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Linearized robust counterparts of two-stage robust optimization problems with applications in operations management

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Abstract: In this article, we discuss an alternative method for deriving conservative approximation models for two-stage robust optimization problems. The method extends in a natural way a linearization scheme that was recently proposed to construct tractable reformulations for robust static problems involving profit functions that decompose as a sum of piecewise linear concave expressions. Given that this generalized method mainly relies on a linearization scheme employed in bilinear optimization problems, we will say that it gives rise to the "linearized robust counterpart" model. We identify a close relation between this linearized robust counterpart model and the popular affinely adjustable robust counterpart model. We also describe a simple way of modifying both types of models in order to make these approximations less conservative. We finally demonstrate how to employ this new scheme in a set of operations management problems in order to improve the performance and guarantees of robust optimization.

Keywords: Two-stage adjustable robust optimization, linear programming relaxation, affinely adjustable robust counterpart, bilinear optimization

1 Introduction

Classical robust optimization (RO) assumes that all decisions are here-and-now, *i.e.,* they must be made before the realization of uncertainty. However this assumption is not realistic in many real-world problems. In many location-transportation problems, for example, transportation decisions can be delayed until the uncertain demand of customers is revealed. To address the uncertainty in such problems, Ben-Tal et al. (2004) introduced an adjustable robust optimization (ARO) problem that takes the following form in a two-stage setting when the uncertainty can be captured on the right-hand side of the constraint set:

$$\text{(ARO)} \quad \underset{x\in\mathcal{X},y(\zeta)}{\text{maximize}} \quad \min_{\zeta\in\mathcal{U}} c^T x + d^T y(\zeta) \tag{1a}$$

$$\text{subject to} \quad Ax + By(\zeta) \leq \Psi(x)\zeta\,,\ \forall\,\zeta\in\mathcal{U}\,, \tag{1b}$$

where $A\in\mathbb{R}^{m\times n_x}$, $B\in\mathbb{R}^{m\times n_y}$, $c\in\mathbb{R}^{n_x}$, $d\in\mathbb{R}^{n_y}$, $\Psi(x):\mathbb{R}^{n_x}\to\mathbb{R}^{m\times n_\zeta}$ such that $\Psi(x)$ is an affine mapping of variable x, where $\mathcal{U}$ is an uncertainty set for ζ, and where $y:\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_y}$ is an adjustable decision. Since problem (1) is shown to be computationally intractable, Ben-Tal et al. suggested instead solving the affinely adjustable robust counterpart (AARC) of the problem, wherein adjustable decisions are forced to be affine adjustments of the observed uncertain vector ζ, *i.e.,* $y(\zeta):=Y\zeta+y$, for some $Y\in\mathbb{R}^{n_y\times n_\zeta}$ and $y\in\mathbb{R}^{n_y}$; therefore, problem (1) is conservatively approximated with

$$\text{(AARC)} \quad \underset{x\in\mathcal{X},y,Y}{\text{maximize}} \quad \min_{\zeta\in\mathcal{U}} \ c^T x + d^T(Y\zeta+y) \tag{2a}$$

$$\text{subject to} \quad Ax + B(Y\zeta+y) \leq \Psi(x)\zeta,\ \forall\,\zeta\in\mathcal{U}. \tag{2b}$$

In recent years, the AARC framework has been successfully applied in a number of fields, including energy planning (Jabr et al., 2015; Babonneau et al., 2010), production planning (Melamed et al., 2016), telecommunications (Ouorou, 2013), emergency logistics planning (Ben-Tal et al., 2011), and supply chain management (Ben-Tal et al., 2005).

