CIREQ

Centre interuniversitaire de recherche en économie quantitative

Cahier 04-2020

Blocking Pairs versus Blocking Students :
Stability Comparisons in School Choice

Battal Doĝan and Lars Ehlers

Université Concordia UNIVERSITY McGill Université de Montréal Fonds de recherche Société et culture Québec

Le **Centre interuniversitaire de recherche en économie quantitative (CIREQ)** regroupe des chercheurs dans les domaines de l'économétrie, la théorie de la décision, la macroéconomie et les marchés financiers, la microéconomie appliquée ainsi que l'économie de l'environnement et des ressources naturelles. Ils proviennent principalement des universités de Montréal, McGill et Concordia. Le CIREQ offre un milieu dynamique de recherche en économie quantitative grâce au grand nombre d'activités qu'il organise (séminaires, ateliers, colloques) et de collaborateurs qu'il reçoit chaque année.

*The **Center for Interuniversity Research in Quantitative Economics (CIREQ)** regroups researchers in the fields of econometrics, decision theory, macroeconomics and financial markets, applied microeconomics as well as environmental and natural resources economics. They come mainly from the Université de Montréal, McGill University and Concordia University. CIREQ offers a dynamic environment of research in quantitative economics thanks to the large number of activities that it organizes (seminars, workshops, conferences) and to the visitors it receives every year.*

Cahier 04-2020

Blocking Pairs versus Blocking Students : Stability Comparisons in School Choice

Battal Doĝan and Lars Ehlers

Blocking Pairs versus Blocking Students: Stability Comparisons in School Choice[*]

Battal Doğan[†]   Lars Ehlers[‡]

April 8, 2020

Abstract

It is known that there are school choice problems without an *efficient* and *stable* assignment. We consider comparing assignments in terms of their stability by comparing their sets of blocking (student-school) pairs or comparing their sets of blocking students who are involved in at least one blocking pair. Although there always exists a Pareto improvement over the student-optimal stable (DA) assignment which is *minimally unstable among efficient assignments* when the stability comparison is based on comparing the sets of blocking pairs in the set-inclusion sense, we show that this is not necessarily true when the stability comparison is based on comparing the sets of blocking pairs in the cardinal sense, or when it is based on comparing sets of blocking students (in the set-inclusion or cardinal sense). Given the latter impossibilities, we characterize the priority profiles where there exists a Pareto improvement over the DA mechanism which is cardinally minimally stable among efficient assignments when counting blocking pairs or counting blocking students. The resulting domain restrictions suggest to take with caution school choice analysis which relies on a particular stability comparison method.

JEL Classification: C70, D47, D61, D63

Keywords: School choice, stability comparisons, minimal instability, deferred acceptance.

[†]Department of Economics, University of Bristol, 3B13 The Priory Road Complex, Priory Road, Clifton, BS8 1TU. Email: battal.dogan@bristol.ac.uk.

[‡]Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, QC H3C 3J7, Canada. Email: lars.ehlers@umontreal.ca.

1   Introduction

A school choice problem consists of a set of students and a set of schools such that each student has a preference ordering over schools, and each school has a capacity and a priority ordering over students. How to assign students to schools in a desirable way turns out to be a nontrivial question, which has led to an extensive school choice literature starting with the seminal study by Abdulkadiroğlu and Sönmez (2003).

A clear indication that school choice problems are nontrivial is the incompatibility of (Pareto) *efficiency* and *stability*—two natural and desirable properties. An assignment is *efficient* if there is no other assignment at which a student is better off while no student is worse off. An assignment is *stable* if it does not involve a "blocking pair" of a student and a school such that the student prefers the school to his assigned school and he has a higher priority than another student who is assigned to that school. Although an *efficient* assignment can always be found, for example, by a serial dictatorship algorithm (Svensson, 1999), and a *stable* assignment can always be found, for example, by the *deferred acceptance algorithm* (DA) (Gale and Shapley, 1962), unfortunately, there exist school choice problems without an assignment that is both *efficient* and *stable* (Roth, 1982).

In this paper, we insist on *efficiency* and investigate assignments which are *minimally unstable among efficient assignments*: such assignments are *efficient* and there is no other *efficient* assignment which is *more stable*. To formulate what it means to be *more stable*, we consider comparing assignments by comparing their sets of blocking (student-school) pairs or comparing their sets of blocking students who are involved in at least one blocking pair. According to the first method, an assignment is *more stable* than another assignment if the set of blocking pairs in the former is a proper subset of the set of blocking pairs in the latter assignment. This method has a corresponding cardinal version such that an assignment is *cardinally more stable* than another assignment if the number of blocking pairs in the former is less than the number of blocking pairs in the latter assignment.[1] According to the second method, an assignment is *blocking-student–wise (BS-wise) more stable* than another assignment if the set of blocking students in the former is a proper subset of the set of blocking students in the latter assignment. This method also has a corresponding cardinal version such that an assignment is *BS-wise cardinally more stable* than another assignment if the number of blocking students in the former is less than the number of blocking students in the latter assignment.

Stability has been a central desirable property in school choice, both in theory and in

---

[1]Clearly, if an assignment is *more stable* than another assignment, than it is also *cardinally more stable*.

applications, especially because a student who is involved in a blocking pair is being treated unfairly which makes the assignment open to criticism on fairness grounds, and even on legal grounds since the student may pursue legal action against the school district. Although stability and efficiency are not always compatible, the DA mechanism always chooses the student-optimal stable assignment, which is a reasonable solution if stability concerns come first and efficiency second. If efficiency concerns come first and instability is inevitable, comparing assignments in terms of their stability by comparing their sets of blocking pairs or blocking students, both in set-inclusion and cardinal ways, are all reasonable methods.[2]

The efficiency adjusted deferred acceptance mechanism (EADA) due to Kesten (2010), which Pareto improves over the DA mechanism, turns out to be *minimally unstable among efficient assignments*, i.e., at each problem, it produces an assignment that is *minimally unstable among efficient assignments*.[3] However, we show that EADA is not *cardinally minimally unstable among efficient assignments*. Even more, there is no mechanism which Pareto improves over DA and which is *cardinally minimally unstable among efficient assignments* (Proposition 2). It turns out that there is no implication relation between any of the blocking–student-wise notions and any of the blocking–pair-wise notions. More interestingly, the EADA mechanism is not *BS-wise minimally unstable among efficient assignments*, and therefore also not *BS-wise cardinally minimally unstable among efficient assignments*. Even more, there is no mechanism which Pareto improves over DA and which is *BS-wise (cardinally) minimally unstable among efficient assignments* (Proposition 1).

Above results suggest that the EADA mechanism's failure of cardinal minimal instability while satisfying minimal instability cannot be solely attributed to the cardinal feature of the comparison method. We also show that there exist priority profiles for which the EADA mechanism is BS-wise cardinally minimally unstable but not necessarily cardinally minimally unstable, and whenever the EADA mechanism is cardinally minimally unstable, then it is BS-wise cardinally minimally unstable. Thus, the conclusions are opposite for the inclusion comparison and the cardinality comparison (of blocking pairs and blocking students). All these results suggest that several conclusions related to minimal instability are sensitive to the choice of the stability comparison method.[4]

We also investigate restricting the domain of priority profiles. For the unit-capacity case, we characterize the priority profiles for which there exists a Pareto improvement over the DA

---

[2]Ehlers and Morrill (2020) provide a thorough analysis of legal assignments in school choice.

[3]This result is also proven in Tang and Zhang (2020). We present our proof which is independent and different.

[4]Not all conclusions have to be sensitive, however. In Doğan and Ehlers (2020), we show that a result by Abdulkadiroğlu et al. (2020), which relies on a particular stability comparison method, is in fact robust to the choice of the stability comparison method.

assignment which is *cardinally minimally unstable among efficient assignments* (Theorem 1).[5] Our characterization result provides two important insights.

1. For any priority profile for which there exists a Pareto improvement over the DA assignment that is *cardinally minimally unstable among efficient assignments*, the DA assignment always includes at most one *improvement cycle*[6] and there is a unique efficient Pareto improvement over the DA assignment. This result suggests that such priority profiles are quite restricted. In other words, if one restricts himself to Pareto improvements over the DA mechanism, it is essentially impossible to guarantee an assignment that is *cardinally minimally unstable among efficient assignments*.

2. Our characterization result fully uncovers the three potential reasons why the EADA assignment may fail to be *cardinally minimally unstable among efficient assignments*. Each potential reason corresponds to the violation of one of the three conditions in Theorem 1.

Also, for the unit-capacity case, we characterize the priority profiles for which there exists a Pareto improvement over the DA assignment which is *BS-wise cardinally minimally unstable among efficient assignments* (Theorem 2), which results in similar insights as above: For any "possibility priority profile", there is a unique efficient Pareto improvement over the DA assignment, which includes exactly one blocking student; and our characterization result again fully uncovers all potential reasons why the EADA assignment fails.

2 Related Literature

Ergin (2002) derived necessary and sufficient conditions (on the capacity-priority profile) for the efficiency of the DA mechanism. The EADA mechanism (Kesten, 2010) is efficient[7] but not necessarily cardinally minimally unstable when counting blocking pairs or counting blocking students. Here we asked a parallel question to Ergin (2002): when is the EADA-mechanism cardinally minimally unstable?

Our methods to compare assignments by their stability are inspired by comparison methods in recent studies (here, we provide a non-exhaustive list). In Pathak and Sönmez (2013),

---

[5]The characterizing conditions are still necessary conditions for the EADA assignment to be *cardinally minimally unstable among efficient assignments* in the general multi-capacity case (Appendix A.2), and therefore the main insights extend to the general setup.

[6]See Section 4 for a definition of an improvement cycle

[7]Doğan and Yenmez (2020), Dur et al. (2019), Ehlers and Morrill (2020), Kwon and Shorrer (2019), Tang and Zhang (2020), and Troyan et al. (2020) provide different other justifications for EADA.

school choice mechanisms are compared in terms of their *manipulability* by comparing the sets of problems, in the set inclusion sense, at which they are manipulable. In Andersson et al. (2014), resource allocation mechanisms (in the model of allocating objects with monetary transfers) are compared also in terms of their *manipulability*, but by comparing *cardinalities* of the sets of problems at which they are manipulable. Similar comparison methods have been used to compare manipulability of social choice functions (Maus et al., 2007b,a). In Doğan et al. (2018), probabilistic assignments are compared in terms of their *efficiency* by comparing the sets of consistent utility profiles, in the set inclusion sense, at which they are ex-ante efficient. Although our study uses similar comparison methods, we depart from these studies by focusing on *stability*.

The closest studies to ours are Abdulkadiroğlu et al. (2020),[8] Kwon and Shorrer (2019), and Tang and Zhang (2020), which also compare school choice mechanisms, and assignments, in terms of their stability. Although these studies also consider *minimally unstable assignments*, to our knowledge, our study is the first to consider *cardinally minimally unstable assignments* in school choice.

Tang and Zhang (2020) introduce the notion of *self-constrained optimality* for assignments, which requires that the assignment Pareto dominates any other assignment that is *more stable*, and show that the EADA assignment is *self-constrained optimal* at each problem. This result also implies that the EADA mechanism is *minimally unstable among efficient assignments*. Kwon and Shorrer (2019) introduce the notion of a blocking triplet which includes, in addition to a blocking pair, a student who violates the priority of the student in the blocking pair. Kwon and Shorrer (2019) show that the EADA mechanism is *minimally unstable among efficient assignments* also when stability comparison is based on comparing (in the set-inclusion sense) sets of blocking triplets. Different from Tang and Zhang (2020) and Kwon and Shorrer (2019), we also study *cardinally minimally unstable efficient assignments* and in particular show that the EADA mechanism may fail to choose such an assignment. Moreover, we also consider two alternative methods to compare assignments by their stability, based on comparing sets of blocking students (the set of students who are involved in at least one blocking pair) instead of blocking pairs or blocking triplets.

In another recent study, Combe et al. (2017) consider a teacher assignment problem where each teacher is initially endowed with a position at a school, and individual rationality and stability are incompatible. Combe et al. (2017) also compare assignments in terms of their stability by comparing the sets of blocking pairs in the set inclusion sense.

---

[8]We say more about Abdulkadiroğlu et al. (2020) in Section 6.

