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Maximal area of equilateral small polygons

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Abstract: The paper shows that among all equilateral polygons with a given number of sides and the same diameter, the regular polygon has the maximal area.

Key Words: Polygon, area, diameter, equilateral.

Résumé : Ce travail montre que parmi tous les polygones équilatéraux convexes avec le même nombre de côtés et le même diamètre, le polygone régulier possède l'aire maximale.

Mots clés : Polygone, aire, diamètre, équilatéral.

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1 Introduction

Let $n \geq 3$ be a fixed integer. We answer the following question: Which n-sided equilateral small polygon has the greatest area ? The term *small* was coined by Graham [11] and signifies that the diameter of the polygon is equal to one, i.e., that the largest Euclidian distance between pairs of its vertices is one. The main result of the present paper is that the regular polygon is the equilateral small n-sided polygon with the greatest area.

One might be tempted to think that this is an obvious result, since it is well known [9] that among all n-sided polygons sharing the same perimeter, the regular one has maximal area. However, for related extremal problems, the optimal polygons are not necessarily regular. When n is an odd number, the n-sided small polygon with maximal area is the regular polygon [16]. This result holds over all small polygons, including non-convex and non-equilateral ones. But when the number of sides is even, the n-sided small polygon with maximal area is not the regular one. This follows from the observation that when $n \geq 6$ is an even number, the area of the small regular $n-1$-sided polygon is larger than that of the small regular n-sided polygon [5]. The optimal figures for n varying from 6 to 12 are given in [11, 7, 12], lower and upper upper bounds on the area and perimeter are presented in [13, 14] for larger values of n.

Similar questions are raised in the context of finding the small polygon with maximal perimeter. There is a small equilateral hexagon whose perimeter is 3.5% larger than that of the unit-diameter hexagon [17], and there is a non-equilateral small hexagon [11] whose area and perimeter are 3.9% and 3.3% larger, respectively, than that of the regular small hexagon. Figure 1 illustrates these three small hexagons, where the line segments represent unit-length diagonals. Equilateral and non-equilateral small octagons with maximal perimeter are presented in [6] and in [2].

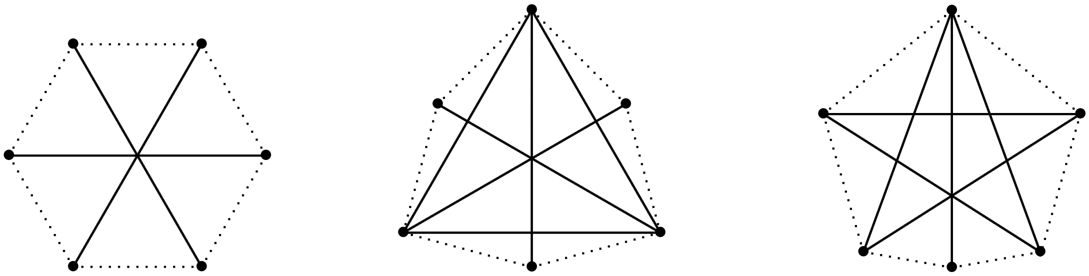

Figure 1: Three small hexagons.

In [1, 3] the authors survey families of extremal problems for convex polygons. They consider the following five attributes: area, perimeter, diameter, width and sum of distances between all pairs of vertices. Then they present a first class of optimization problem that consists in fixing one of the attribute, while optimizing over another one. This results in 10 minimization and 10 maximization problems. A second class of problems is obtained by adding the requirement that the polygon be equilateral. The number of optimal solutions to some of the isodiametric and isoperimetric problems is studied in [15].

Since the publication of these surveys, the three questions of finding the equilateral unit-width convex polygons that maximize the perimeter, diameter or area were solved in [8]. Also, the survey [1] states that when n is not a power of 2, isoperimetric n-sided polygons of maximum width were first found by [4], but the result was published 24 years earlier in a russian journal [10].

The present note studies the question of finding the small equilateral polygon with greatest area. The main result is shown in the next section, based on two preliminary lemmas.

2 The largest small equilateral polygon

The following notation is adopted: The area and perimeter of a polygon $\mathcal{P}$ are denoted $A(\mathcal{P})$ and $P(\mathcal{P})$, respectively. A_n^{reg}, P_n^{reg} and c_n^{reg} are the area, perimeter and side length of the small regular n-sided polygon.

The main result is shown by contraction, by developing a connection between the area of a polygon with that of another with twice as many vertices. The next lemma states this connection for regular polygons with an even number of sides.

Lemma 2.1 *If $n \geq 4$ is an even number, then*

$$A_{2n}^{reg} = A_n^{reg} + \frac{nc_n^{reg}}{4}\left(1 - \sqrt{1-(c_n^{reg})^2}\right).$$

Proof. When n is even, the side length and area of the small regular n-sided polygon satisfy $c_n^{reg} = \sin(\frac{\pi}{n})$ and $A_n^{reg} = \frac{n}{8}\sin(\frac{2\pi}{n})$ [5]. It follows that

$$\begin{aligned} A_{2n}^{reg} - A_n^{reg} &= \frac{n}{4}\left(\sin\left(\frac{\pi}{n}\right) - \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)\right) \\ &= \frac{n}{4}\left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\right) \\ &= \frac{n}{4}\left(c_n^{reg} - c_n^{reg}\sqrt{1-(c_n^{reg})^2}\right), \end{aligned}$$

and the result follows. ■

We now show that if the area of a small polygon exceeds that of the regular one, then there exists another polygon with twice as many vertices with an area exceeding that of the regular one by the same amount.