In a seemingly unrelated article, Ardestani-Jaafari and Delage (2016) recently proposed a scheme for creating conservative approximation models for static robust optimization problems that involve a sum of piecewise linear concave functions. Specifically, they study the problem of

$$\underset{x\in\mathcal{X}}{\text{maximize}} \quad g(x) := \min_{\zeta\in\mathcal{U}} \ \sum_{i=1}^{N} \min_{k=1,\ldots,K} c_\zeta^{i,k}(x)^T\zeta + d_\zeta^{i,k}(x)\,, \tag{3}$$

where $c_\zeta^{i,k}:\mathbb{R}^{n_x}\to\mathbb{R}^{n_\zeta}$ and $d_\zeta^{i,k}:\mathbb{R}^{n_x}\to\mathbb{R}$ are affine mappings of variable x. Broadly speaking, this scheme consists in linearizing the objective function of a mixed-integer programming representation of the worst-case analysis problem before relaxing the integrality constraints as shown below:

$$g(x) = \begin{array}{rl} \displaystyle\min_{\zeta\in\mathcal{U},\{\lambda^k\}_{k=1}^K} & \displaystyle\sum_{ik} c_\zeta^{i,k}(x)^T\zeta\lambda_k^i + d_\zeta^{i,k}(x)\lambda_k^i \\ \text{subject to} & \displaystyle\sum_k \lambda_k^i = 1\,,\ \forall\,i=1,\ldots,N \\ & \lambda^k\in\{0,1\}^N\,,\ \forall\,k \end{array} \ \geq \ \begin{array}{rl} \displaystyle\min_{\zeta\in\mathcal{U},\{\lambda^k\}_{k=1}^K,\{\Delta^{ik}\}_{ik}} & \displaystyle\sum_{ik} c_\zeta^{i,k}(x)^T\Delta^{ik} + d_\zeta^{i,k}(x)\lambda_k^i \\ \text{subject to} & \displaystyle\sum_k \lambda_k^i = 1\,,\ \forall\,i=1,\ldots,N \\ & \lambda^k\in[0,1]^N\,,\ \forall\,k \\ & \displaystyle\sum_k \Delta^{ik} = \zeta_i\,,\ \forall\,i\,, \end{array}$$

where each $\Delta^{ik}\in\mathbb{R}^{n_\zeta}$ is introduced to capture the relation $\Delta^{ik}=\zeta\lambda_i^k$. Note that the right-hand side model is a linear program that will lead to a compact conservative approximation of problem (3) when the dual maximization model associated with this linear program is reintroduced into problem (3). The authors show that, for a specific choice of a mixed-integer programming formulation, the resulting "linearized robust counterpart" (LRC) model is equivalent to employing affine adjustments in the following two-stage representation of (3):

$$\underset{x\in\mathcal{X},y(\zeta)}{\text{maximize}} \quad \min_{\zeta\in\mathcal{U}} \ \sum_{i=1}^{N} y_i(\zeta) \tag{4a}$$

$$\text{subject to} \qquad y_i(\zeta) \le c_\zeta^{i,k}(x)^T\zeta + d_\zeta^{i,k}(x)\,,\ \forall\, i\,,\ \forall\, k\,,\ \forall\, \zeta \in \mathcal{U}\,. \tag{4b}$$

They also show that, under some conditions on the structure of the uncertainty set and of the mappings $c_\zeta^{i,k}(\cdot)$ and $d_\zeta^{i,k}(\cdot)$, this conservative approximation is exact. Furthermore, they propose a way of improving this approximation using a semi-definite programming formulation.

In this paper, we extend the scope of the linearization scheme proposed in Ardestani-Jaafari and Delage (2016) to the set of two-stage ARO problems that take the form described in (1). In doing so, we offer the following contributions:

- We introduce a new scheme for constructing conservative approximation models (called linearized robust counterpart models) of two-stage ARO problems with right-hand side uncertainty.
- We establish a new interpretation for the conservative approximation models obtained by employing AARC on a two-stage ARO problem. This interpretation will be based on popular relaxation methods that are used (e.g. in Sherali and Alameddine (1992)) for approximating bilinear optimization problems.
- We provide a methodology to improve LRC, using linear and conic valid inequalities. This will lead to a simple procedure that can be used to improve the approximation obtained using AARC and to guarantee the feasibility of the approximation model.
- Finally, we discuss the application of the LRC and AARC models in four types of operations management problems wherein, it is possible to either demonstrate the exactness of these methods or improve the quality of their solution using valid inequalities. In particular, the LRC model we propose is used to disprove the claim that a model found in Denton et al. (2010) provides an exact reformulation for the robust version of a classical surgery block allocation problem.