3   Model

Let N denote a finite set of students and C denote a finite set of schools. Each student $i \in N$ has a **preference ordering** R_i over $C \cup \{\emptyset\}$,[9] where $\emptyset$ represents an outside option for the student. The strict part of the preference ordering R_i is denoted by P_i, so if $c_1, c_2 \in C \cup \{\emptyset\}$, $c_1 \neq c_2$, and $c_1\, R_i\, c_2$, then $c_1\, P_i\, c_2$. School c is **acceptable** to student i if the student prefers it to the outside option, that is, $c\, P_i\, \emptyset$. Each school $c \in C$ has a **capacity** $q_c \in \mathbb{N}$, which is the maximum number of students that the school can admit, and a **priority ordering** $\succeq_c$ over the set of students N.[10] The strict part of the priority ordering $\succeq_c$ is denoted by $\succ_c$.

An assignment is a mapping $\mu : N \cup C \to N \cup C \cup \{\emptyset\}$ such that

(i) for each $i \in N$, $\mu(i) \in C \cup \{\emptyset\}$,

(ii) for each $c \in C$, $\mu(c) \subseteq N$ such that $|\mu(c)| \leq q_c$, and

(iii) for each $i \in N$ and each $c \in C$, $i \in \mu(c)$ if and only if $c = \mu(i)$.

Let $\mathcal{A}$ denote the set of all assignments.

An assignment μ is **individually rational** if for each $i \in N$, $\mu(i)\, R_i\, \emptyset$.

An assignment μ **Pareto improves** an assignment μ' if for each $i \in N$, $\mu(i)\, R_i\, \mu'(i)$ and there exists $j \in N$ such that $\mu(j)\, P_j\, \mu'(j)$. An assignment μ is **efficient** if it cannot be Pareto improved.

A pair $(i, c) \in N \times C$ **blocks** μ if $c\, P_i\, \mu(i)$ and $[|\mu(c)| < q_c$ or there exists $j \in \mu(c)$ such that $i \succ_c j]$. Let

$$B(\mu) = \{(i, c) \in N \times C : (i, c) \text{ blocks } \mu\}$$

denote the set of blocking pairs at μ and let $B_i(\mu) = B(\mu) \cap (\{i\} \times C)$ denote the set of blocking pairs at μ containing student i.

An assignment μ is **stable** if it is individually rational and includes no blocking pair. Unfortunately, there exist school choice problems without an assignment that is both *efficient* and *stable* (Roth, 1982). We investigate assignments which are *minimally unstable among efficient assignments* based on methods to compare assignments by their stability.

---

[9]Formally, a preference ordering over $C \cup \{\emptyset\}$ is a complete, transitive, and anti-symmetric binary relation over $C \cup \{\emptyset\}$. Binary relation R_i over $C \cup \{\emptyset\}$ is *complete* if, for every $c_1, c_2 \in C \cup \{\emptyset\}$, $c_1 R_i c_2$ or $c_2 R_i c_1$. It is *transitive* if, for every $c_1, c_2, c_3 \in C \cup \{\emptyset\}$, $c_1 R_i c_2$ and $c_2 R_i c_3$ imply $c_1 R_i c_3$. It is *anti-symmetric* if, for every $c_1, c_2 \in C \cup \{\emptyset\}$, $c_1 R_i c_2$ and $c_2 R_i c_1$ imply $c_1 = c_2$.

[10]The priority ordering $\succeq_c$ is a complete, transitive, and anti-symmetric binary relation over N. Our results extend to the more general setup where some students may be unacceptable for some schools.

A (school choice) **problem** P is a quintuple $(N, C, R, q, \succeq)$ where $R = (R_i)_{i \in N}$ denotes the (student) preference profile, $q = (q_c)_{c \in C}$ denotes the (school) capacity profile, and $\succeq = (\succeq_c)_{c \in C}$ denotes the (school) priority profile. We keep everything except the preference profile fixed, and for short a problem is denoted by R. Let $\mathcal{P}$ denote the set of all problems.

3.1 Stability Comparisons

A **stability comparison** is a function f associating with each problem $P \in \mathcal{P}$ a binary relation $f(P)$ over assignments.[11] Instead of $f(P)$, we write $\gtrsim_f^P$ (where $\mu \gtrsim_f^P \nu$ means that μ is f-more stable than ν at P). We say that μ is f-**minimally unstable at** P **among efficient assignments** if μ is efficient and there exists no efficient assignment ν such that $\nu \gtrsim_f^P \mu$. We will use the abbreviation **a.e.a.** for "among efficient assignments".

Below, we describe two natural methods for stability comparisons based on blocking pairs and blocking students. Each of them has an inclusion method and a (corresponding) cardinal method.

3.1.1 Blocking Pairs

The blocking pairs inclusion comparison ($pincl$) is defined as follows. For each problem $P \in \mathcal{P}$ and $\mu, \nu \in \mathcal{A}$,

$$\mu \gtrsim_{pincl}^P \nu \Leftrightarrow B(\mu) \subseteq B(\nu).$$

Among others, Abdulkadiroğlu et al. (2020) and Tang and Zhang (2020) study this stability comparison.

The blocking pairs cardinality comparison ($pcard$) is defined as follows. For each problem $P \in \mathcal{P}$ and $\mu, \nu \in \mathcal{A}$,

$$\mu \gtrsim_{pcard}^P \nu \Leftrightarrow |B(\mu)| \leq |B(\nu)|.$$

Obviously, for any problem P: (i) $\gtrsim_{pincl}^P \subseteq \gtrsim_{pcard}^P$, (ii) $\gtrsim_{pincl}^P$ is transitive but not complete, and (iii) $\gtrsim_{pcard}^P$ is complete (as any two assignments can be compared) and transitive.

We will use the convention to write *more stable* instead of *pincl*-more stable, *minimally unstable* instead of *pincl*-minimally unstable, *cardinally more stable* instead of *pcard*-more

---

[11]A binary relation over assignments is a subset $\gtrsim \subseteq \mathcal{A} \times \mathcal{A}$. We write $\mu \gtrsim \nu$ instead of $(\mu, \nu) \in \gtrsim$, and $[\mu \gtrsim \nu \Leftrightarrow \mu \gtrsim \nu \& \text{ not } \nu \gtrsim \mu]$. Let $\mathcal{L}$ denote the set of all binary relations. Given $\gtrsim \in \mathcal{L}$, (i) $\gtrsim$ is complete if for all $\mu, \nu \in \mathcal{A}$ we have $\mu \gtrsim \nu$ or $\nu \gtrsim \mu$ and (ii) $\gtrsim$ is transitive if $\mu \gtrsim \nu$ and $\nu \gtrsim \eta$ imply $\mu \gtrsim \eta$. Furthermore, given $\gtrsim, \gtrsim' \in \mathcal{L}$ such that $\gtrsim \subseteq \gtrsim'$ we say that $\gtrsim$ is coarser than $\gtrsim'$ and $\gtrsim'$ is finer than $\gtrsim$.

stable, and *cardinally minimally unstable* instead of *pcard*-minimally unstable.

3.1.2   Blocking Students

The blocking students inclusion comparison (*sincl*) is defined as follows. Let $BS(\mu) = \{i \in N : B_i(\mu) \neq \emptyset\}$. For each problem $P \in \mathcal{P}$ and $\mu, \nu \in \mathcal{A}$,

$$\mu \succsim_{sincl}^{P} \nu \Leftrightarrow BS(\mu) \subseteq BS(\nu).$$

The blocking students cardinality comparison (*scard*) is defined as follows. For each $P \in \mathcal{P}$ and $\mu, \nu \in \mathcal{A}(P)$,

$$\mu \succsim_{scard}^{P} \nu \Leftrightarrow |BS(\mu)| \leq |BS(\nu)|.$$

Obviously, (i) $\succsim_{sincl}^{P} \subseteq \succsim_{scard}^{P}$, (ii) $\succsim_{sincl}^{P}$ is transitive but not complete and (iii) $\succsim_{scard}^{P}$ is complete and transitive.

We will use the convention to write *BS-wise more stable* instead of *sincl*-more stable, *BS-wise minimally unstable* instead of *sincl*-minimally unstable, *BS-wise cardinally more stable* instead of *scard*-more stable, and *BS-wise cardinally minimally unstable* instead of *scard*-minimally unstable.

Remark 1 *For any stable assignment μ, there is no other assignment which is (cardinally) more stable or BS-wise (cardinally) more stable than μ. However, μ is not necessarily minimally unstable among efficient assignments since μ may not be efficient.*

3.2   Mechanisms: DA and EADA

A **mechanism** associates each problem with an assignment. When we say that a mechanism satisfies a certain assignment property (such as *efficiency* or *minimal instability among efficient assignments*), we mean that at each problem, the assignment prescribed by the mechanism satisfies the property.

The deferred acceptance (DA) mechanism due to Gale and Shapley (1962) is used in many school districts that have reformed their school choice systems. The **DA mechanism** associates each problem P with the assignment determined by the following **deferred acceptance algorithm**.

Deferred Acceptance (DA) Algorithm:

Step 1. Each student proposes to her top-ranked acceptable school. If there is no such school, then she is assigned to her outside option. Each school c considers the set of proposals that it receives. Among them, it tentatively accepts the highest priority students up to its capacity and rejects the others. If there is no rejection, then stop.

Step $t \geq 2$. Each student who is rejected at Step $t-1$ proposes to her top-ranked acceptable school among the ones that have not rejected her yet. If there is no such school, then she is assigned to her outside option. Each school c considers the set of students that it tentatively accepted at Step $t-1$ together with students that have proposed at Step t. Among them, it tentatively accepts the highest priority students up to its capacity and rejects the others. If there is no rejection, then stop. Otherwise, move to Step $t+1$.

The DA algorithm stops in finitely many steps and the DA assignment, which we denote by $DA(P)$, is defined by the acceptances at the last step. At each problem, the DA assignment is *stable* but not necessarily *efficient* (Abdulkadiroğlu and Sönmez, 2003).

The **efficiency-adjusted deferred acceptance (EADA) mechanism** (Kesten, 2010) is based on the EADA algorithm which works by iteratively removing certain schools from the preference orderings of certain students and rerunning the DA algorithm. Instead of providing Kesten's original definition of the EADA algorithm, we provide a simplified version due to Tang and Yu (2014). Given an assignment μ, a school $c \in C$ is **underdemanded** at μ if no student strictly prefers it to his assigned school. We adopt the convention that for each student $i \in N$, his outside option is underdemanded at any assignment.

Efficiency-Adjusted Deferred Acceptance (EADA) Algorithm:

Round 0. Run DA for the problem $P = (N, C, R, q, \succeq)$.

Round $r \geq 1$. Identify underdemanded schools at the outcome of Round $r-1$. Let I_r denote the set of students who are assigned to underdemanded schools (including the students who are assigned to their outside options). Let μ^r denote the restriction of the outcome of Round $r-1$ to I_r (note that μ^r includes only the underdemanded schools and students in I_r.) Remove these schools and I_r from the problem. Stop if there are no remaining schools. Otherwise, run DA for the reduced problem. Move to the next round, Round $r+1$.

The EADA algorithm stops in finitely many rounds, say in m rounds. The EADA assignment is defined as the collection of $\mu^1, \ldots, \mu^m$. That is, $EADA(P) = \mu^1 \cup \cdots \cup \mu^m$. For

each $t \in \{1, \ldots, m\}$, let μ_t denote the assignment obtained by collecting $\mu^1, \ldots, \mu^t$ together with the DA assignment for the reduced problem at the end of Round t. In other words, μ_t is the assignment obtained by iterating the EADA algorithm for only t rounds. Note that $\mu_m = EADA(P)$, and μ_m contains at least as many blocking pairs as steps where at least one student is assigned a different school than at the previous step: if there is an additional step and a student is assigned a different school, then there is a Pareto improvement which was blocked by some student-school pair at the previous step.

At each problem, the EADA assignment is *efficient* and Pareto improves the DA assignment (Tang and Yu, 2014; Kesten, 2010).

Remark 2 *Given a problem, if the EADA assignment coincides with the DA assignment, then the EADA assignment is the unique assignment that is (BS-wise) (cardinally) minimally unstable among efficient assignments.*

4 Minimal Instability of Efficient Improvements of DA

We explore the minimal instability of Pareto improvements of DA according to our stability comparisons based on blocking pairs and blocking students. We do this for inclusion comparisons and the corresponding cardinal comparisons.

The following result shows that there exists a mechanism which is a Pareto improvement over the DA mechanism and minimally unstable among efficient assignments for the blocking pairs inclusion comparison. In particular, the EADA mechanism is minimally unstable among efficient assignments for the blocking pairs inclusion comparison.[12] Unfortunately, this is not true for the blocking students inclusion comparison.

