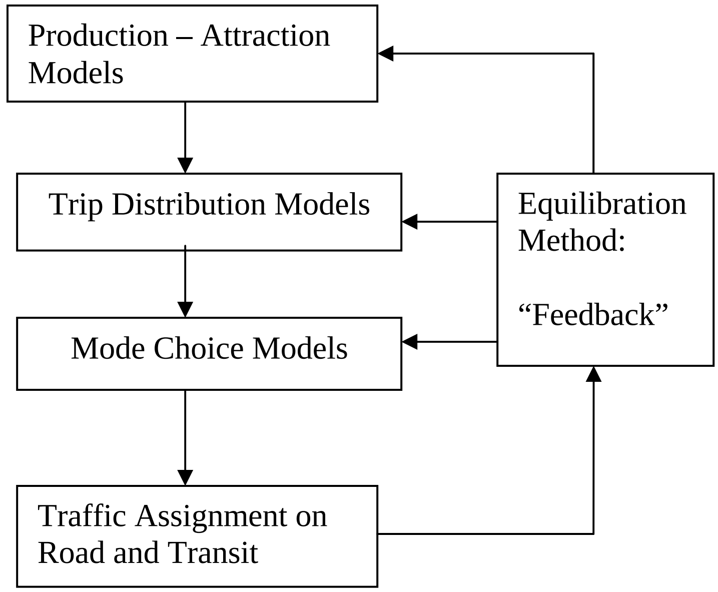

Figure 4. The four-step transportation planning paradigm with equilibration

This was first approached by stating models that combined the entropy type trip distribution model and the network equilibrium model for one trip purpose and one class of traffic on the road mode. It is perhaps one of the simplest combined models and was first proposed by Evans. Its mathematical formulation is:

$$\min \sum_{a\in A}\int_0^{v_a} s_a(x)dx + \rho\sum_p\sum_q g_{pq}\ln g_{pq} \tag{45}$$

subject to

$$\sum_{q\in Q} g_{pq} = O_p,\ p\in P$$

$$\sum_{p\in P} g_{pq} = D_q,\ q\in Q$$

$$g_{pq} \geq 0,\ \ p\in P,\ \ q\in Q.$$

$$\sum_{k\in K_i} h_k = g_i \quad \forall i\in I$$

$$h_k \ \ \geq 0,\ k\in K_i$$

Florian, Nguyen and Ferland (1975) developed the adaptation of the linear approximation method for this model. Evans (1973, 1976) proposed a solution method, which can be interpreted as a partial linear approximation algorithm. It has a better empirical convergence than the former.

A more elaborate combined model was formulated and analyzed by Florian and Nguyen (1978). It is a combined distribution-assignment modal choice model based on entropy type trip distribution models for two modes: auto, au, and transit, tr. The travel costs on the transit network are not flow dependent while the travel times on the auto network are flow dependent. The convex cost optimization problem:

$$\min \sum_{a\in A}\int_0^{v_a^{au}} s_a(x)dx + \rho\sum_p\sum_q g_{pq}^{au}\ \ln g_{pq}^{au}\ + + \rho\sum_p\sum_q g_{pq}^{tr}\ (\ln g_{pq}^{tr} + u_{pq}^{tr}) \tag{46}$$

subject to

$$\sum_{q\in Q}(g_{pq}^{au} + g_{pq}^{tr}) = O_p,\ p\in P \tag{47}$$

$$\sum_{p\in P}(g_{pq}^{au} + g_{pq}^{tr}) = D_q,\ q\in Q \tag{48}$$

$$g_{pq}^{au} \geq 0,\ g_{pq}^{tr} \geq 0\ p\in P,\ q\in Q. \tag{49}$$

$$\sum_{k\in K_i} h_k^{au} = g_i^{au},\quad \forall i\in I \tag{50}$$

$$v_a = \sum_{i\in I}\sum_{k\in K_p}\delta_{ak}h_k^{au} + v_a^{tr},\quad \forall a\in A \tag{51}$$

$$h_k\ \geq 0,\ k\in K_i\quad , \tag{52}$$

yields unique solutions and has the property that the resulting mode choice is given by a logit function:

$$p(\,au_{pq}\,) = \frac{\exp(1/\rho)u_{pq}^{au}}{\exp(1/\rho)u_{pq}^{au} + \exp(1/\rho)u_{pq}^{tr}} \tag{53}$$

This is a relatively simple mode choice function, since it does not include explanatory variables other than the travel times from origins to destinations u_{pq}^{au}, u_{pq}^{tr}. This motivated the formulation of models that would consider mode choice functions calibrated with data originating from surveys. Thus, a demand function is integrated with network assignment models.

One of the first such combined formulations was contributed by Florian (1977). It considered the dependency between modes of traffic sharing the same facility, e.g. buses slow down the speed of cars and cars slow down the buses. This model was reformulated as a variational inequality by Florian and Spiess (1982).