The remainder of the paper is organized as follows. In Section 2, we introduce the linearized robust counterpart model associated to an ARO problem with a polyhedral uncertainty set and we define the notion of a relaxation gap that can be used to bound the sub-optimality of solutions. Next, in Section 3, we establish the equivalence between the LRC and AARC models. Section 4 describes the two methods that can be used to tighten the approximation obtained through LRC or AARC. These methods are heavily inspired from the use of linear and conic valid inequality in the linearization process for bilinear models. Section 5 briefly describes how one might extend our results to general convex sets. Finally, we present four applications to practical operations management problems in Section 6, and we conclude in Section 7.

2 The linearized robust counterpart model

In order to present the LRC model, we need to make the following three assumptions.

Assumption 1 *Let $\mathcal{U}$ be a bounded and non-empty polyhedral set defined as $\mathcal{U} := \{\zeta | P\zeta \le q\}$ where $P \in \mathbb{R}^{n_\mathcal{U}\times n_\zeta}, q \in \mathbb{R}^{n_\mathcal{U}}$.*

Assumption 2 *Let the ARO model possess relatively complete recourse, namely, that*

$$\forall\, x \in \mathcal{X}\,,\ \exists\, y(\zeta) :\ Ax + By(\zeta) \le \Psi(x)\zeta \quad \forall\, \zeta \in \mathcal{U}\,.$$

Assumption 3 *For all $x \in \mathcal{X}$, there exists a feasible ζ, such that the recourse problem is bounded. In other words, let problem (1) be bounded above.*

The three assumptions described above should not be considered limiting. Considering Assumption 1, it is typically the case that $\mathcal{U}$ includes, at very least, a nominal, or most-likely, scenario, and that all possible scenarios reside in a bounded set. Satisfying Assumption 2 is mostly a matter of formulating $\mathcal{X}$ so that it does not include any solutions for which there might be no feasible second-stage solutions, a situation that is typically associated with an infinite loss. Finally, it is reasonable to assume that problem (1) is bounded above in realistic practical problems.

Let us now consider the fact that the ARO model can be formulated as

$$\underset{x\in\mathcal{X}}{\text{maximize}} \qquad g(x) \tag{5a}$$

where $g(x)$ is defined as

$$g(x) := \min_{\zeta\in\mathcal{U}} \quad \max_{y} \quad c^T x + d^T y \tag{6a}$$

$$\text{subject to} \quad Ax + By \leq \Psi(x)\zeta\,. \tag{6b}$$

Based on Assumption 2, one can apply duality theory on the inner maximization problem to show that $g(x)$ is exactly equal to

$$g(x) = \min_{\zeta,\lambda} \quad c^T x + (\Psi(x)\zeta)^T\lambda - (Ax)^T\lambda \tag{7a}$$

$$\text{subject to} \quad B^T\lambda = d \tag{7b}$$

$$P\zeta \leq q \tag{7c}$$

$$\lambda \geq 0\,, \tag{7d}$$

where $\lambda \in \mathbb{R}^m$ is the dual variable associated with constraint (6b).