Proposition 1 (Inclusion Comparisons and EADA) *(i) The EADA mechanism is minimally unstable among efficient assignments.*

(ii) The EADA mechanism is NOT BS-wise minimally unstable among efficient assignments. In particular, there is no efficient mechanism which is both a Pareto improvement over the DA mechanism and BS-wise minimally unstable among efficient assignments.

Proof. We show (i) in Appendix A.1.

---

[12]Part (i) also follows from Tang and Zhang (2020). We thank Szilvia Papai for bringing this to our awareness. We present our proof, which is independent and different than the one in Tang and Zhang (2020), in Appendix A.1.

In showing (ii), consider the following problem P: let $N = \{1,2,3,4,5,6,7\}$, $C = \{c_1,c_2,c_3,c_4,c_5,c_6\}$ (all with unit capacities), and

R_1	R_2	R_3	R_4	R_5	R_6	R_7	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\succeq_{c_4}$	$\succeq_{c_5}$	$\succeq_{c_6}$
c_1	c_1	c_3	c_3	c_4	c_1	c_6	7	2	5	4	1	3
c_5	c_2	c_2	c_4	c_3	$\emptyset$	c_5	6	3	3	5	7	7
$\emptyset$	$\emptyset$	c_1	$\emptyset$	$\emptyset$		c_1	3	$\vdots$	4	$\vdots$	$\vdots$	$\vdots$
		c_6				$\emptyset$	1		$\vdots$			
		$\emptyset$					2					
							$\vdots$					

Note that
$$DA(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ c_5 & c_2 & c_6 & c_4 & c_3 & \emptyset & c_1 \end{pmatrix}$$

$$EADA(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ c_5 & c_1 & c_2 & c_3 & c_4 & \emptyset & c_6 \end{pmatrix}$$

and $BS(EADA(P)) = \{1,3,6\}$ (since $B(EADA(P)) = \{(1,c_1),(3,c_3),(6,c_1)\}$).

Consider the following assignment μ:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ c_1 & c_2 & c_6 & c_3 & c_4 & \emptyset & c_5 \end{pmatrix}$$

where $BS(\mu) = \{3,6\}$ (since $B(\mu) = \{(3,c_1),(6,c_1)\}$). Note that μ is efficient and Pareto improves over DA. Moreover, μ is BS-wise more stable than $EADA(P)$ since $BS(\mu) \subsetneq BS(EADA(P))$. Furthermore, note that μ and $EADA(P)$ are the only efficient Pareto imrpovements over $DA(P)$.[13]

Consider the following assignment ν:

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ c_5 & c_2 & \emptyset & c_3 & c_4 & c_1 & c_6 \end{pmatrix}$$

where $BS(\nu) = \{3\}$ (since $B(\nu) = \{(3,c_3),(3,c_6)\}$). Note that ν is efficient and BS-wise more stable than μ and $EADA(P)$. ∎

Now in Proposition 1 (i) is encouraging, but (ii) already shows that no efficient mechanism,

---

[13]This follows because for any efficient Pareto improvment η over $DA(P)$, we must have $\eta(6) = \emptyset$, $\eta(4) = c_3$, $\eta(5) = c_4$ and $\eta(7) \neq c_1$. If $\eta(7) = c_5$, then $\eta = \mu$ and if $\eta(7) = c_6$, then $\eta = \mu$.

which is a Pareto improvement over DA, is BS-wise minimally unstable a.e.a.. Below we show that the last result extends to the two cardinal comparison methods.

Proposition 2 (Cardinal Comparisons and Efficient Assignments) *(i) The EADA mechanism is NOT (BS-wise) cardinally minimally unstable among efficient assignments.*

(ii) There is no efficient mechanism which is a Pareto improvement over the DA mechanism and (BS-wise) cardinally minimally unstable among efficient assignments.

(iii) For any $n \geq 3$, there exists a unit-capacity problem P with $|N| = n$ and $|C| = n - 1$ and an efficient assignment μ such that $EADA(P)$ is the unique efficient Pareto improvement over $DA(P)$ and

$$|B(EADA(P))| = n - 1 \text{ and } |B(\mu)| = 1.$$

Proof. Note that (i) follows from (ii) as EADA is an efficient mechanism which is a Pareto improvement over DA. Furthermore, for BS-wise cardinal minimal instability, (ii) follows from (ii) of Proposition 1 as BS-wise cardinal minimal instability implies BS-wise minimal instability. For cardinal minimal instability, (ii) follows from (iii) as $n \geq 3$.

In showing (iii), consider the following problem P: let $N = \{1, 2, 3, \ldots, n\}$, $C = \{c_1, c_2, c_3, \ldots, c_{n-1}\}$ (all with unit capacities) and

R_1	R_2	R_3	$\cdots$	R_{n-1}	R_n	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\cdots$	$\succeq_{c_{n-2}}$	$\succeq_{c_{n-1}}$
c_1	c_1	c_2		c_{n-2}	c_{n-1}	n	2	3	$\cdots$	$n-2$	$n-1$
c_2	c_2	c_3		c_{n-1}	c_1	1	1	1	$\cdots$	1	1
c_3	$\emptyset$	$\emptyset$		$\emptyset$	$\emptyset$	2	3	4	$\cdots$	$n-1$	n
c_4						$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$											
c_{n-1}											
$\emptyset$											

Note that

$$DA(P) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \emptyset & c_2 & c_3 & \cdots & c_{n-1} & c_1 \end{pmatrix}$$

$$EADA(P) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \emptyset & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}$$

12

and $B(EADA(P)) = \{(1, c_1), (1, c_2), \ldots, (1, c_{n-2}), (1, c_{n-1})\}$. Note that the $EADA$ assignment is the unique efficient Pareto improvement over the DA assignment.

Consider the following assignment μ.

$$\mu = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & \emptyset \end{pmatrix}$$

where $B(\mu) = \{(n, c_1)\}$. Note that μ is *efficient* and is cardinally more stable than $EADA(P)$. ∎

Note that in Proposition 2 we considered minimal instability among the whole set of efficient assignments. In particular, in the proof of (iii) of Proposition 2, the efficient assignment, which is cardinally more stable than the EADA assignment, is not a Pareto improvement over the DA assignment. However, we show next that (i) and (ii) of Proposition 2 remain unchanged when we restrict ourselves to efficient Pareto improvements over DA.

Proposition 3 (Efficient Pareto Improvements over DA)   *(i) The EADA mechanism is NOT BS-wise (cardinally) minimally unstable among efficient Pareto improvements over the DA mechanism.*

  (ii) The EADA mechanism is NOT cardinally minimally unstable among efficient Pareto improvements over the DA mechanism.

 (iii) For any $n \geq 5$, there exists a unit-capacity problem P with $|N| = n$ and $|C| = n-1$ and an efficient assignment μ such that μ Pareto improves over $DA(P)$ and

$$|B(EADA(P))| = n - 2 \text{ and } |B(\mu)| = 2.$$

Proof. In showing (i), note that in the example for the proof of (iii) of Proposition 1 we have $BS(\mu) = \{3, 6\} \subsetneq \{1, 3, 6\} = BS(EADA(P))$.

Note that (ii) follows from (iii) as $n \geq 5$.

In showing (iii), consider the following problem P: let $N = \{1, 2, 3, \ldots, n\}$, $C =$

$\{c_1, c_2, c_3, \ldots, c_{n-1}\}$ (all with unit capacities) and

R_1	R_2	R_3	$\cdots$	R_{n-2}	R_{n-1}	R_n	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\cdots$	$\succeq_{c_{n-3}}$	$\succeq_{c_{n-2}}$	$\succeq_{c_{n-1}}$
c_1	c_1	c_2	$\cdots$	c_{n-3}	c_1	c_{n-1}	n	2	3	$\cdots$	$n-3$	1	$n-2$
c_2	c_2	c_3	$\cdots$	c_1	$\emptyset$	c_{n-2}	$n-1$	1	1	$\cdots$	1	n	n
c_3	$\emptyset$	$\emptyset$	$\cdots$	c_{n-1}		c_1	$n-2$	3	4	$\cdots$	$n-2$	$\vdots$	1
$\vdots$				$\emptyset$		$\emptyset$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$		$\vdots$
c_{n-3}							3						
c_{n-2}							1						
$\emptyset$							2						

.

Note that

$$DA(P) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ c_{n-2} & c_2 & c_3 & \cdots & c_{n-1} & \emptyset & c_1 \end{pmatrix}$$

$$EADA(P) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ c_{n-2} & c_1 & c_2 & \cdots & c_{n-3} & \emptyset & c_{n-1} \end{pmatrix}$$

and $B(EADA(P)) = \{(1,c_1),(1,c_2),\ldots,(1,c_{n-3}),(n-1,c_1)\}$ and $|B(EADA(P))| = n-2$.

Consider the following assignment μ.

$$\mu = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & \emptyset & c_{n-2} \end{pmatrix}$$

where $B(\mu) = \{(n-2,c_1),(n-1,c_1)\}$ and $|B(\mu)| = 2$. Note that μ is efficient, μ Pareto improves over DA, and μ is cardinally more stable than $EADA(P)$. $\blacksquare$

5   Domain Restrictions

Recall that the DA mechanism is stable but not necessarily efficient. In an important contribution, Ergin (2002) derived necessary and sufficient conditions (on the capacity-priority profile) for the efficiency of the DA mechanism.

The EADA mechanism is efficient but not necessarily cardinally minimally unstable. Here we ask a parallel question to Ergin (2002) for cardinal minimal instability: when is the EADA-mechanism cardinally minimally unstable? More generally, for which priority profiles, does there exist a Pareto improvement over the DA mechanism which is cardinally minimally unstable among efficient assignments at each problem? We investigate this question under

the following two assumptions:

A1. (Unit capacities) Each school has unit capacity, i.e., $q_c = 1$ for each $c \in C$.

A2. (At least five schools) There are at least five schools, i.e., $|C| \geq 5$.

The following notions will be useful. A priority profile $(\succeq_c)_{c \in C}$ includes an **Ergin-cycle** (Ergin, 2002) if there exist a list of three students (i_1, i_2, i_3) and a pair of schools (c_1, c_2) such that $i_3 \succ_{c_1} i_1 \succ_{c_1} i_2$ and $i_2 \succ_{c_2} i_3$. We call student i_1 as the **initiator** of the cycle.

We say that a list of three students (i_1, i_2, i_3) and a pair of schools (c_1, c_2) constitute a **tight Ergin-cycle** if they constitute an Ergin-cycle, and both

I. there is no $m \in N \setminus \{i_2, i_3\}$ such that $i_2 \succ_{c_2} m \succ_{c_2} i_3$ and

II. there is no $m \in N \setminus \{i_1, i_2, i_3\}$ such that $i_3 \succ_{c_1} m \succ_{c_1} i_2$.

Given two Ergin-cycles consisting of (i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c'_1, c'_2), respectively, we say that the Ergin-cycles are **distinct** if all the students and schools in the two Ergin-cycles are distinct, i.e., $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \emptyset$ and $\{c_1, c_2\} \cap \{c'_1, c'_2\} = \emptyset$. We say that the two Ergin-cycles are **distinct except for the initiator** if $i_1 = j_1$ and all the other students and schools in the two Ergin-cycles are distinct.

We introduce the following three conditions on a priority profile $(\succeq_c)_{c \in C}$.

Condition C1: All Ergin-cycles are tight.

Condition C2: There are no two Ergin-cycles that are distinct except for the initiator.

Condition C3: There are no two distinct Ergin-cycles.

Our main result of this section, Theorem 1, is that these three conditions are necessary for the EADA assignment to be *cardinally minimally unstable among efficient assignments*; even more, each of them is necessary for the existence of a Pareto improvement over DA which is *cardinally minimally unstable among efficient assignments*.[14] Moreover, the three conditions are sufficient for the EADA assignment to be *cardinally minimally unstable among efficient assignments*.

Theorem 1 *Suppose each school has unit capacity and there are at least five schools. Then the following are equivalent:*

---

[14]In Appendix A.2, we show that the counterparts of the characterizing conditions in the multi-capacity case are still necessary conditions for the EADA assignment to be cardinally minimally unstable a.e.a..

(i) *The EADA mechanism is cardinally minimally unstable among efficient assignments.*

(ii) *There exists an efficient mechanism which is both a Pareto improvement over DA and cardinally minimally unstable among efficient assignments.*

(iii) *The priority profile satisfies C1, C2 and C3.*

We next investigate the same questions as in Theorem 1 for BS-wise cardinal minimal instability (instead of cardinal minimal instability. It turns the corresponding result is almost identical except for one new cycle condition.