The fundamental result of Smith (1979), that the network equilibrium model can be formulated as the variational inequality

$$\sum_a s_a(v)(v_a - v_a^*) \geq 0 \tag{54}$$

subject to

$$\sum_{k \in K_i} h_k = g_i, \quad \forall i \in I \tag{11}$$

$$h_k \geq 0 \quad \forall k \in K. \tag{12}$$

and

$$v_a = \sum_{i \in I} \sum_{k \in K_p} \delta_{ak} h_k \quad \forall a \in A \tag{13}$$

influenced the research carried out at the CRT in this area in a significant way. The formulation of combined models that would no longer satisfy the property that there exists an equivalent convex cost minimization problem was given a new theoretical framework. The uniqueness property of the equilibrium solutions would be satisfied only if the cost (travel time) functions satisfied monotonicity conditions and the convergence of algorithms would be ensured if certain sufficient conditions were satisfied.

A combined mode choice-assignment with dependency between modes may be formulated by postulating that a demand function has been calibrated. It is monotone decreasing with travel time and its inverse depends on the travel times of the two modes referred to above as auto, au , and transit, tr .

$$u_i^{au} - u_i^{tr} = w_i(g_i^{au}), i \in I \tag{55}$$

The travel times by auto depend on the volume of transit vehicles and the travel time by transit depends on the volume of cars. Let (v_a) be the vector of cars and buses on a link $(v_a) = (v_a^{au}, v_a^{tr})$. The total demand is fixed, hence $g_i^{au} + g_2^{tr} = \overline{g}_i, i \in I$, and it is assumed that the origin to destination travel times $u_m^i(v) = \min_{k \in K_m^i} s_k(v), i \in I, m = \{au, tr\}$ satisfy Wardrop's user optimal principle. The variational inequality

$$\sum_a s_a^{au}(v_a^*)(v_a^{au} - v_a^{au*}) + \sum_a s_a^{tr}(v_a^{tr} - v_a^{tr*}) - \sum_i w_i(g_i^{au*}) - (g_i^{au} - g_i^{au*}) \geq 0 \tag{56}$$

Subject to:

$$\sum_{k \in K_i^m} h_k = g_i^m, \quad \forall i \in I, m = au, tr \tag{57}$$

$$v_a = \sum_{i \in I} \sum_{k \in K_m^i} \delta_{ak} h_k, \quad \forall a \in A, m = au, tr \tag{58}$$

$$g_i^m \geq 0, \; h_k \geq 0, \; k \in K_i, i \in I, m = au, tr \tag{59}$$

$$s_k(v) = \sum_{a \in A} \delta_{ak} s_a^m(v) \quad \forall k \in K_i, i \in I, m = au, tr , \tag{60}$$

yields a model that has the desired properties. An equilibrium flow is established on both the road and transit networks and the mode choice is consistent with the origin to destination travel times on both networks.

Due to the dependence between modes, this model is no longer equivalent to a convex cost minimization model and must be solved by algorithms designed for variational inequalities. Florian and Spiess (1983) suggested the Jacobi method and provided a sufficient condition for its local convergence (see Florian and Spiess, 1982).

Since 1979, the literature on the solution of variational inequalities has practically exploded and a variety of algorithms are available for solving such models. For some other contributions from the CRT, see Nguyen and Dupuis (1984), Marcotte (1986), Marcotte and Dussault (1987), Dussault and Marcotte (1989), Marcotte (1991), Wu, Florian and Marcotte (1991), Marcotte and Zhu (1993), Zhu and Marcotte (1994), Marcotte and Wu (1995), Marcotte and Zhu (1995), Zhu and Marcotte (1996), Goffin, Marcotte and Zhu (1997), Crouzeix, Marcotte and Zhu (2000) and Marcotte and Zhu (2001). It is somewhat regrettable that these contributions and those of other researchers in this area cannot be applied directly to the solution of more complex combined models because the complex multi-modal models formulated in practice are rather “ad-hoc” and not amenable to a neat mathematical model formulation. This will be further discussed below.

Other multi-modal models consider trips by “combined modes”. These are trips that start on one mode, say auto, and end on another mode, say bus. These are quite common in cities that have a developed transit system and adequate parking lots for parking the car before taking a transit service. Fernandez, DeCea and Florian (1994) formulated and solved a bi-modal model that considered pure modes and combined modes. Such models are quite common in practice now and have been enhanced to consider parking restrictions (see Florian and Los, 1980a). They require the computation of intermediate origin-destination matrices (see Florian and Los, 1980b): from an origin to a transfer point on a first mode, and from the transfer point to the destination on another mode. Florian, Wu and He (2002) re-formulated and developed an algorithm for a multi-class multi-modal planning model for the City of Santiago, Chile.