Lemma 1 *Problem (7) possesses a feasible solution.*

Proof. Assumption 3 guarantees that, for all $x \in \mathcal{X}$, there exists a feasible $\bar{\zeta}$ for which the maximization problem in y has a finite optimal value. By the strong duality property, this indicates that, for this same $\bar{\zeta}$, the minimization problem in λ also has a finite optimal value and must therefore have a feasible solution $\bar{\lambda}$. Together, the pair $(\bar{\zeta}, \bar{\lambda})$ constitutes a feasible solution for problem (7). □

In Sherali and Alameddine (1992), the authors employ a linearization scheme that exploits a set of valid inequalities for a bilinear optimization problem similar to problem (7). In the context that we study here, this scheme leads us to consider that

$$g(x) = \min_{\zeta,\lambda} \quad c^T x + \mathrm{tr}(\Psi(x)\zeta\lambda^T) - (Ax)^T\lambda \tag{8a}$$

$$\text{subject to} \quad B^T\lambda = d \tag{8b}$$

$$P\zeta \leq q \tag{8c}$$

$$\lambda \geq 0 \tag{8d}$$

$$\zeta\lambda^T B = \zeta d^T \tag{8e}$$

$$P\zeta\lambda^T \leq q\lambda^T \tag{8f}$$

$$B^T\lambda\lambda^T = d\lambda^T \tag{8g}$$

$$\lambda\lambda^T \geq 0\,, \tag{8h}$$

where $\mathrm{tr}(\cdot)$ stands for the trace operator, and where, for any two matrices A and B of the same dimension $n \times m$, a constraint $A \leq B$ stands for $A_{ij} \leq B_{ij}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and similarly for the constraint $A = B$. Note that, in this model, constraints (8e) to (8h) are a set of redundant constraints that were added to problem (7). In particular, constraint (8e) is implied from

$$(7b) \Rightarrow B^T\lambda\zeta^T = d\zeta^T \Rightarrow \zeta\lambda^T B = \zeta d^T\,.$$

Moreover, constraints (8g)–(8h) can be similarly derived:

$$(7c)\&(7d) \Rightarrow P\zeta\lambda^T \leq q\lambda^T\,,$$

$$(7b) \Rightarrow B^T\lambda\lambda^T = d\lambda^T\,,$$

$$(7d) \Rightarrow \lambda\lambda^T \geq 0\,.$$

We next linearize problem (8) by introducing the variables $\Delta \in \mathbb{R}^{n_\zeta \times m}$ and $\Lambda \in \mathbb{R}^{m\times m}$, respectively defined as $\Delta := \zeta\lambda^T$ and $\Lambda := \lambda\lambda^T$, such that

$$g(x) = \min_{\zeta,\lambda,\Delta,\Lambda} \quad c^T x + \mathrm{tr}(\Psi(x)\Delta) - (Ax)^T\lambda \tag{9a}$$

$$\begin{aligned}
\text{subject to} \quad & B^T\lambda = d && \text{(9b)}\\
& P\zeta \leq q && \text{(9c)}\\
& \lambda \geq 0 && \text{(9d)}\\
& \Delta B = \zeta d^T && \text{(9e)}\\
& P\Delta \leq q\lambda^T && \text{(9f)}\\
& B^T\Lambda = d\lambda^T && \text{(9g)}\\
& \Lambda \geq 0 && \text{(9h)}\\
& \Lambda = \lambda\lambda^T && \text{(9i)}\\
& \Delta = \zeta\lambda^T\,. && \text{(9j)}
\end{aligned}$$

A simple relaxation of problem (9) will lead to the linearized robust counterpart model for problem (1).