We say that a list of three students (i_1, i_2, i_3) and a pair of schools (c_1, c_2) constitute a **weakly tight Ergin-cycle** if they constitute an Ergin-cycle, and there is no $m \in N \setminus \{i_1, i_2, i_3\}$ such that $i_3 \succ_{c_1} m \succ_{c_1} i_2$ or $i_2 \succ_{c_2} m \succ_{c_2} i_3$.[15]

Condition C4: All Ergin-cycles are weakly tight.

Theorem 2 *Suppose each school has unit capacity and there are at least five schools. Then the following are equivalent:*

(i) *The EADA mechanism is BS-wise cardinally minimally unstable among efficient assignments.*

(ii) *There exists an efficient mechanism which is both a Pareto improvement over DA and BS-wise cardinally minimally unstable among efficient assignments.*

(iii) *The priority profile satisfies C2, C3 and C4.*

Remark 3 *(a) The proof of Theorem 1 (and, respectively, of Theorem 2) shows that if the priority profile satisfies C1, C2 and C3, then for each problem either the DA assignment is efficient or there exists a unique Pareto improvement of the DA assignment which contains exactly one blocking pair (and, respectively, one blocking student). Therefore, the latter assignment coincides with EADA assignment.*

(b) By (a), under C1, C2 and C3 the EADA assignment contains no blocking student if the DA assignment is efficient and otherwise exactly one blocking student. Thus, the EADA mechanism is BS-wise cardinally minimally unstable among efficient assignments if the

---

[15]Note that every tight Ergin-cycle is also weakly tight.

priority profile satisfies C1, C2 and C3. However, the following priority profile

$\succeq_{c_1}$	$\succeq_{c_2}$
3	2
1	1
2	3

satisfies C2, C3 and C4 but violates C1. Hence, the EADA mechanism is BS-wise cardinally minimally unstable among efficient assignments but not cardinally minimally unstable among efficient assignment. This is in contrast to the inclusion stability comparisons where the EADA mechanism is minimally unstable among efficient assignments but not necessarily BS-wise minimally unstable among efficient assignments.

(c) *As the set of efficient Pareto improvements over DA is a subset of the efficient assignments, the conditions C1, C2 and C3 are sufficient for the EADA mechanism to be cardinally minimally stable among efficient Pareto improvements over DA: when counting blocking pairs this follows from Theorem 1 and when counting blocking students this follows from (b). It is an open question to determine necessary and sufficient conditions for the EADA mechanism to be cardinally minimally unstable among efficient Pareto improvements over DA (where blocking pairs or blocking students are counted).*

5.1 Proof of Theorem 1

(i)$\Rightarrow$(ii): This follows from the fact that EADA is an efficient mechanism which is a Pareto improvement over DA.

(ii)$\Rightarrow$(iii): We show that if one of the conditions C1, C2 or C3 is violated, then there exist problems where no efficient Pareto improvement over DA is cardinally minimally unstable.

Proposition 4 *Suppose that the priority profile violates Condition C1, i.e., it includes an Ergin-cycle that is not tight. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Suppose that a list of students $(1, 2, 3)$ and a list of schools (c_1, c_2) constitute an Ergin-cycle of $(\succeq_c)_{c \in C}$ that is not tight.

Case 1: Suppose that there is $m \in N \setminus \{2, 3\}$ such that $2 \succ_{c_2} m \succ_{c_2} 3$ and also suppose that

$m = 1$. Below, we only depict the relative positions of the students $\{1, 2, 3\}$ in $\succeq$.

$\succeq_{c_1}$	$\succeq_{c_2}$
3	2
1	1
2	3

Let R be a preference profile such that the preference orderings of students in $\{1, 2, 3\}$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

R_1	R_2	R_3
$\underline{c_1}$	$\boxed{c_1}$	$\boxed{c_2}$
c_2	c_2	c_1
$\boxed{\emptyset}$	$\emptyset$	$\underline{\emptyset}$

Let μ be the assignment where the assignments of the students in $\{1, 2, 3\}$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $\{1, 2, 3\}$ are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over the DA assignment that is efficient and μ' is an efficient assignment. Moreover, $B(\mu) = \{(1, c_1), (1, c_2)\}$ and $B(\mu') = \{(3, c_1)\}$.

Case 2: Suppose that there is $m \in N \setminus \{2, 3\}$ such that $3 \succ_{c_2} m \succ_{c_2} 2$ and also suppose that $m \neq 1$. Below, we only depict the relative positions of the students $\{1, 2, 3, m\}$ in $\succeq$.

$\succeq_{c_1}$	$\succeq_{c_2}$
3	2
1	m
2	3

Let R be a preference profile such that the preference orderings of $\{1, 2, 3, m\}$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

18

R_1	R_2	R_m	R_3
$\underline{c_1}$	$\boxed{c_1}$	c_2	$\boxed{c_2}$
$\boxed{\underline{\emptyset}}$	c_2	$\boxed{\underline{\emptyset}}$	c_1
	$\emptyset$		$\underline{\emptyset}$

Let μ be the assignment where the assignments of the students in $\{1, 2, 3, m\}$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $\{1, 2, 3, m\}$ are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over the DA assignment that is efficient and μ' is an efficient assignment. Moreover, $B(\mu) = \{(1, c_1), (m, c_2)\}$ and $B(\mu') = \{(3, c_1)\}$.

Case 3: Suppose that there is $m \in N \setminus \{1, 2, 3\}$ such that $3 \succ_{c_1} m \succ_{c_1} 2$. Without loss of generality, suppose that $m \succ_{c_1} 1$. Below, we only depict the relative positions of the students $\{1, 2, 3, m\}$ in $\succeq$.

$\succeq_{c_1}$	$\succeq_{c_2}$
3	2
m	3
1	
2	

Let R be a preference profile such that the preference orderings of $\{1, 2, 3, m\}$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

R_1	R_2	R_3	R_m
c_1	$\boxed{c_1}$	$\boxed{c_2}$	$\underline{c_1}$
$\boxed{\underline{\emptyset}}$	c_2	c_1	$\boxed{\emptyset}$
	$\emptyset$	$\underline{\emptyset}$	

Let μ be the assignment where the assignments of the students in $\{1, 2, 3, m\}$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $\{1, 2, 3, m\}$are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over the DA assignment that is efficient and μ' is an efficient assignment. Moreover, $B(\mu) = \{(1, c_1), (m, c_1)\}$ and $B(\mu') = \{(3, c_1)\}$. ∎

Proposition 5 *Suppose that the priority profile violates Condition C2, i.e., it includes two Ergin-cycles that are distinct except for the initiator. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Suppose that there are two Ergin-cycles of $(\succeq_c)_{c \in C}$ consisting of (i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c_1', c_2'), that are distinct except for the initiator. Let $i_1 = j_1 \equiv i$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_1'}$	$\succeq_{c_2'}$
i_3	i_2	j_3	j_2
i	i_3	i	j_3
i_2		j_2	

Let R be a preference profile such that the preference orderings of students in $\{i, i_2,, i_3, j_2, j_3\}$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

R_i	R_{i_2}	R_{i_3}	R_{j_2}	R_{j_3}
$\underline{c_1}$	$\boxed{c_1}$	$\boxed{c_2}$	$\boxed{c_1'}$	$\boxed{c_2'}$
c_1'	$\underline{c_2}$	c_1	c_2'	c_1'
$\boxed{\underline{\emptyset}}$	$\emptyset$	$\underline{\emptyset}$	$\emptyset$	$\emptyset$

Let μ be the assignment where the assignments of the students in $\{i, i_2,, i_3, j_2, j_3\}$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $\{i, i_2,, i_3, j_2, j_3\}$ are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over DA that is efficient and μ' is an efficient assignment. Moreover, $B(\mu) = \{(i, c_1), (i, c_1')\}$ and $B(\mu') = \{(i_3, c_1)\}$. ■

Proposition 6 *Suppose that the priority profile violates Condition C3, i.e., it includes two distinct Ergin-cycles. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Suppose that there are two distinct generalized cycles of $(\succeq_c)_{c \in C}$ consisting of (i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c_1', c_2'). Below, we only depict the relevant relative positions

20

of the students $(i_1, i_2, i_3, j_1, j_2, j_3)$ in $\succeq$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_1'}$	$\succeq_{c_2'}$
i_3	i_2	j_3	j_2
i_1	i_3	j_1	j_3
i_2		j_2	

Since $|C| \geq 5$, there exists $c \in C \setminus \{c_1, c_2, c_1', c_2'\}$. Without loss of generality, suppose that $j_1 \succ_c i_1$.

Let R be a preference profile such that the preference orderings of students in $(i_1, i_2, i_3, j_1, j_2, j_3)$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

R_{i_1}	R_{i_2}	R_{i_3}	R_{j_1}	R_{j_2}	R_{j_3}
$\underline{c}$	$\boxed{c_1}$	$\boxed{c_2}$	c_1'	$\boxed{c_1'}$	$\boxed{c_2'}$
c_1	c_2	c_1	$\boxed{c}$	c_2'	c_1'
$\boxed{\emptyset}$	$\emptyset$	$\emptyset$	$\emptyset$	$\underline{\emptyset}$	$\emptyset$

Let μ be the assignment where the assignments of the students in $(i_1, i_2, i_3, j_1, j_2, j_3)$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $(i_1, i_2, i_3, j_1, j_2, j_3)$ are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over DA that is efficient and μ' is an efficient assignment. Moreover, $B(\mu) = \{(i_1, c_1), (j_1, c_1')\}$ and $B(\mu') = \{(j_2, c_2')\}$. ∎

<u>(iii)⇒(i)</u>: We will now show that C1, C2 and C3 are sufficient for the EADA assignment to be *cardinally minimally unstable among efficient assignments*. First, we define some auxiliary notions and prove some auxiliary results.

A priority profile $(\succeq_c)_{c \in C}$ includes a **generalized cycle** (of length $n-1$) if there exist a list of students $(1, \ldots, n)$ and a list of schools $(c_1, \ldots, c_{n-1})$ such that $n \succ_{c_1} 1 \succ_{c_1} 2$ and for each $i \in \{2, \ldots, n-1\}$, $i \succ_{c_i} i+1$.[16] Given a generalized cycle consisting of $(1, \ldots, n)$ and $(c_1, \ldots, c_{n-1})$, we call the first student, student 1, as the **initiator** of the generalized cycle.[17]

Given two generalized cycles consisting of $(i_1, \ldots, i_n)$, $(c_1, \ldots, c_{n-1})$, and $(j_1, \ldots, j_m)$, $(c_1', \ldots, c_{m-1}')$, respectively, we say that the two generalized cycles are **distinct** if all the

---

[16]The notion of a generalized cycle was first introduced in Ergin (2002).

[17]Note that in the definition of a generalized cycle, the first school c_1 and the first student 1 have particular roles; in that sense, rotating the elements of a generalized cycle would not necessarily result in a new generalized cycle, in contrast to what the word "cycle" would normally indicate.

students and schools in the two generalized cycles are distinct, i.e., $\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_m\} = \emptyset$ and $\{c_1, \ldots, c_{n-1}\} \cap \{c'_1, \ldots, c'_{m-1}\} = \emptyset$. We say that the two generalized cycles are **distinct except for the initiator** if $i_1 = j_1$ and all the other students and schools in the two generalized cycles are distinct.

We say that a list of students $(1, \ldots, n)$ and a list of schools $(c_1, \ldots, c_{n-1})$ constitute a **tight generalized cycle** (of length $n-1$) if they constitute a generalized cycle and

I. there is no $m \in N \setminus \{2, \ldots, n\}$ and $k \in \{2, \ldots, n-1\}$ such that $k \succ_{c_k} m \succ_{c_k} k+1$

II. and there is no $m \in N \setminus \{1, 2, \ldots, n\}$ such that $n \succ_{c_1} m \succ_{c_1} 2$.

Lemma 1 *If the priority profile satisfies C3, then there are no two distinct generalized cycles.*

Proof. Note that Ergin-cycles are generalized cycles of length 2. Thus, if there are two distinct Ergin-cyles, then there are two distinct generalized cycles. Furthermore, if there are two distinct generalized cycles, then by Ergin (2002) (Step 2 in the proof of Theorem 1), there are two distinct Ergin-cycles. ∎

Lemma 2 *If the priority profile satisfies C2 and C3, then there are no two generalized cycles that are distinct except for the initiator.*

Proof. Consider any two generalized cycles consisting of $(i_1, \ldots, i_n)$, $(c_1, \ldots, c_{n-1})$, and $(j_1, \ldots, j_m)$, $(c'_1, \ldots, c'_{m-1})$, that are distinct except for the initiator. Then $i_1 = j_1$ and all the other students and schools in the two generalized cycles are distinct. But then, by Ergin (2002) (Step 2 in the proof of Theorem 1), either there exist two distinct Ergin-cycles, which implies that C3 is violated, or there exist two Ergin-cycles that are distinct except for the initiator, which implies that C2 is violated. ∎

Lemma 3 *If the priority profile satisfies C1, C2, and C3, then all generalized cycles are tight.*

Proof. Suppose that there is a generalized cycle which is not tight. Then there are a list of students $(1, \ldots, n)$ and a list of schools $(c_1, \ldots, c_{n-1})$ such that

(1) there is $m \in N \setminus \{1, 2, \ldots, n\}$ such that $n \succ_{c_1} m \succ_{c_1} 2$ or

(2) there is $m \in N \setminus \{2, \ldots, n\}$ and $k \in \{2, \ldots, n-1\}$ such that $k \succ_{c_k} m \succ_{c_k} k+1$.