6.2 Large-Scale Combined Models

The development of transportation planning models was greatly influenced by the increasing speed of various computing platforms. In particular, greater speed, RAM availability and disk storage capacity led to more large-scale models. The need to better model the demand led to the proliferation of trip distribution models, mode choice models and multi-class network equilibrium models to achieve travel demand forecasting in an urban area. For instance, the Southern California Association of Governments SCAG) model of the year 2000 has the following characteristics:

- 13 categories for trip generation and attraction (by income, by trip purpose)

- 7 trip distribution models (purpose)
- 7 mode choice models (by purpose) for 11 modes (and combined modes)
- 6 classes of vehicular traffic (single occupancy, multiple occupancy, light trucks, heavy trucks, etc
- 6 transit classes (workers, school children, etc.)
- 2 combined modes: auto to local bus, local to express bus

There are 3339 zones, 30,678 nodes, 109,770, links and 1093 transit lines with 65,417 transit line segments in this network. The complexity of the models used is illustrated in the block diagram of Figure 5. Figures 6 and 7 are network plots of the SCAG road network used in the planning model. The algorithm implied by the block diagram in Figure 5 computes initial travel times from origins to destinations for all the modes considered, then average trip travel times are determined by using the mode choice models (log sums); then the trip distribution models are executed to obtain total travel demand matrices; then the mode choice models are applied to determine the demand by mode; then the resulting demand matrices for each mode are used for the auto and transit assignment. The procedure is repeated by using a ‘feed back” scheme in the search of an approximate equilibrium solution.

The only way to equilibrate such a complex model is by using a heuristic method such as the Method of Successive Averages (MSA) to dampen the oscillations in the link flows and origin to destination travel times. Such a heuristic method is referred to as “feedback” since the link flows and origin to destination travel times are “fed back” from one iteration to the next in an averaging scheme.

There are many variants of the MSA that are used in practice. The basic method starts from a feasible solution; the travel times are updated and a new auxiliary solution is computed. An updated solution is computed by combining the current solution and the auxiliary solution by using a heuristic step size λ .

New solution = Current solution *(1- λ step size)+ Auxiliary Solution * λ

The most common choices for the step size used are: $1/k$, where k is the iteration number and $1/const$. where $const$ is a predetermined number (2,3,..5). The latter is referred to as exponential smoothing since the early solutions have an exponentially decreasing weight. The convergence of these “feedback” methods are monitored by using a measure of gap which is inspired from the measures of gap used in optimization and/or variational inequality models. If v is the vector of link flows for all classes, s is the vector of link travel times for all classes, g is the demand vector for all classes and u is the vector of origin to destination travel times for all classes, then a measure of relative gap is $(\sum_a s_a v_a - \sum_i g_i u_i / \sum_a s_a v_a)$. When an equilibrium solution is reached, this measure of gap is zero. In practice, one accepts solutions that are of the order of 0.01 to 0.05.

The convergence of averaging schemes has been proven under some sufficient conditions, but is nearly impossible to verify on large-scale models. The important point to note is that all such complex models are based on the trip distribution, mode choice, road and transit assignment models described earlier.