Proposition 1 *The following linearized robust counterpart model is a conservative approximation of problem* (1)*:*

$$\begin{aligned}
\textit{(LRC)} \qquad \underset{x\in\mathcal{X},Y,y,S,s}{\text{maximize}} \quad & c^Tx + d^Ty - q^Ts && \text{(10a)}\\
\text{subject to} \quad & P^TS = Y^TB^T - \Psi(x)^T && \text{(10b)}\\
& Ax + By + S^Tq \leq 0 && \text{(10c)}\\
& P^Ts = -Y^Td && \text{(10d)}\\
& s \geq 0\,,\ S \geq 0\,, && \text{(10e)}
\end{aligned}$$

where $Y \in \mathbb{R}^{n_y\times n_\zeta}, y \in \mathbb{R}^{n_y}, S \in \mathbb{R}^{n_\mathcal{U}\times m}$*, and* $s \in \mathbb{R}^{n_\mathcal{U}}$*.*

Proof. We start by relaxing problem (9) by removing constraint (9j) to get a lower bound for $g(x)$. Next, we consider that since, when constraints (9b) to (9f) are satisfied, one can simply let $\hat{\Lambda} := \hat{\lambda}\hat{\lambda}^T$ in order to satisfy constraints (9g)–(9i), the problem stays the same when disregarding Λ and the three constraints (9g)–(9i). Hence, we obtain a lower bound for $g(x)$ in the form

$$\begin{aligned}
g(x) \geq g_{LRC}(x) := \qquad \min_{\zeta,\lambda,\Delta} \quad & c^Tx + \text{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && \text{(11a)}\\
\text{subject to} \quad & B^T\lambda = d && \text{(11b)}\\
& P\zeta \leq q && \text{(11c)}\\
& \Delta B = \zeta d^T && \text{(11d)}\\
& P\Delta \leq q\lambda^T && \text{(11e)}\\
& \lambda \geq 0\,. && \text{(11f)}
\end{aligned}$$

Since, based on Lemma 1, there exists a solution $(\hat{\zeta}, \hat{\lambda})$ that satisfies constraints (11b), (11c), and (11f), one can confirm that the triplet $(\hat{\zeta}, \hat{\lambda}, \hat{\Delta})$, with $\hat{\Delta} := \hat{\zeta}\hat{\lambda}^T$, is a feasible solution to problem (11). Hence, strong duality applies for problem (11) so that it can be equivalently represented as

$$\begin{aligned}
g_{LRC}(x) = \qquad \max_{Y,y,S,s} \quad & c^Tx + d^Ty - q^Ts && \text{(12a)}\\
\text{subject to} \quad & P^TS = Y^TB^T - \Psi(x)^T && \text{(12b)}\\
& Ax + By + S^Tq \leq 0 && \text{(12c)}\\
& -P^Ts = Y^Td && \text{(12d)}\\
& s \geq 0\,,\ S \geq 0\,, && \text{(12e)}
\end{aligned}$$

where the variables $y \in \mathbb{R}^{n_y}$, $s \in \mathbb{R}^{n_\mathcal{U}}$, $Y \in \mathbb{R}^{n_y\times n_\zeta}$, and $S \in \mathbb{R}^{n_\mathcal{U}\times m}$ are the dual variables associated with constraints (11b), (11c), (11d), and (11e) respectively. We next combine problem (12) with the maximization over variable $x \in \mathcal{X}$, which leads to LRC (10). □

In order to provide a relation between the optimal solution of ARO and the optimal solution of LRC, we first introduce the notion of a relaxation gap.

Definition 1 *Assuming that, for all $x \in \mathcal{X}$, $g_{LRC}(x)$ is positive definite, i.e., $g_{LRC}(x) \geq 0$, the relaxation gap γ of $g_{LRC}(x)$ is defined as the bound on the largest achievable ratio between the value obtained by $g_{LRC}(x)$ and the one obtained by $g(x)$ for any feasible solution x, i.e.,*

$$g_{LRC}(x) \leq g(x) \leq \gamma g_{LRC}(x)\,,\ \forall\, x \in \mathcal{X}\,. \tag{13}$$

Next, we establish that, if one can identify a bound on the relaxation gap of $g_{LRC}(x)$, then this automatically provides a bound on the sub-optimality of the solution returned by this model.