If $n - 1 = 2$, then there is an Ergin-cycle which is not tight, which implies that C1 is violated. So, suppose that $n - 1 > 2$.

Step 1: Suppose that (1) is satisfied. Without loss of generality, suppose that there is no shorter generalized cycle which satisfies (1). If $2 \succ_{c_2} n$, then there exists an Ergin-cycle that is not tight, consisting of $(1, 2, n)$ and (c_1, c_2), which violates C1. Similarly, if there exists $l \in \{1, \ldots, n - 1\}$ such that $2 \succ_{c_l} n$, then there exists an Ergin-cycle that is not tight. So suppose that $n \succ_{c_l} 2$ for all $l \in \{1, \ldots, n - 1\}$.

Now, if $n - 2 \succ_{c_{n-1}} n - 1 \succ_{c_{n-1}} n \succ_{c_{n-1}} 2$, then there is a shorter generalized cycle which satisfes (1), consisting of $(n, 2, 3, \ldots, n - 3, n - 2)$ and $(c_{n-1}, c_2, \ldots, c_{n-2})$, a contradiction. If $n - 1 \succ_{c_{n-1}} n - 2 \succ_{c_{n-1}} n \succ_{c_{n-1}} 2$, then, again, there is a shorter generalized cycle which satisfes (1), consisting of $(1, 2, \ldots, n - 2, n)$ and $(c_1, \ldots, c_{n-3}, c_{n-1})$, a contradiction. If $n - 1 \succ_{c_{n-1}} n \succ_{c_{n-1}} n - 2 \succ_{c_{n-1}} 2$, then, again, there is a shorter generalized cycle which satisfes (1), consisting of $(n - 2, 2, 3, \ldots, n - 2, n - 1)$ and $(c_{n-1}, c_2, c_3, \ldots, c_{n-2}, c_{n-1})$, a contradiction. If $n - 1 \succ_{c_{n-1}} n \succ_{c_{n-1}} 2 \succ_{c_{n-1}} n - 2$, then, there exists an Ergin-cycle that is not tight, consisting of $(2, n - 2, n - 1)$ and (c_{n-1}, c_{n-2}), which is a violation of C1, a contradiction. Hence (1) cannot be satisfied.

Step 2: Suppose that (1) is not satisfied but (2) is satisfied. Without loss of generality, suppose that there is no shorter generalized cycle which satisfies (2). Note that $m \neq 1$ or $m = 1$.

Suppose that $m \neq 1$. If $k \succ_{c_k} 1 \succ_{c_k} k + 1$, clearly there is a generalized cycle which satisfes (1), a contradiction. Thus, $1 \succ_{c_k} k$ or $k + 1 \succ_{c_k} 1$. If $1 \succ_{c_k} k$, then the generalized cycle consisting of $(k, k + 1, \ldots, n, 1)$ and $(c_k, \ldots, c_{n-1}, c_1)$ satisfies (1) by $1 \succ_{c_k} m \succ_{c_k} k + 1$, a contradiction. If $k + 1 \succ_{c_k} 1$, then the generalized cycle consisting of $(k + 1, 1, 2, \ldots, k)$ and $(c_k, c_1, \ldots, c_{k-1})$ satisfies (1) by $k \succ_{c_k} m \succ_{c_k} 1$, a contradiction.

Suppose that $m = 1$. If $k + 1 \succ_{c_k} n$, then the generalized cycle consisting of $(k + 1, n, 2, 3, \ldots, k)$ and $(c_k, c_1, c_2, \ldots, c_{k-1})$ satisfies (1) by $k \succ_{c_k} 1 \succ_{c_k} k + 1$, a contradiction. Similarly, if $n \succ_{c_k} k$, then the generalized cycle consisting of $(k, k + 1, \ldots, n)$ and $(c_k, c_{k+1}, \ldots, c_n)$ satisfies (1) by $k \succ_{c_k} 1 \succ_{c_k} k + 1$, a contradiction. Thus, $k \succ_{c_k} n \succ_{c_k} k + 1$. If $k + 1 \succ_{c_1} 2$, then the generalized cycle consisting of $(n, k + 1, 2, \ldots, k)$ and $(c_k, c_1, \ldots, c_{k-1})$ satisfies (1) by $k \succ_{c_k} 1 \succ_{c_k} k + 1$, a contradiction. Thus, $2 \succ_{c_1} k + 1$. But then the generalized cycle consisting of $(2, k + 1, k + 2, \ldots, n)$ and $(c_1, c_{k+1}, \ldots, c_{n-1})$ satisfies (1) by $n \succ_{c_1} 1 \succ_{c_1} k + 1$, a contradiction. ∎

Given an assignment μ, a list of students $(i_1, \ldots, i_k)$ is called an **improvement cycle**[18]

---

[18]The notion of an *improvement cycle* is from Dur et al. (2019).

if $\mu(i_{t+1}) \, P_{i_t} \, \mu(i_t)$ for each $t \in \{1, \ldots, k-1\}$ and $\mu(i_1) \, P_{i_k} \, \mu(i_k)$.[19] In this case, we say that student i_t **precedes** student i_{t+1} in the improvement cycle. We say that an assignment μ' is obtained by **implementing an improvement cycle** at μ if μ' is obtained from μ by simply assigning each student in the improvement cycle to the school of the student whom he precedes, keeping the assignments of the students who do not belong to the improvement cycle the same.

Proposition 7 *Suppose that every generalized cycle of the priority profile is tight. Then, at each problem where the DA assignment is not efficient, there is a unique efficient Pareto improvement over the DA assignment.*

Proof. Let $P = (N, C, R, q, \succeq)$ be a problem such that every generalized cycle of $(\succeq_c)_{c \in C}$ is tight. Let μ denote the DA assignment.

Step 1: We will first show that there is no student who is included in two different "improvement cycles" at μ.

We claim that any two different improvement cycles cannot have a common student. Suppose not, i.e. suppose that $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_q)$ are improvement cycles such that $(i_1, \ldots, i_k) \neq (j_1, \ldots, j_q)$ and $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_q\} \neq \emptyset$. We claim that there exists $i \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_q\}$ such that the student preceding i in the cycle $(i_1, \ldots, i_k)$, say student j, and the student preceding i in the cycle $(j_1, \ldots, j_k)$, say student j', are different students, i.e. $j \neq j'$. To see this, consider any student $i \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_q\}$. If the students preceding i in the two cycles are different, then we are done. Otherwise, the student j who precedes i in both cycles satisfies $j \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_q\}$. Proceeding similarly, our claim follows from the facts that the two cycles are finite and different.

Now, without loss of generality, suppose that the two cycles are of the form $(j, i_1, \ldots, i_k)$ and $(j', i_1, j_1, \ldots, j_q)$ such that $j \neq j'$. Let $c_1 \equiv \mu(i_1)$. Note that $c_1 \, P_j \, \mu(j)$ and $c_1 \, P_{j'} \, \mu(j')$. Then, by the stability of μ, we have $i_1 \succ_{c_1} \{j, j'\}$. Without loss of generality, suppose that $i_1 \succ_{c_1} j' \succ_{c_1} j$. Let $c_t \equiv \mu(i_t)$ for each $t \in \{1, \ldots, k\}$, and let $c_0 \equiv \mu(j)$. Since $(j, i_1, \ldots, i_k)$ is an improvement cycle, by Ergin (2002),[20] there exist $t \in \{0, 1, \ldots, k\}$ and $i \in N \setminus \{i_1, \ldots, i_k, j\}$ such that $(c_t, c_{t-1}, \ldots, c_0, c_k, c_{k-1}, \ldots, c_{t+1})$ and $(i, i_{t-1}, i_{t-2}, \ldots, i_1, j, i_k, i_{k-1}, \ldots, i_{t+1}, i_t)$ con-

---

[19]Note that for any $t \in \{1, \ldots, k\}$, $(i_t, i_{t+1}, \ldots, i_k, i_1, \ldots, i_{t-1})$ is also an improvement cycle including the same set of students.

[20]More precisely, it follows from arguments in the proof of Theorem 1 in Ergin (2002).

stitute a cycle of $(\succeq_c)_{c \in C}$.

$\succeq_{c_0}$	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\cdots$	$\succeq_{c_t}$	$\cdots$	$\succeq_{c_k}$
j	i_1	i_2	i_3		i_t		i_k
i_k	j'	i_1	i_2	$\cdots$	i	$\cdots$	i_{k-1}
	j				i_{t-1}		

However, note that whether $t = 1$ or $t \neq 1$, $(\succeq_c)_{c \in C}$ includes a generalized cycle that is not tight, which is a contradiction. Hence, no two improvement cycles share a common student.

Step 2: By Lemma 6 of Dur et al. (2019), for any Pareto improvement μ' over the DA assignment μ, there exist a set of disjoint improvement cycles (that is, no two improvement cycles share a common student) such that μ' can be obtained from μ by implementing the improvement cycles. Since any two improvement cycles are disjoint, there is a unique efficient Pareto improvement over the DA assignment. ∎

Proposition 8 *Suppose that every generalized cycle of the priority profile is tight. Then, at each problem, any assignment obtained from the DA assignment by implementing an improvement cycle includes a unique blocking pair.*

Proof. Let $P = (N, C, R, q, \succeq)$ be a problem such that every generalized cycle of $(\succeq_c)_{c \in C}$ is tight. Let μ denote the DA assignment. Let μ' be obtained from μ by implementing the improvement cycle $(i_1, \ldots, i_k)$. Let $c_t \equiv \mu(i_{t-1})$ for each $t \in \{2, \ldots, k\}$ and $c_1 \equiv \mu(i_k)$. Suppose that μ' includes two blocking pairs $(i, c) \neq (j, c')$.

Since $(i_1, \ldots, i_k)$ is an improvement cycle, by Ergin (2002) there exist $t \in \{1, \ldots, k\}$ (without loss of generality, let $t = 1$) and $i_0 \in N \setminus \{i_1, \ldots, i_k\}$ such that $(c_1, \ldots, c_k)$ and $(i_0, i_1, \ldots, i_k)$ constitute a cycle of $(\succeq_c)_{c \in C}$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\cdots$	$\succeq_{c_k}$
i_k	i_1	$\cdots$	i_{k-1}
i_0	i_2	$\cdots$	i_k
i_1			

Note that since $(i, c), (j, c') \in B(\mu') \setminus B(\mu)$, $c, c' \in \{c_1, \ldots, c_k\}$.

Case 1: Suppose that $c = c_1$. Since $(i, c) \in B(\mu') \setminus B(\mu)$, $i_k \succ_{c_1} i \succ_{c_1} i_1$. If $i \neq i_0$, this contradicts that every generalized cycle is tight. Suppose that $i = i_0$. Now, either $j \neq i_0$ or $c' \neq c_1$. In either case, by similar arguments, it is easy to see that there is a generalized cycle is that is not tight.

Case 2: Suppose that $c = c_t$, $t \neq 1$. Since $(i, c) \in B(\mu') \setminus B(\mu)$, $i_{t-1} \succ_{c_t} i \succ_{c_t} i_t$, which indicates that there is a generalized cycle that is not tight. ∎

Let $P = (N, C, R, q, \succeq)$ be a problem. By Lemma 3, every generalized cycle of $(\succeq_c)_{c \in C}$ is tight. Then, by Proposition 7 all the improvement cycles at $DA(P)$ are distinct and the assignment obtained from $DA(P)$ by implementing the improvement cycles, let us call it μ, is the unique efficient Pareto improvement over $DA(P)$. Now, there can be at most one improvement cycle, since otherwise there must exist two generalized cycles that are either distinct, which would be a violation of Condition C3 by Lemma 1, or distinct except for the initiator, which would be a violation of Condition C2 or C3 by Lemma 2. Now, by Proposition 8, μ includes at most one blocking pair.