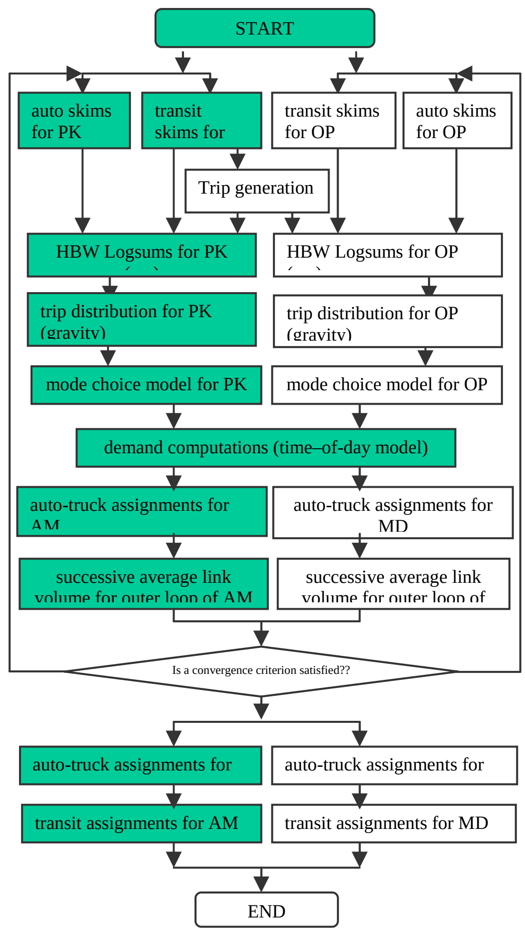

Figure 5. The SCAG multi-modal multi-period model

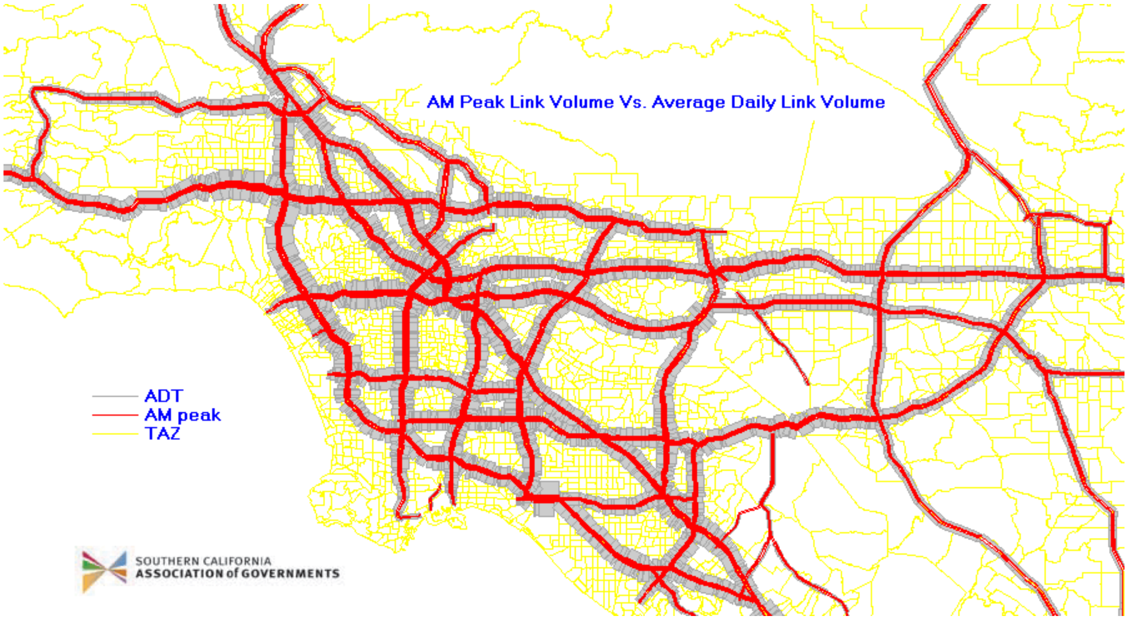

Figure 6. AM peak auto volumes on the SCAG network

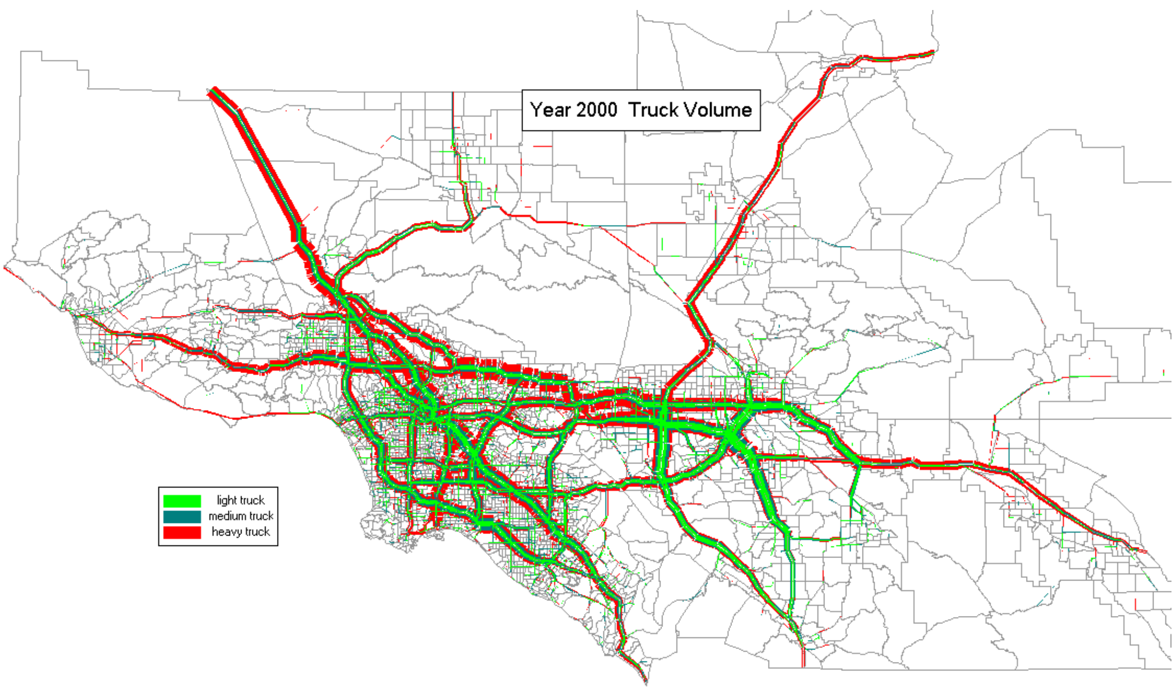

Figure 7. AM Peak truck flows on the SCAG network

7. O-D matrix adjustment

Transportation planning models are very data hungry. The calibration of demand models is usually done with data originating from home surveys on the travel patterns of a sample of households. This data is quite expensive to collect in the developed world and is not, in general, done on a regular basis (the Montreal area is an exception as home travel surveys are done on a regular basis). Often, the origin-destination matrices that are used for the network assignments are out of date. However, traffic counts on the road network and on transit vehicles are carried out on a regular basis and are much less costly than a home survey. Hence the interest to use count data to adjust existing origin-destination matrices. The adjustment of an origin-destination (O-D) matrix by using observed flows (counts) on the links and turns of a transportation planning network has attracted the attention of many researchers. The methods proposed may be subdivided into two categories depending on whether the network considered is assigned constant travel times or flow-dependent travel times.

Some of the contributions made at the CRT for origin-destination matrix adjustment on uncongested networks include those of Spiess (1987). However, the more important contributions were made for the situation that the underlying route choice is that of a network equilibrium model. Nguyen (1977) proposed a method that was a precursor to future work as it only required the adjusted matrix to replicate the origin to destination travel times. Jornsten and Nguyen (1979) explored this approach as well. A survey paper by Cascetta and Nguyen (1988), which resulted from a collaboration between the CRT and several Italian Universities, provided a framework for this class of problems. However, some significant contributions were still to come.

This problem was formulated by Spiess (1990) as a bi-level optimization problem (or Mathematical Program with Equilibrium Constraints, MPEC). The multi-class O-D adjustment problem is given by:

$$Min\ Z(g) = \frac{1}{2}\sum_{m\in M}\sum_{a\in \hat{A}}(v_a - \hat{v}_a)^2 \tag{61}$$

Subject to

$$v = assign(g) \tag{62}$$