Proposition 2 *Let $\hat{x}$ and $g_{LRC}(\hat{x})$ respectively be the optimal solution and optimal value of LRC (10); then, both the actual worst-case value of $\hat{x}$, i.e., $g(\hat{x})$, and the estimated worst-case value $g_{LRC}(\hat{x})$ are less than a factor of γ away from the optimal value of problem (1), where γ is the relaxation gap of $g_{LRC}(x)$.*

Proof. Given that x^* is the optimal solution of problem (1) and $\hat{x}$ is the optimal solution of problem (10), this indicates that

$$g_{LRC}(x^*) \leq g_{LRC}(\hat{x}) \leq g(\hat{x}) \leq g(x^*)\,, \tag{14}$$

where the second inequality is implied from the fact that $g_{LRC}(x)$ is a conservative approximation of $g(x)$ for all $x \in \mathcal{X}$. Combining (13) and (14) results in

$$g_{LRC}(x^*) \leq g_{LRC}(\hat{x}) \leq g(\hat{x}) \leq g(x^*) \leq \gamma g_{LRC}(x^*) \leq \gamma g_{LRC}(\hat{x})\,,$$

and consequently, it results in

$$g_{LRC}(\hat{x}) \leq g(\hat{x}) \leq g(x^*) \leq \gamma g_{LRC}(\hat{x})\,.$$

□

Furthermore, while it is known that evaluating the worst-case value of a given first-stage solution, *i.e.*, evaluating $g(x)$, is computationally intractable, we can show that it is possible to efficiently evaluate a bound on the worst-case value of the solution of the LRC model.

Lemma 2 *Given that $\hat{x}$ is the optimal solution of LRC (10) and $(\hat{\lambda},\, \hat{\zeta},\, \hat{\Delta})$ is the optimal solution of problem (11), when x is fixed to $\hat{x}$, we have that*

$$0 \leq g(\hat{x}) - g_{LRC}(\hat{x}) \leq \hat{\lambda}^T \Psi(x)\hat{\zeta} - \mathrm{tr}(\Psi(x)\hat{\Delta})\,.$$

Proof. Given that $(\hat{\lambda},\, \hat{\zeta})$ is a feasible solution of problem (6) when x is fixed to $\hat{x}$, and since $g_{LRC}(\hat{x}) = c^T\hat{x} + \sum_{ij} \Psi(x)_{ij}\hat{\Delta}_{ji} - (A\hat{x})^T\hat{\lambda}$, this indicates that

$$0 \leq g(\hat{x}) - g_{LRC}(\hat{x}) \leq (\Psi(x)\hat{\zeta})^T\hat{\lambda} - \mathrm{tr}(\Psi(x)\hat{\Delta})\,.$$

□

3 Relation to AARC

In this section, we explain how the LRC model can be considered equivalent to the conservative approximation model obtained with AARC.

Proposition 3 *LRC (10) is equivalent to AARC (2).*

Proof. As a first step, we reformulate problem (11) in terms of an inner and an outer minimization operations

$$\begin{aligned} g_{LRC}(x) = \min_{\zeta\in\mathcal{U}} \quad & \min_{\lambda,\Delta} \quad && c^T x + \mathrm{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && (15a)\\ & \text{subject to} \quad && B^T\lambda = d && (15b)\\ & && \Delta B = \zeta d^T && (15c)\\ & && P\Delta \leq q\lambda^T && (15d)\\ & && \lambda \geq 0\,. && (15e) \end{aligned}$$