If μ includes no blocking pair, then it is trivially *cardinally minimally unstable among efficient assignments* (in fact, this means that the DA assignment is efficient at this problem). Suppose that μ includes a unique blocking pair. Suppose that μ is not *cardinally minimally unstable among efficient assignments*. Then, there exist an efficient assignment μ' without a blocking pair. But then, $DA(P)$ does not include any improvement cycle and therefore μ cannot have a blocking pair, a contradiction.

Thus, $\mu = EADA(P)$, and the EADA mechanism is cardinally minimally unstable among efficient assignments.

5.2 Proof of Theorem 2

(i)⇒(ii): This follows from the fact that EADA is an efficient mechanism which is a Pareto improvement over DA.

(ii)⇒(iii): We show that if one of the conditions C2, C3 or C4 is violated, then there exist problems where no efficient Pareto improvement over DA is BS-wise cardinally minimally unstable.

Note that in the proof of Proposition 4, Case 2 and Case 3 remain valid for BS-wise cardinal minimal stability, i.e. C4 has to be satisfied. Similarly, the proof of Proposition 6 remains valid for BS-wise cardinal minimal stability, i.e. C3 has to be satisfied. In showing C2, we make the following détour with another condition.

Condition C5: If two Ergin-cycles share exactly one common student and no common schools, and if the common student is the initiator of one of the two cycles, then he must be the initiator of the other cycle as well. Formally, for any two Ergin-cycles consisting of

(i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c'_1, c'_2), respectively, if $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \{i_1\}$ and $\{c_1, c_2\} \cap \{c'_1, c'_2\} = \emptyset$, then $i_1 = j_1$.

Proposition 9 *Suppose that the priority profile violates C5. Then, there exists a problem such that there is no Pareto improvement over DA which is BS-wise cardinally minimally unstable among efficient assignments.*

Proof. Suppose that there are two Ergin-cycles of $(\succeq_c)_{c \in C}$ consisting of (i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c'_1, c'_2), that violates C5. Note that $\{i_1\} = \{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}$.

Case 1: $i_1 = j_2 = i$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c'_1}$	$\succeq_{c'_2}$
i_3	i_2	j_3	i
i	i_3	j_1	j_3
i_2		i	

Let R be a preference profile such that the preference orderings of students in $\{i, i_2, , i_3, j_2, j_3\}$ over their acceptable schools are as depicted below, and each other student finds no school acceptable.

R_i	R_{i_2}	R_{i_3}	R_{j_1}	R_{j_3}
c_1	$\boxed{c_1}$	$\boxed{c_2}$	c'_1	$\boxed{c'_2}$
$\boxed{c'_1}$	c_2	c_1	$\boxed{\emptyset}$	c'_1
$\underline{c'_2}$	$\emptyset$	$\emptyset$		$\emptyset$

Let μ be the assignment where the assignments of the students in $\{i, i_2, , i_3, j_1, j_3\}$ are as depicted above in boxes, and each other student is assigned to his outside option.

Let μ' be the assignment where the assignments of the students in $\{i, i_2, , i_3, j_2, j_3\}$ are as underlined above, and each other student is assigned to his outside option.

Note that μ is the unique Pareto improvement over DA that is efficient and μ' is an efficient assignment. Moreover, $BS(\mu) = \{i, j_1\}$ and $BS(\mu') = \{i\}$.

Case 2: $i_1 = j_3 = i$. This case is very similar to the previous case. We omit the detailed arguments and just depict the priority and preference profiles that yield the desired conclusion.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c'_1}$	$\succeq_{c'_2}$
i_3	i_2	i	j_2
i	i_3	j_1	i
i_2		j_2	

27

R_i	R_{i_2}	R_{i_3}	R_{j_1}	R_{j_3}
c_1	$\boxed{c_1}$	$\boxed{c_2}$	c_1'	$\boxed{c_1'}$
$\boxed{c_2'}$	c_2	c_1	$\boxed{\emptyset}$	$\underline{c_2'}$
$\underline{c_1'}$	$\emptyset$	$\emptyset$		$\emptyset$
$\underline{\emptyset}$				

Let μ be the boxed assignment and μ' be the underlined assignment. Note that μ is the unique Pareto improvement over DA that is efficient and μ' is an efficient assignment. Moreover, $BS(\mu) = \{i, j_1\}$ and $BS(\mu') = \{i\}$. ■

Lemma 4 *If a priority profile satisfies C4, and C5, then it also satisfies C2.*

Proof. Suppose that $\succeq$ satisfies C4 and C5 but violates C2. Then, there are two Ergin-cycles of $(\succeq_c)_{c \in C}$ consisting of (i_1, i_2, i_3), (c_1, c_2), and (j_1, j_2, j_3), (c_1', c_2'), that are distinct except for the initiator. Let $i_1 = j_1 \equiv i$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_1'}$	$\succeq_{c_2'}$
i_3	i_2	j_3	j_2
i	i_3	i	j_3
i_2		j_2	

Consider the position of i in $\succeq_{c_2'}$. Observe that if $i \succeq_{c_2'} j_2$ or $j_3 \succeq_{c_2'} i$, then C5 is violated. Thus, $j_2 \succeq_{c_2'} i \succeq_{c_2'} j_3$. Similarly, $i_2 \succeq_{c_2} i \succeq_{c_2} i_3$.

$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_1'}$	$\succeq_{c_2'}$
i_3	i_2	j_3	j_2
i	i	i	i
i_2	i_3	j_2	j_3

Now consider the position of j_2 in $\succeq_{c_2}$. Note that either $j_2 \succeq_{c_2} i_2$ or $i_3 \succeq_{c_2} j_2$, otherwise $C4$ is violated. Note that the same is true for j_3 by similar arguments.

Suppose that $j_2 \succeq_{c_2} i_2$. Consider the position of j_3 in $\succeq_{c_2}$. If $j_3 \succeq_{c_2} j_2$, then C4 is violated: consider the cycle consisting of (i, j_3, j_2) and (c_2', c_2). If $j_2 \succeq_{c_2} j_3 \succeq_{c_2} i_2$, then C4 is violated: consider the cycle consisting of (j_3, i, j_2) and (c_2, c_1'). If $i_3 \succeq_{c_2} j_3$, then C4 is violated: consider the cycle consisting of (i_3, j_3, j_2) and (c_2, c_1').

Note that we have exhausted all possible configurations, and every possible configuration results in a contradiction. ■

(iii)$\Rightarrow$(i): We will now show that C2, C3 and C4 are sufficient for the EADA assignment to

be *BS-wise cardinally minimally unstable among efficient assignments.* First, we define a weakly tight generalized cycle.

We say that a list of students $(1, \ldots, n)$ and a list of schools $(c_1, \ldots, c_{n-1})$ constitute a **weakly tight generalized cycle** (of length $n-1$) if they constitute a generalized cycle and

I. there is no $m \in N \setminus \{1, 2, \ldots, n\}$ and $k \in \{2, \ldots, n-1\}$ such that $k \succ_{c_k} m \succ_{c_k} k+1$

II. and there is no $m \in N \setminus \{1, 2, \ldots, n\}$ such that $n \succ_{c_1} m \succ_{c_1} 2$.

Because C3 is satisfied, Lemma 1 remains valid and there are no two distinct generalized cycles.

Because C2 and C3 are satisfied, Lemma 2 remains valid and there are no two generalized cycles that are distinct except for the initiator.

Because C2, C3 and C4 are satisfied, the proof of Lemma 3 shows that all generalized cycles are weakly tight.

Lemma 5 *If the priority profile satisfies C2, C3, and C4, then any weakly tight generalized cycle is either tight or an Ergin-cycle (i.e. a cycle of length two).*

Proof. Suppose that there is a weakly generalized cycle which is neither tight nor of length two. Then there are a list of students $(1, \ldots, n)$ and a list of schools $(c_1, \ldots, c_{n-1})$ (with $n - 1 > 2$) such that $n \succ_{c_1} 1 \succ_{c_1} 2$ and for some $k \in \{2, \ldots, n-1\}$, $k \succ_{c_k} 1 \succ_{c_k} k+1$.

Also suppose that this a shortest instance and $k < n - 1$ (as otherwise we relabel). Consider n's position in $\succeq_{c_k}$.

If $k \succ_{c_k} 1 \succ_{c_k} k+1 \succ_{c_k} n$, then the generalized cycle consisting of $(1, \ldots, k, n)$ and $(c_1, \ldots, c_k)$ is not weakly tight, a contradiction.

If $n \succ_{c_k} k \succ_{c_k} 1 \succ_{c_k} k+1$, then the generalized cycle consisting of $(1, k+1, \ldots, n)$ and $(c_k, \ldots, c_n)$ is not weakly tight, a contradiction.

If $k \succ_{c_k} 1 \succ_{c_k} n \succ_{c_k} k+1$, then the generalized cycle consisting of $(1, \ldots, k, n)$ and $(c_1, \ldots, c_k)$ is a shorter instance where we have a weakly tight cycle which is not tight or of length two, a contradiction.[21]

Thus, $k \succ_{c_k} 1 \succ_{c_k} n \succ_{c_k} k+1$. Analogous arguments (where k is in the role of n and c_1 is in the role of c_k) show $n \succ_{c_1} k \succ_{c_1} 1 \succ_{c_1} 2$. Consider k's position in $\succeq_{c_{n-1}}$.

---

[21]In case of $n - 1 = 3$, one can also check that we have a contradiction to C4.

If $n - 1 \succ_{c_{n-1}} k$, then the generalized cycle consisting of $(1, k+1, \ldots, n-1, k)$ and $(c_k, \ldots, c_n)$ is not weakly tight, a contradiction.

If $k \succ_{c_{n-1}} n - 1$, then the generalized cycle consisting of $(1, \ldots, k, n)$ and $(c_1, \ldots, c_{k-1}, c_n)$ is not weakly tight, a contradiction. ∎

But now Proposition 7 remains unchanged, and Proposition 8 shows that any assignment obtained from the DA assignment by implementing an improvement cycle includes a unique blocking student (as weakly tight generalized cycles, which are not tight, must be of the form

$$
\begin{array}{cc}
\succeq_{c_1} & \succeq_{c_2} \\
\hline
3 & 2 \\
1 & 1 \\
2 & 3
\end{array}
).
$$

6 Concluding Remarks

Another important property for the design of mechanisms is strategy-proofness: a mechanism is strategy-proof if at any problem, no student can be better off by reporting a preference relation different than his true preference relation. Proposition 1 in Kesten (2010) shows that there is no efficient and strategy-proof mechanism that selects the efficient and stable assignment whenever it exists. Hence, there exists no mechanism which is strategy-proof and minimally unstable among efficient assignments for any of our stability comparisons based on blocking pairs or blocking students. Even if the priority profile satisfies the conditions C1, C2 and C3 and the EADA mechanism does not coincide with DA, the EADA mechanism is not strategy-proof: this follows from Abdulkadiroğlu et al. (2009).

The top trading cycles (TTC) mechanism (Abdulkadiroğlu and Sönmez, 2003), which is based on Gale's TTC algorithm (Shapley and Scarf, 1974), is another well-known efficient mechanism. The TTC mechanism is not a Pareto improvement over the DA mechanism, but it is *strategy-proof*. The TTC mechanism is not *minimally unstable among efficient assignments* simply because, TTC may not choose the efficient and stable assignment when it exists. Yet, Abdulkadiroğlu et al. (2020) has shown that, if we fix a set of agents and a set of schools with unit capacities, then TTC is *minimally unstable among efficient and strategy-proof mechanisms.* In our companion paper Doğan and Ehlers (2020), we show that TTC is also *minimally unstable among efficient and strategy-proof mechanisms* for any of our stability comparisons if we fix a set of agents and a set of schools with unit capacities.

Another known efficient mechanism is the *top trading cycles over DA mechanism* (DA⊕TTC),

which is based on applying Gale's TTC procedure over the DA outcome, i.e., first run the DA algorithm and obtain the DA assignment; then, using the DA assignment as the endowment profile, calculate the TTC assignment as in Shapley and Scarf (1974). DA⊕TTC mechanism is *efficient* and Pareto improves over DA. DA⊕TTC assignments are also not necessarily *minimally unstable among efficient assignments*, since there exist problems where the EADA assignment is *more stable* (see Example 8 in Kesten (2010)). Hence, DA⊕TTC is also not cardinally minimally unstable among efficient assignments, which also follows from our Proposition 2.