where $\hat{v}_a$ are the observed flows and $assign(g)$ is the notation used to indicate that the vector of flows v is the result of the equilibrium assignment of demand g. This assignment problem is:

$$Min\ F(v) = \sum_{a\in A}\int_0^{v_a} s_a(v)dv \tag{63}$$

Subject to

$$v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} h_k \qquad a \in A, \tag{64}$$

$$\sum_{k \in K_i} h_k = g_i \qquad i \in I, \tag{65}$$

$$h_k^m \geq 0 \qquad k \in K_i \ , \tag{66}$$

$$\delta_{ak}^m = \begin{cases} 1 \text{ if } a \in k \text{ for mode } m \\ 0 \text{ otherwise} \end{cases} \tag{67}$$

Spiess (1990) developed an approximate gradient method for the solution of this problem that has very good empirical convergence properties. The gradient method adjusts the origin-destination matrix to best fit the counts while introducing small changes to the matrix, which is a very desirable property. Chen (1994), Florian and Chen (1995) and Florian and Chen (1998) studied optimality conditions for bi-level optimization problems and experimented with other algorithms for this problem. None were as efficient as the gradient method. However, this research revealed the mathematical structure of this model.

The gradient method has been implemented in practice and was recently extended for the simultaneous adjustment of the origin-destination matrices of several classes of traffic by Noriega and Florian (2007).

8. Dynamic Network Equilibrium

8.1 The formulation of dynamic network equilibrium models.

The applications of network equilibrium models in practice have been successful. However, this large body of practical experience has revealed an important shortcoming of this class of models for analyzing temporal phenomena. The flows that result from a network equilibrium model describe the average flows during a time period and do not model the formation and dissipation of queues since, for a given link, the outflow equals the inflow. Hence, there was always a need to extend the methodology to describe the evolution of traffic over time while adhering to an equilibrium concept of route choice.

The paper by Friesz et al (1993) provides a formulation of an equilibrium dynamic traffic model. Even though the formulation of the dynamic network equilibrium model was originally stated in continuous time, the model presented here adopts a temporal discretization into periods $\tau = 1, 2, ..., \left| \frac{T_d}{\Delta t} \right|$, where Δt is the chosen duration of a departure time interval. Traffic is assumed to depart origins within each time period, hence the demand is time varying. This results in a time discrete model.