We next derive the dual formulation of the inner minimization over λ and Δ as

$$\begin{aligned} \max_{y,Y,S} \quad & c^T x + d^T(y + Y\zeta) && (16a)\\ \text{subject to} \quad & Ax + By + S^T q \leq 0 && (16b)\\ & P^T S = Y^T B^T - \Psi(x)^T && (16c)\\ & S \geq 0\,, && (16d) \end{aligned}$$

where $y \in \mathbb{R}^{n_y}, Y \in \mathbb{R}^{n_y \times n_\zeta}$ and $S \in \mathbb{R}^{n_\mathcal{U} \times m}$ are the dual variables associated with constraints (15b), (15c), and (15d) respectively. Based on Sion's minimax theorem, since $\mathcal{U}$ is bounded, the same value for $g_{LRC}(x)$ can be obtained by reversing the order of minimization over ζ and maximization over y, Y, and S. Equivalently, we have that

$$\begin{aligned} g_{LRC}(x) = \quad & \max_{y,Y,S} \quad && \min_{\zeta\in\mathcal{U}} c^T x + d^T(Y\zeta + y) && (17a)\\ & \text{subject to} \quad && Ax + By + S^T q \leq 0 && (17b)\\ & && P^T S = Y^T B^T - \Psi(x)^T && (17c)\\ & && S \geq 0\,. && (17d) \end{aligned}$$

We next consider the i^{th} row of constraint (17b) and the i^{th} column of constraint (17c):

$$\begin{aligned} & A_{i:}x + B_{i:}w + (S_{:i})^T q \leq 0, && (18a)\\ & P^T S_{:i} = Y^T (B_{i:})^T - (\Psi(x)_{i:})^T && (18b) \end{aligned}$$

where $A_{i:}$, $B_{i:}$, and $\Psi(x)_{i:}$ denote the i^{th} row of matrices A, B, and $\Psi(x)$ respectively, and $S_{:i}$ denotes the i^{th} column of matrix S. We show that constraints (18a) and (18b) are equivalent to

$$A_{i:}x + B_{i:}(Y\zeta + y) \leq \Psi(x)_{i:}\zeta,\ \forall\,\zeta \in \mathcal{U}\,. \tag{19}$$

We do so by considering that S is not in the objective function so that we can remove S from the set of decision variables and instead replace constraints (18a) and (18b) with

$$\begin{aligned} \min_{S_{:i}} \quad & A_{i:}x + B_{i:}w + (S_{:i})^T q \quad \leq 0\,, && (20a)\\ \text{subject to} \quad & P^T S_{:i} = X^T (B_{i:})^T - (\Psi(x)_{i:})^T\,, && (20b) \end{aligned}$$

where the embedded minimization problem can be replaced by a maximization problem, using duality theory. This leads us to considering constraint (20) as equivalent to

$$\begin{aligned} \max_{\zeta} \quad & A_{i:}x + B_{i:}w + (X^T(B_{i:})^T - (\Psi(x)_{i:})^T)^T\zeta \quad \leq 0\,,\\ \text{subject to} \quad & P\zeta \leq q \end{aligned}$$

with ζ the dual variable of (20b). We have thus confirmed that constraints (18a) and (18b) are equivalent to (19), and likewise that constraints (17b) and (17c) are equivalent to constraint (2b). Therefore, the LRC model (10) is equivalent to the AARC model (2). $\square$

4 Improving LRC and AARC using valid inequalities

In this section, we identify two types of valid inequalities that can be employed to formulate improved versions of LRC that provide tighter conservative approximations. First, we will make use of valid linear inequalities that can be derived from an implicit upper bound on the optimal solution for λ in problem (7). This process will lead to a modified LRC model that preserves the computational complexity of LRC while guaranteeing the feasibility of the resulting approximation model. Secondly, we will identify a set of conic valid inequalities that will lead to a semi-definite programming formulation for LRC.

We start with a simple assumption that can be used to generate helpful valid inequalities for problem (7) and obtain our Modified LRC (MLRC) model.