Another alternative stability comparison is based on blocking triplets. Kwon and Shorrer (2019) study the blocking triplets inclusion comparison (*tincl*) which is defined as follows (given problem P and assignment μ): $(i,j,c) \in T(\mu)$ if and only if $i \succ_c j$, $\mu(j) = c$, and $cP_i\mu(i)$; then for $\mu,\nu \in \mathcal{A}$,

$$\mu \gtrsim^P_{tincl} \nu \Leftrightarrow T(\mu) \subseteq T(\nu).$$

Now the blocking triplets cardinality comparison *tcard* is defined as follows. For each $P \in \mathcal{P}$ and $\mu,\nu \in \mathcal{A}$,

$$\mu \gtrsim^P_{tcard} \nu \Leftrightarrow |T(\mu)| \leq |T(\nu)|.$$

Note that $\gtrsim^P_{tincl} \subseteq \gtrsim^P_{tcard}$.

An immediate observation is that Theorem 1 extends to the blocking triplets cardinality comparison *tcard* as for problems where schools have unit capacity, for any problem P we have $|B(\mu)| = |T(\mu)|$ for any efficient assignment μ. Hence, *tcard* and *scard* coincide on the set of efficient assignments. The same is true for Proposition 2 and (ii) and (iii) of Proposition 3 as in the examples of the proofs all schools have unit capacities.[22]

A   Appendix

A.1   Proof of (i) of Proposition 1

Proof. The following notation will be useful. Given a set of students $I \subseteq N$ and a set of school $S \subseteq C$, let $\mu(I) = \cup_{i \in I}\mu(i)$ and $\mu(S) = \cup_{c \in S}\mu(c)$ denote the aggregate assignments of

---

[22]Furthermore, (i) of Proposition 1 holds for *tincl*: let μ and ν be two efficient assignments such that $\mu \gtrsim^P_{tincl} \nu$; then $T(\mu) \subseteq T(\nu)$; and for all $(i,j,c) \in T(\mu)$ we have (a) $(i,c) \in B(\mu)$ and (b) $(i,j,c) \in T(\nu)$ and $(i,c) \in B(\nu)$. Since $B(\mu) = \{(i,c) : (i,j,c) \in T(\mu) \text{ for some } j \in N\}$ we obtain $B(\mu) \subseteq B(\nu)$ and $\mu \gtrsim^P_{pincl} \nu$. This and (i) of Proposition 1 imply that $EADA(P)$ is *tincl*-minimally unstable a.e.a. (which is also shown by Kwon and Shorrer (2019) in Proposition 4).

I and S at μ, respectively. Given a set of students $I \subseteq N$, let $\mu|_I$ denote the restriction of μ to I. Note that $\mu|_I$ is a mapping $\mu|_I : I \cup \mu(I) \to I \cup \mu(I)$.

The following results from the literature will be useful. The following is Lemma 2 of Tang and Yu (2014), which shows that each step of the EADA algorithm Pareto improves upon the previous step.

Lemma 6 *For each $t \in \{2, \ldots, m\}$, μ_t Pareto improves μ_{t-1}. Also, μ_1 Pareto improves the DA assignment.*

The following lemma follows from Doğan and Yenmez (2020), which shows that the EADA satisfies a particular consistency property.

Lemma 7 *Whenever a student, who is assigned to a school that is underdemanded at DA, is removed with his assigned seat at the EADA assignment, the assignments of the remaining students do not change when the EADA algorithm is run for the reduced problem.*

The proof is in two steps. First, we prove the statement for problems at which each school has unit capacity. Then, we extend it to the entire domain of problems.

Unit-capacity case: Let $P = (N, C, R, q, \succeq)$ be a problem such that for each $c \in C$, $q_c = 1$. Let $\mu = EADA(P)$. Let $\{I_1, \ldots, I_m\}$ denote the partition of N generated by the underdemanded schools algorithm. Let $\nu = DA(P)$. For each $t \in \{1, \ldots, m\}$, let $I_{>t} = I_{t+1} \cup \cdots \cup I_m$ (and similarly we define $I_{\leq t}$ and $I_{<t}$). Let $\mu_1, \ldots, \mu_m$ be as defined above, i.e., for each $t \in \{1, \ldots, m\}$, μ_t is the assignment obtained by iterating the EADA algorithm for only t rounds.

We first claim that $B(\mu_1) \subseteq B(\mu_2) \subseteq \cdots \subseteq B(\mu_m) = B(\mu)$. Note that for each $t \in \{1, \ldots, m-1\}$, μ_{t+1} is a Pareto improvement over μ_t by Lemma 6. Then, for each school $c \in C$, the student who is assigned to c in μ_{t+1} has a weakly lower priority than the student who is assigned to c in μ_t, since otherwise μ_t restricted to $I_{\geq t}$ would be unstable. Thus, for each student $i \in I_{\leq t}$, the set of blocking pairs in μ_t that include i is a subset of the set of the set of blocking pairs in μ_{t+1} that include i. Moreover, since no student in $I_{>t}$ prefers a school that is assigned to a student in $I_{\leq t}$ to his assigned school in μ_{t+1}, no student in $I_{\geq t+1}$ is included in a blocking pair in μ_t or μ_{t+1}, which proves the claim.

Suppose by contradiction that μ' is an efficient assignment such that $B(\mu') \subsetneq B(\mu)$. Then, for some $j \in N$, $\mu'(j) \neq \mu(j)$. Let $j \in I_t$. If $\mu(I_t) = \mu'(I_t)$, then by the underdemanded schools algorithm, $\mu(i) = \mu_t(i)$ for all $i \in I_t$. But then $B_i(\mu) \cap (I_t \times \mu(I_t)) = \emptyset$ for all $i \in I_t$.

By $B(\mu') \subseteq B(\mu)$, we also have $B_i(\mu') \cap (I_t \times \mu(I_t)) = \emptyset$ for all $i \in I_t$. Thus, both $\mu|_{I_t}$ and $\mu'|_{I_t}$ are stable. Because $\mu(I_t) = \mu'(I_t)$, it now follows from efficiency that $\mu(i) = \mu'(i)$ for all $i \in I_t$, a contradiction to $\mu'(j) \neq \mu(j)$ and $j \in I_t$. Thus, $\mu(I_t) \neq \mu'(I_t)$.

Suppose, without loss of generality, that $\mu'(j) \notin \mu(I_t)$. But then going from μ to μ' agent j is involved in a trading cycle $j_1, \ldots, j_r$ such that $\mu'(j_l) = \mu(j_{l+1})$ for each $l \in \{1, \ldots, r-1\}$, and $\mu'(j_r) = \mu(j_1)$. Let $c_l = \mu(j_l)$ for each $l \in \{1, \ldots, r\}$.

We show that there exists $l \in \{1, \ldots, r\}$ such that $j_l \in I_t$, $c_{l+1} \in \mu(I_{>t})$, and $c_{l+1} P_{j_l} c_l$. By efficiency of μ', there exists $l \in \{1, \ldots, r\}$ such that $c_{l+1} P_{j_l} c_l$. Then by the underdemanded schools algorithm, $c_{l+1} \in \mu(I_{\geq t})$. If $c_{l+1} \in \mu(I_{>t})$, then we have the desired l. Suppose not, that is, suppose that $c_{l+1} \in \mu(I_t)$. Then we have $\mu_t(j_l) = \mu(j_l) = c_l$ and $\mu_t(j_{l+1}) = \mu(j_{l+1}) = c_{l+1}$ which implies $j_{l+1} \succ_{c_{l+1}} j_l$. Since $(j_{l+1}, c_{l+1}) \notin B(\mu)$ and $B(\mu') \subseteq B(\mu)$, we have $(j_{l+1}, c_{l+1}) \notin B(\mu')$ and $c_{l+2} = \mu'(j_{l+1}) P_{j_{l+1}} c_{l+1}$. But then again $c_{l+2} \in \mu(I_{\geq t})$. If $c_{l+2} \in \mu(I_{>t})$, we have the desired l. Otherwise, $c_{l+2} \in \mu(I_t)$. Since $j \in \{j_1, \ldots, j_r\}$ and $\mu'(j) \notin \mu(I_t)$, continuing similarly we will eventually reach the desired l.

Now, without loss of generality, let j be such that $j \in I_t$, $\mu'(j) \in \mu(I_{>t})$, and $\mu'(j) P_j \mu(j)$. Let $i_1 = \mu_t(\mu'(j))$. Note that $i_1 \in I_{>t}$ since $\mu'(j) \in \mu(I_{>t})$. By $\mu'(j) P_j \mu(j)$, we have $i_1 \succ_{\mu'(j)} j$. Thus, $\mu(i_1) R_{i_1} \mu'(j)$ and $(i_1, \mu'(j)) \notin B(\mu)$. By $B(\mu') \subseteq B(\mu)$, $(i_1, \mu'(j)) \notin B(\mu')$. Hence, $\mu'(i_1) P_{i_1} \mu'(j)$ and since $i_1 \in I_{>t}$, by the underdemanded schools algorithm, $\mu'(i_1) \in \mu(I_{>t})$.

Let $i_2 = \mu_t(\mu'(j))$. By similar arguments, there exists an agent i_3 such that $i_3 = \mu_t(\mu'(i_2))$, $\mu'(i_3) \in \mu(I_{>t})$, and $\mu'(i_3) P_{i_3} \mu(i_3)$. Continuing similarly, since the number of agents is finite, we will eventually reach an agent i_r such that $j = \mu_t(\mu'(i_r))$, $\mu'(i_r) \in \mu(I_{>t})$, and $\mu'(i_r) P_{i_r} \mu(i_r)$, which is a contradiction since $\mu'(i_r) = \mu(j)$ and $\mu(j) \in \mu(I_t)$.

Extension to general capacities: Let $P = (N, C, R, q, \succeq)$ be a problem (not necessarily unit-capacity). We construct an auxiliary unit-capacity problem $P' = (N, C', R', q', \succeq')$, which has the same set of students, as follows.

(i) For each $c \in C$, we construct q_c unit-capacity schools labelled as $c_1, \ldots, c_{q_c}$ and assign them into C'. Note that $|C'| = \sum_{c \in C} q_c$ and q' is a $|C'|$-tuple of 1's.

(ii) For each $i \in N$ and $c_p, c'_q \in C'$, we have $c_p R'_i c'_q$ if and only if $c P_i c'$ or $[c = c'$ and $p \leq q]$. Also, $c_p P_i i$ if and only if $c P_i i$.

(iii) For each $c_p \in C'$ and $i, j \in N$, we have $i \succeq'_{c_p} j$ if and only if $i \succeq_c j$. Also, $i \succeq_{c_p} c_p$ if and only if $i \succeq_c c$.

We define a mapping φ from the set of assignments in P' to the set of assignments in P as follows. Given an assignment μ in problem P', let $\varphi(\mu')$ be the assignment in problem P such that for each $i \in N$, $\varphi(\mu')(i) = c$ if and only if $\mu'(i) = c_p$ for some $p \in \{1, \ldots, c_{q_c}\}$.

Observation 1: $\varphi(DA(P')) = DA(P)$. To see this, first note that clearly $\varphi(DA(P'))$ is stable at P. Suppose that $\varphi(DA(P')) \neq DA(P)$. Since $DA(P)$ is the student-optimal stable assignment at P, $DA(P)$ Pareto improves $\varphi(DA(P'))$. Let μ be the assignment in problem P' such that for each $i \in N$, $\mu(i) = c_p$ if and only if $DA(P) = c$ and $|j \in DA(c) : j \succeq_c i| = p$. Note that μ is stable and Pareto improves $DA(P')$ at P', which is a contradiction since $DA(P')$ is the student-optimal stable assignment at P'.

Observation 2: Each student $i \in N$ who is assigned to an underdemanded school at $DA(P')$ is assigned to an underdemanded school also at $DA(P)$. (Note that it is not necessarily true that if $c \in C$ is underdemanded at $DA(P)$, each of $c_1, \ldots, c_{q_c}$ is underdemanded at $DA(P')$.) This easily follows from $\varphi(DA(P')) = DA(P)$.

We show that $\varphi(EADA(P')) = EADA(P)$. Let $i_1 \in N$ be a student who is assigned to an underdemanded school c_q at $DA(P')$ (Since all the students who are assigned to a school at the last step of the DA algorithm are assigned to underdemanded schools, there exists such a student). By Observation 1, $\varphi(DA(P'))(i_1) = DA(P)(i) = c$. By Observation 2, c is underdemanded at $DA(P)$. Then, $\varphi(EADA(P'))(i_1) = EADA(P)(i_1)$.