The mathematical statement of a time discrete version of the dynamic equilibrium problem in the space of path flows h_k^τ, for all paths k belonging to the set K_i for an origin-destination $i \in I$, at time τ. The time-varying demands are denoted $g_i^\tau, i \in I, all\ \tau$ .The path flow rates in the feasible region Ω satisfy the conservation of flow and non-negativity constraints

$$\Omega^\tau = \{h_k^\tau : \sum_{k \in K_i} h_k^\tau = g_i^\tau, i \in I, all\ \tau\, ;\ h_k^\tau \geq 0, k_i, i \in I, all\ \tau\}. \qquad (68)$$

A temporal version of Wardrop's (1952) user-optimal route choice results in the model:

$$\begin{aligned} & h_k^\tau \in \Omega, u_i^\tau(t) = \min_{k \in K_i} \left\{ s_k^\tau(t) \right\} \\ & s_k^\tau = u_i^\tau \ if\ h_k^\tau > 0 \qquad\qquad k \in k_i,\ i \in I, \tau = 1, 2, ..., \left| \frac{T_d}{\Delta t} \right| \\ & s_k^\tau \geq u_i^\tau \ if\ h_k^\tau = 0, \end{aligned} \qquad (69)$$

which can be shown to be equivalent to solving the discrete variational inequality.

$$\sum_\tau \sum_{k \in K} s_k^\tau (h^\tau *)\,(h_k^\tau - h_k^\tau *) \geq 0 \qquad (70),$$

where $K = \bigcup_{i \in I} k_i$ where h^τ is the vector of path flows (h_k^τ) for all k and τ.

The demonstration of the existence and uniqueness of a solution to this model depends on the properties of the mapping ***s(h[g]),*** that is the dependence of link and path travel times on the path input flows and the dependence of the path input flows on the link and path travel times. These mappings depend on the models used to propagate the time varying traffic on the links of the network. In general, the properties of these mapping are not easily verified. The equilibrium principle is used as a **guide** in computing an approximate solution of the time discrete variational inequality.

In order to provide a framework for algorithmic approaches to the solution of dynamic traffic assignment (DTA) models, it is convenient to refer to the main components of any dynamic traffic model: the route-set generation method, the determination of the path input flows and the network-loading mechanism. The latter is the method used to represent the evolution of the traffic flow over the links of the network once the route choice and the path input flows have been determined. A schematic way to represent the general algorithmic approach is given in Figure 8 below.

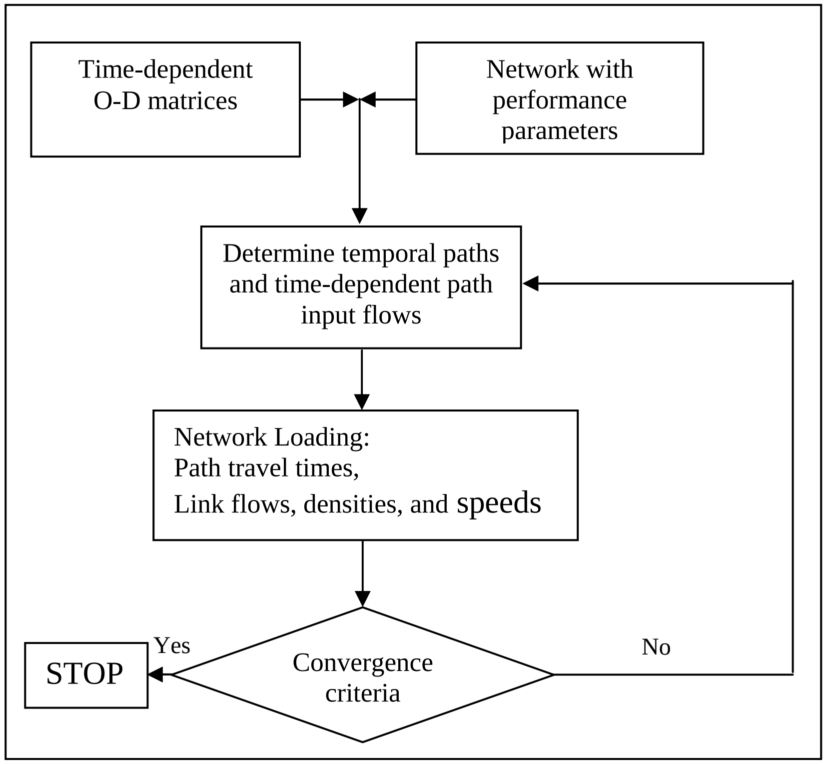

Figure 8. Structure of solution algorithms for the DTA model

8.2 Contributions to the solution of dynamic traffic equilibrium models.

There were several approaches to the solution of the network loading problems that were explored at the CRT in research that lead to several theses: Er-Rafia (2000), who studied the extensions of an analytical network loading model with explicit capacity constraints; Velan (2002), who studied extensions to the cell transmission model, which is a numerical method used to solve the hydrodynamic model of traffic flow; Mahut (2002), who developed an efficient event-based simulation of vehicles by using a simplified car following model; Er-Rafia (2000), Rubio-Ardanaz (2002) who studied analytical methods for dynamic network loading.