Lemma 2.2 *Let $n \geq 4$ be an even number. If $\mathcal{P}$ is a small equilateral n-sided polygon such that $A(\mathcal{P}) \geq A_n^{reg} + \delta$ for some $\delta > 0$, then there exists an equilateral $2n$-sided polygon $\mathcal{Q}$ such that $A(\mathcal{Q}) \geq A_{2n}^{reg} + \delta$.*

Proof. Let $\mathcal{P}$ be a small equilateral n-sided polygon such that $\delta := A(\mathcal{P}) - A_n^{reg} > 0$, where $n \geq 4$ is an even number. Define $c = \frac{1}{n}P(\mathcal{P})$ to be the length of the equilateral sides of the polygon $\mathcal{P}$.

The equilateral polygon $\mathcal{Q}$ is constructed form $\mathcal{P}$ by adding one vertex near the center of each side of $\mathcal{P}$. Each added vertex is at distance h away from the center of one side of $\mathcal{P}$, where $h > 0$ is taken to be as large as possible so that the diameter of $\mathcal{Q}$ remains equal to one. The initial polygon $\mathcal{P}$ is represented in Figure 2 by the full lines, the added vertices are depicted by white circles and $\mathcal{Q}$ is delimited by the dotted lines.

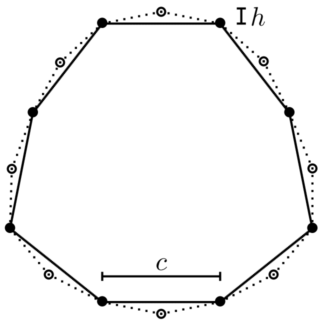

Figure 2: The dotted equilateral polygon $\mathcal{Q}$ is obtained by adding n vertices at a the same distance h from each side of the equilateral n-sided polygon $\mathcal{P}$.

The area of both polygons are related as follows:

$$A(\mathcal{Q}) \;=\; A(\mathcal{P}) + \frac{nch}{2} \;=\; A_n^{reg} + \frac{nch}{2} + \delta. \tag{1}$$

We next compute a valid lower bound $\underline{h}$ on the distance h from each added vertex to the polygon $\mathcal{P}$. The left part of Figure 3 illustrates two opposite sides of the n-sided polygon, together with the added vertices labeled A and B. By construction, the distance between them satisfies $|AB| \leq 1$. The value of h diminishes when these sides are moved away from each other. The minimal value of h occurs when both sides are parallel, and when the two pairs of vertices are at unit distance, as illustrated in the right part of the Figure 3. The distance $\underline{h}$ from the added vertices A' and B' to the polygon satisfies

$$h \;\geq\; \underline{h} \;:=\; \frac{1}{2}\left(1-\sqrt{1-c^2}\right).$$

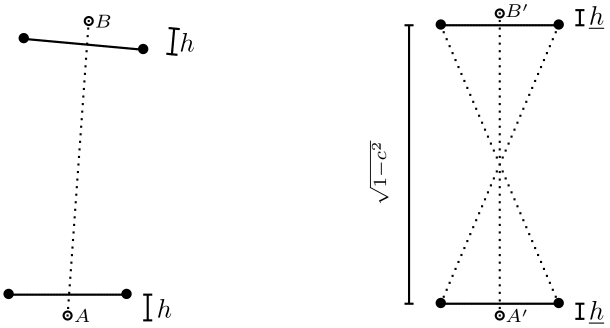

Figure 3: The distance from an added vertex to the polygon is larger when the corresponding sides are not parallel to each other: $h \geq \underline{h}$.

Recall that the maximal area enclosed by an n-sided polygon of a given perimeter is achieved by the regular polygon [9]. This implies that $P(\mathcal{P}) > P_n^{reg}$ and $c > c_n^{reg}$ because $A(\mathcal{P}) > A_n^{reg}$. Combining this into Equation (1), using Lemma 2.2 and using the fact that the function $c\left(1-\sqrt{1-c^2}\right)$ is monotone increasing ensure that

$$\begin{aligned} A(\mathcal{Q}) &\geq A_n^{reg} + \frac{nc\underline{h}}{2} + \delta \\ &\geq A_n^{reg} + \frac{nc}{4}\left(1-\sqrt{1-c^2}\right) + \delta \\ &\geq A_n^{reg} + \frac{nc_n^{reg}}{4}\left(1-\sqrt{1-(c_n^{reg})^2}\right) + \delta \\ &= A_{2n}^{reg} + \delta. \end{aligned}$$

■

Repeated applications of this last lemma allows us to prove the main result.

Theorem 2.3 *For any integer $n \geq 3$, the small n-sided equilateral polygon with the greatest area is the regular small polygon.*

Proof. Suppose by contradiction that there exists a small equilateral n-sided polygon $\mathcal{P}$ for which $A(\mathcal{P}) = A_n^{reg} + \delta$ for some scalar $\delta > 0$. The integer n is necessarily even, because the largest small polygon is the regular one when it is odd [16].

Applying Lemma 2.2 repeatedly yields a sequence of polygons with $2n, 4n, 8n, \ldots$ sides, each of which with an area exceeding that of the small regular one by the value δ. However

$$\lim_{m\to\infty} A_{2m}^{reg} + \delta \;=\; \lim_{m\to\infty} \frac{m}{4}\sin\left(\frac{\pi}{m}\right) + \delta \;=\; \frac{\pi}{4} + \delta$$

implies a contradiction: there exists a small polygon whose area exceeds $\frac{\pi}{4}$, the area of the circle with unit diameter. ■

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