By Lemma 7, at the problem P', if student i_1 is removed with his assigned seat at the $EADA(P')$ assignment, the assignments of the remaining students do not change when the $EADA$ algorithm is run for the reduced problem. Again by Lemma 7, at the problem P, if student i is removed with his assigned seat at the $EADA(P)$ assignment, the assignments of the remaining students do not change when the $EADA$ algorithm is run for the reduced problem.

Now, we proceed likewise with the reduced problems. Let $i_2 \in N \setminus \{i_1\}$ be a student who is assigned to an underdemanded school c_q at the DA assignment for the problem reduced from P' (As long as $N \setminus \{i_1\} \neq \emptyset$, there exists such a student). By similar arguments as above, $\varphi(EADA(P'))(i_2) = EADA(P)(i_2)$. Proceeding likewise, we will eventually exhaust all the students, which concludes that $\varphi(EADA(P')) = EADA(P)$.

Suppose that $EADA(P)$ is not minimally unstable among efficient assignments. Let μ be an efficient assignment that is more stable than $EADA(P)$ at P. Let μ' be the assignment in problem P' such that for each $i \in N$, $\mu(i) = c_p$ if and only if $DA(P) = c$ and $|j \in DA(c) : j \succeq_c i| = p$. Note that μ' is efficient and more stable than $EADA(P')$, which is a contraction since the $EADA$ assignment is minimally unstable among efficient assignments

when each school has unit capacity. ∎

A.2 Necessary Conditions in the Multi-Capacity Case

A capacity-priority profile $(q_c, \succeq_c)_{c \in C}$ includes an (multi-capacity) **Ergin-cycle** (Ergin, 2002) if there exist a list of three students (i_1, i_2, i_3), a pair of schools (c_1, c_2), and a pair of (possibly empty) disjoint sets of students (N_{c_1}, N_{c_2}) such that such that $i_3 \succ_{c_1} i_1 \succ_{c_1} i_2$ and $i_2 \succ_{c_2} i_3$.

   i. $i_3 \succ_{c_1} i_1 \succ_{c_1} i_2$ and $i_2 \succ_{c_2} i_3$,

   ii. for each $t \in \{1, 2\}$, $N_{c_t} \subset N \setminus \{1, 2, 3\}$ and $|N_{c_t}| = q_{c_t} - 1$,

   iii. $N_{c_1} \subset \{i \in N : i \succ_{c_t} 1\}$ and $N_{c_2} \subset \{i \in N : i \succ_{c_2} 3\}$.

    Given an Ergin-cycle consisting of $(1, 2, 3)$, (c_1, c_2), and (N_{c_1}, N_{c_2}), we call the first student, student 1, as the **initiator** of the Ergin-cycle.

    Given an Ergin-cycle consisting of (i_1, i_2, i_3), (c_1, c_2), and (N_{c_1}, N_{c_2}), and another Ergin-cycle consisting of (j_1, j_2, j_3), (c'_1, c'_2), and $(N'_{c'_1}, N'_{c'_2})$, we say that the two generalized cycles are **distinct** if all the students, schools, and sets of students in the two generalized cycles are distinct; we say that the two generalized cycles are **distinct except for the initiator** if $i_1 = j_1$ and all the other students, schools, and sets of students in the two generalized cycles are distinct.

    We say that a list (i_1, i_2, i_3), (c_1, c_2), and (N_{c_1}, N_{c_2}) constitute a **tight** Ergin-cycle if they constitute an Ergin-cycle and

   iv. there is no $m \in N \setminus \{2, 3\}$ such that $2 \succ_{c_2} m \succ_{c_2} 3$

   v. and there is no $m \in N \setminus \{1, 2, 3\}$ such that $3 \succ_{c_1} m \succ_{c_1} 2$.

    Condition C1*: All the Ergin-cycles are tight.

    Condition C2*: There are no two Ergin-cycles that are distinct except for the initiator.

    Condition C3*: For any two Ergin-cycles consisting of (i_1, i_2, i_3), (c_1, c_2), and (N_{c_1}, N_{c_2}), and (j_1, j_2, j_3), (c'_1, c'_2), and $(N'_{c'_1}, N'_{c'_2})$, there does not exist any student $i \in \{i_1, i_2, i_3\}$, any school $c \in C \setminus \{c_1, c_2, c'_1, c'_2\}$ and any set of students N_c distinct from the students in the two Ergin-cycles (including the sets of students in the two Ergin-cycles) such that $N_c \subset \{j \in N : j \succ_c i\}$ and $|N_c| = q_c - 1$.

Proposition 10 *Suppose that $(q_c, \succeq_c)_{c \in C}$ violates Condition C1*, i.e., it includes an Ergin-cycle that is not tight. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Follows from the same arguments as in the proof of Proposition 4 with the following modifications.

- In the preference profile R, for each $t \in \{1, 2\}$, each student in N_{c_t} top ranks c_t.

- In the assignments μ and μ', for each $t \in \{1, 2\}$, each student in N_{c_t} is assigned to c_t. ∎

Proposition 11 *Suppose that $(q_c, \succeq_c)_{c \in C}$ violates Condition C2*, i.e., it includes two Ergin-cycles that are distinct except for the initiator. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Follows from the same arguments as in the proof of Proposition 5 with the following modifications.

- In the preference profile R, for each $t \in \{1, 2\}$, each student in N_{c_t} top ranks c_t and each student in $N'_{c'_t}$ top ranks c'_t.

- In the assignments μ and μ', for each $t \in \{1, 2\}$, each student in N_{c_t} is assigned to c_t and each student in $N'_{c'_t}$ is assigned to c'_t. ∎

Proposition 12 *Suppose that $(q_c, \succeq_c)_{c \in C}$ violates Condition C3*. Then, there exists a problem such that there is no Pareto improvement over DA which is cardinally minimally unstable among efficient assignments.*

Proof. Follows from the same arguments as in the proof of Proposition 6 with the following modifications.

- Note that in the proof of Proposition 6, existence of a school c with the desired property follows from the unit-capacity assumption and the assumption that $|C| \geq 5$. Here, it directly follows from the violation of C3*.

- In the preference profile R, each student in N_c top ranks c and for each $t \in \{1, 2\}$, each student in N_{c_t} top ranks c_t and each student in $N'_{c'_t}$ top ranks c'_t.

- In the assignments μ and μ', each student in N_c is assigned to c and for each $t \in \{1, 2\}$, each student in N_{c_t} is assigned to c_t and each student in $N'_{c'_t}$ is assigned to c'_t. ∎

References

Abdulkadiroğlu, Atila and Tayfun Sönmez, "School choice: A mechanism design approach," *American Economic Review*, June 2003, *93* (3), 729–747.

__ , **Parag Pathak, and Alvin Roth**, "Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC High School Match," *American Economic Review*, 2009, *99*, 1954–1978.

__ , **Yeon-Koo Che, Parag A. Pathak, Alvin E. Roth, and Olivier Tercieux**, "Efficiency, Justified Envy, and Incentives in Priority-Based Matching," *American Economic Review: Insights, forthcoming*, 2020.

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson, "Budget-balance, fairness and minimal manipulability," *Theoretical Economics*, 2014, *9* (3), 753–777.

Combe, Julien, Olivier Tercieux, and Camille Terrier, "The Design of Teacher Assignment: Theory and Evidence," 2017. Working Paper.

Doğan, Battal and Lars Ehlers, "Robust Minimal Instability of the Top Trading Cycles Mechanism," *Working paper*, 2020.

__ **and M. Bumin Yenmez**, "Consistent Pareto improvement over the student-optimal stable mechanism," 2020. Economic Theory Bulletin, forthcoming.

__ , **Serhat Doğan, and Kemal Yıldız**, "A new ex-ante efficiency criterion and implications for the probabilistic serial mechanism," *Journal of Economic Theory*, 2018, *175*, 178–200.

Dur, Umut, A. Arda Gitmez, and Özgür Yılmaz, "School choice under partial fairness," *Theoretical Economics*, 2019, *14*, 1309–1346.

Ehlers, Lars and Thayer Morrill, "(Il)legal assignments in school choice," *The Review of Economic Studies, forthcoming*, 2020.

Ergin, Haluk I., "Efficient resource allocation on the basis of priorities," *Econometrica*, 2002, *70* (6), 2489–2497.

Gale, David and Lloyd S. Shapley, "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, jan 1962, *69* (1), 9–15.

Kesten, Onur, "School choice with consent," *The Quarterly Journal of Economics*, 2010, *125* (3), 1297–1348.

Kwon, Hyukjun and Ran I. Shorrer, "Justified-Envy Minimal Mechanisms in School Choice," *Working Paper Available at SSRN: https://ssrn.com/abstract=3495266*, 2019.

Maus, Stefan, Hans Peters, and Ton Storcken, "Anonymous voting and minimal manipulability," *Journal of Economic Theory*, 2007, *135*, 533 – 544.

_ , _ , **and** _ , "Minimal manipulability: Anonymity and unanimity," *Social Choice and Welfare*, 2007, *29*, 247 – 269.

Pathak, Parag A. and Tayfun Sönmez, "School Admissions Reform in Chicago and England: Comparing Mechanisms by Their Vulnerability to Manipulation," *American Economic Review*, September 2013, *103* (1), 80–106.

Roth, Alvin E., "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 1982, *7*, 617–628.

Shapley, Lloyd and Herbert Scarf, "On cores and indivisibility," *Journal of Mathematical Economics*, 1974, *1* (1), 23 – 37.

Svensson, Lars-Gunnar, "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 1999, *16*, 557–567.

Tang, Qianfeng and Jingsheng Yu, "A new perspective on Kesten's school choice with consent idea," *Journal of Economic Theory*, 2014, *154*, 543–561.

_ **and Yongchao Zhang**, "Weak Stability and Pareto Efficiency in School Choice," *Economic Theory, forthcoming*, 2020.

Troyan, Peter, David Delacrétaz, and Andrew Kloosterman, "Essentially stable matchings," *Games and Economic Behavior*, 2020, *120*, 370–390.

Récents cahiers de recherche du CIREQ
Recent Working Papers of CIREQ

La liste complète des cahiers de recherche du CIREQ est disponible à l'adresse suivante : http://www.cireqmontreal.com/cahiers-de-recherche .

The complete list of CIREQ's working papers is available at the following address : http://www.cireqmontreal.com/cahiers-de-recherche .

03-2019	Galindo da Fonseca, J.A., P. Pannella, "The Housing Boom and Selection into Entrepreneurship", mai 2019, 30 pages
04-2019	Galindo da Fonseca, J.A., "Unemployment, Entrepreneurship and Firm Outcomes", mai 2019, 96 pages
05-2019	Galindo da Fonseca, J.A., C. Berubé, "Spouses and Entrepreneurship", mai 2019, 45 pages
06-2019	Galindo da Fonseca, J.A., I. Snoddy, "Entrepreneurship, Outside Options and Constrained Efficiency", mai 2019, 22 pages
07-2019	Benchekroun, H., A.R. Chaudhuri, D. Tanseem, "On the Impact of Trade in a Common Property Renewable Resource Oligopoly", mai 2019, 28 pages
08-2019	Duffy, J., J.H. Jiang, H. Xie, "Experimental Asset Markets with an Indefinite Horizon", juillet 2019, 58 pages
09-2019	Rauh, C., "Measuring Uncertainty at the Regional Level Using Newspaper Text", août 2019, 28 pages
10-2019	Aguirregabiria, V., M. Marcoux, "Imposing Equilibrium Restrictions in the Estimation of Dynamic Discrete Games", septembre 2019, 45 pages
11-2019	Pelli, M., J. Tschopp, N. Bezmaternykh, K.M. Eklou, "In the Eye of the Storm : Firms and Capital Destruction in India", octobre 2019, 43 pages
12-2019	Amodio, F., G. Chiovelli, S. Hohmann, "The Employment Effects of Ethnic Politics", décembre 2019, 65 pages
13-2019	Amodio, F., Martinez-Carrasco, M.A., "Inputs, Incentives, and Self-selection at the Workplace", décembre 2019, 55 pages
14-2019	Degan, A., M. Li, H. Xie, "Persuasion Bias in Science : An Experiment on Strategic Sample Selection", novembre 2019, 47 pages
01-2020	Gupta, R., M. Pelli, "Electrification and Cooking Fuel Choice in Rural India", janvier 2020, 34 pages
02-2020	Doĝan, B., L. Ehlers, "Robust Minimal Instability of the Top Trading Cycles Mechanism", mars 2020, 21 pages
03-2020	Dufour, J.-M., E. Flachaire, L. Khalaf, A. Zalghout, "Identification-Robust Inequality Analysis", avril 2020, 42 pages