The results of this line of research were presented in several articles: Wu, Chen and Florian (1997), Adamo et al (1999), Astarita et al (2001), Rubio-Ardanaz, Wu and Florian (2003), Velan and Florian (2002). The network loading model of Mahut (2002) was integrated in an algorithm for DTA that proved to provide good approximations to dynamic user equilibrium with reasonable computing times. The early work with this method was reported in Mahut, Florian and Tremblay (2002) and Mahut et al (2004).

The approach taken in the DTA method based on Mahut's network loading model uses a detailed lane based representation of the traffic model. The physical network is defined by links and nodes. Each link is defined by its length, number of lanes and free-flow speed. Additional lanes on intersection approaches, for left and right turns, bus stops, etc.

are required and are appropriate to the fidelity of the traffic simulation model. Similarly, at each node, each turning movement is defined by the lanes on its upstream (incoming) and downstream (outgoing) links that are permitted for the movement, along with a maximum turning speed. Maximum (free-flow) speeds on links and turning movements, when combined with the physical parameters of vehicle length and driver response time, produce the well-known fundamental relationship of traffic flow for the car-following model used in the traffic simulator. As a result, per lane flow capacities, storage (density) capacities, and negative (backward moving) shock-wave speeds are all determined by the specification of the maximum speed, vehicle length, and driver response time.

The model does not use geometrical information such as intersection size and shape, or the radii of curvature of the turning movements. Each lane of a link, and each turning movement, can be restricted to a subset of the vehicle classes, permitting the modeling of HOV (high-occupancy vehicle) lanes, or reserved lanes for buses and/or taxis, etc. The model also permits the specification of detailed traffic control information such as (pre-timed) signal timing and ramp metering plans.

The demand is defined by a time-sliced O-D matrix for each vehicle class. Each vehicle class is comprised of one or more vehicle *types*, which are distinguished by the physical attributes of the vehicle *effective length* and the driver/vehicle *response time*, as discussed above.

The traffic simulation model used in this model was designed to produce reasonably accurate results with a **minimum** number of parameters and a **minimum** of computational effort (Astarita et al., 2001). The underlying structure of the network model and the car moving logic have more in common with microscopic than with mesoscopic approaches, as it is designed to capture the effects of car following, lane changing and gap acceptance. This method could be characterized as a *simplified* microscopic model, as it employs less complex variants of the car-following, lane-changing and gap-acceptance models implemented in micro-simulation software packages that are intended for more detailed traffic modeling. The route finding part of the algorithm is based on computing temporal shortest path and distributing flow among the kept paths by using a variant of the method of successive averages (MSA and a flow balancing method) among the used paths which is akin to the projected gradient method in static network equilibrium models.

The complexity of the traffic simulation, which is motivated by a desire for a realistic representation of the system, results in an assignment map that is discontinuous and difficult to characterize analytically. Nevertheless, the algorithm has been found to work well in practice for finding approximate dynamic equilibrium conditions on real-world networks of significant size.

Several applications have been carried out with this model, now called Dynameq (for DYNamic Equilibrium).

9. Software development

With the establishment of the CRT, the research areas were expanded to include pilot computer-based implementations of the methods developed and applications on data that originated from practice. The equilibrium route choice models were tested first with data from the City of Winnipeg. The code that was first used to solve the NEM model with the linear approximation method was called TRAFIC and was authored by Sang Nguyen and Linda James. It served to carry out the calibration and validation of the model with the data originating from the City of Winnipeg, mentioned above. At about the same time, Robert Chapleau developed a transit assignment model called TRANSCOM that implemented a route choice method mentioned above as well. Chapleau developed a transit assignment code named TRANSCOM (see Chapleau and Trottier, 1978) that implemented the method developed in his doctoral thesis and was used by the Montreal Transit Commission. Chapleau went on to establish the MADITUC research group at Ecole Polytechnique in Montreal and continued research on transit assignment and related topics.

The computer software of that time (1972-1976) considered the various modes of transportation in an urban area as separate models. The integration of demand models with network models in a single model that would describe all choices made in a city regarding destination, mode and route choice was an innovative idea at the time. In order to test such an integrated model, the CRT received a grant form the Transportation Development Agency of Transport Canada for a demonstration project that would test its efficiency, validity and applicability.

At the time, there was considerable skepticism regarding the use of rigorous algorithms for congested traffic assignment as heuristic methods such as incremental assignment and variants of the so-called “capacity constrained” method were used in practice. This motivated the development of experimental software that would consider all the components of the traditional planning process as a single integrated model that could be solved with an iterative method, referred to colloquially as “feedback”, which would achieve a properly equilibrated solution.

The software was named EMME, for Equilibre Multimodal-Multimodal Equilibrium, a research model that resulted from a project that lasted from 1976 to 1979. It was implemented computationally as a FORTRAN batch code for validation and calibration of the model. The algorithm for solving the network assignment was based on TRAFIC and the transit assignment was based on TRANSCOM. The project was completed in 1979 and a report of the results of the project was published. The results were summarized in Florian et al. (1979). This contribution, although initially judged as being of purely academic interest, eventually made its way into practice.

A second stage in the development of the methodology and software was initiated in 1980 when Heinz Spiess started the work leading up to his doctoral thesis. The development team also included Andre Babin and Linda James-Lefebvre. The aim was to develop robust software for transportation planning that could be used outside the university lab. The EMME experimental code was entirely rewritten and implemented for interactive-graphic use on a CDC Cyber mainframe by using Tektronix graphic terminals. It was renamed EMME/2 to distinguish it from the earlier EMME batch code and was presented in Babin et al (1982). Heinz Spiess (see Spiess, 1983) was a principal contributor to the interactive-graphic version of EMME.

A first version of EMME/2 was developed between 1980 and 1983. The dependence on mainframe computing was a limitation that was addressed by writing all the code for 32 bit computing platforms, which were yet to emerge. The dependence on the IBM mainframe computers of the time, which would be replaced by personal computers, provided an opportunity to develop software that, in the near future, would no longer depend on expensive installations.

During this period, a new transit route choice method was developed that was based on the notion of “strategies” (see Spiess, 1983 and Spiess and Florian, 1989). It was implemented in EMME/2. At that time, Isabelle Constantin joined the EMME/2 team and contributed by developing a disaggregate transit route choice method that analyses individual trips from any origin to any destination given by their coordinates (see Constantin, 1986). Diane Larin joined the team as well and worked on the handling of polygon objects (see Larin, 1988).

Throughout its history, EMME/2 has been offered for a variety of operating systems; recently it was offered on four flavors of UNIX, as well as MS DOS (on Definicon co-processor boards), MS Windows 2000 and XP and Linux. In contrast, most other software is only available for MS operating systems. The early versions of the IBM PC did not support 32 bit computing and offered very limited RAM. In order to address larger scale problems, without the restriction of the INTEL 286 computing platform, EMME/2 was adapted to run on Definicon co-processor boards which were essentially 32 bit mini-computers on a PC-AT expansion card. EMME/2 was used on these co-processor boards until the emergence of the Intel 386 chip and its 32 bit successors. A version of EMME/2 which could be run on an IBM PC-AT was made available in 1985 but was quickly superseded by the 32 bit version.

INRO (www.inro.ca), which was founded in 1976, eventually obtained the distribution rights for the EMME/2 software in 1984 and continued its development and support independently of the Centre for Research of Transportation of the University of Montreal.

EMME/2 evolved rapidly during 1983-2004 (from Release 2 to Release 9) into a toolkit for modeling multimodal transportation networks, with all modes integrated into a consistent network with full integration of transit and car modes. EMME/2 featured: matrix manipulation tools that allowed implementation of a wide variety of travel demand models; assignment methods based on sound theories; interactive calculators for implementation of evaluation and impact analysis methods; a powerful macro language for automating repetitive procedures; comprehensive graphic display capabilities; and new interactive/graphic network editors in Emme 3 (INRO, 2007).

Specific examples of additions to its modeling and graphic capabilities include:, the implementation of virtually any spatial interaction model; multi-class assignment with generalized costs; comprehensive path analysis capabilities; triple index matrix operations; O-D matrix adjustment for highway and transit; congested and capacitated transit assignments; and stochastic road and transit assignments. The modular nature of the code permitted the development of many model variants, without writing new code, by developing macro procedures which use the computational building blocks of the software. For instance, the transit assignment methods that take into account congestion were implemented as macro procedures as well as the gradient method for adjusting origin-destination matrices by using counts. Also the combined distribution-assignment model was implemented as a macro (see Metaxatos et al, 1995). In general, very complex multi-mode, multi-class models have been implemented in EMME/2 in many important cities in 5 continents.

From 2005 to the present, INRO undertook the development of Emme 3, which provides the same modeling capabilities as EMME/2 with improved path analysis capabilities for multi-class assignments, and adds new interactive graphic editors, the use of media (images, shape files, dbf files, etc.) and interfaces with various GIS data. Emme 3 was released as a beta version software in March 2006 and the formal release occurred in February 2007.

By the end of 2007, Emme 3 was being used in 77 countries by over 900 organizations, including cities, metropolitan areas, and various levels of public administration, transit agencies, consulting firms and universities in intra-city as well as inter-city applications.

The dynamic traffic assignment method based on Mahut's (2002) network loading model and associated research on empirical convergence of various solution methods (see Mahut et al, 2007) was integrated by INRO into a software package called Dynameq. The software was distributed first in March 2006 and is now used by approximately 50 urban areas in various countries.

Conclusions

Some of the principal contributions made by CRT researchers to advance the state-of-the-art of the models and algorithms used for travel demand forecasting in an urban transportation planning context have been presented. The contributions of the CRT are not only theoretical. Some of the new theories developed have been implemented in software that is used outside the university laboratories for planning urban areas.

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