Assumption 4 *There exists a bounding vector $u \in \mathbb{R}^m$ such that, for all $x \in \mathcal{X}$ and for all $\zeta \in \mathcal{U}$, there exists an optimal solution $\lambda^* \leq u$ for the problem*

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & (\Psi(x)\zeta - Ax)^T \lambda && (21a)\\ \text{subject to} \quad & B^T\lambda = d && (21b)\\ & \lambda \geq 0\,. && (21c) \end{aligned}$$

Proposition 4 *Given Assumption 4, the following modified linearized robust counterpart model is a conservative approximation to problem (1):*

$$\begin{aligned} (MLRC) \quad \underset{x\in\mathcal{X},Y,y,S,s,W,w}{\text{maximize}} \quad & c^Tx + d^Ty - q^Ts - u^Tw - u^TWq && (22a)\\ \text{subject to} \quad & Y^TB^T - P^T(S-W) = \Psi(x)^T && (22b)\\ & Ax + By + (S-W)^Tq - w \leq 0 && (22c)\\ & -P^T(s+Wu) = Y^Td && (22d)\\ & s \geq 0\,,\ S \geq 0\,,\ w \geq 0\,,\ W \geq 0\,, && (22e) \end{aligned}$$

where $Y \in \mathbb{R}^{n_y \times n_\zeta}$, $y \in \mathbb{R}^{n_y}$, $S \in \mathbb{R}^{n_\mathcal{U} \times m}$, $s \in \mathbb{R}^{n_\mathcal{U}}$, $W \in \mathbb{R}^{n_\mathcal{U} \times m}$, and $w \in \mathbb{R}^m$. Furthermore, the optimal value of problem (22) is necessarily larger than or equal to the optimal value of LRC (10).

Proof. Given that Assumption 4 is satisfied, the following constraints are valid inequalities for problem (7) in the sense that they can be added to this problem without affecting its optimal value:

$$\begin{aligned} & \lambda \leq u && (23a)\\ & (q - P\zeta)(u-\lambda)^T \geq 0 && (23b) \end{aligned}$$

where constraint (23b) can be linearized by replacing $\Delta := \zeta\lambda^T$ as

$$P\Delta \geq q\lambda^T - (q - P\zeta)u^T\,. \tag{24}$$

Adding constraints (23a) and (24) to problem (11) leads to MLRC after applying duality theory. □

One might consider that Assumption 4 is difficult to exploit in practice since λ is a variable that does not have a clear physical meaning. For this reason, we propose a numerical procedure that can be used, in a specific problem instance, to identify a u that satisfies this useful assumption. We will later show in two practical examples presented in Section 6 that a good value for u can also often be obtained analytically.

Proposition 5 *For any fixed $k = 1, 2, \ldots, m$, one can identify an upper bound u_k that satisfies Assumption 4 by solving the following mixed-integer program*

$$\begin{aligned} \underset{x\in\mathcal{X},\zeta\in\mathcal{U},y,\lambda,v}{\max} \quad & \lambda_k && (25a)\\ \text{subject to} \quad & Ax + By \leq \Psi(x)\zeta && (25b) \end{aligned}$$

$$A^T\lambda = d \tag{25c}$$
$$\lambda_i \leq Mv_i\,,\ \forall\, i \tag{25d}$$
$$-a_i^T x - b_i^T y + \psi_i^T\zeta \leq M(1-v_i)\,,\ \forall\, i \tag{25e}$$
$$\lambda \geq 0\,,\ v \in \{0,1\}^m\,, \tag{25f}$$

where M is a large enough constant. Furthermore, this optimization problem reduces to a mixed-integer linear program when $\Psi(x) = \Psi$, i.e., it is independent of x, and the feasible set $\mathcal{X}$ can be represented using linear constraints.

Proof. For any fixed $x \in \mathcal{X}$ and $\zeta \in \mathcal{U}$, it is well known that a certain λ^* is optimal in problem (21) if and only if it can be paired to some y^* so that together they satisfy the following KKT conditions:

$$Ax + By^* \leq \Psi(x)\zeta \tag{26a}$$
$$A^T\lambda^* = d \tag{26b}$$
$$\lambda^* \geq 0 \tag{26c}$$
$$\lambda_i^*(-a_i^T x - b_i^T y^* + \Psi(x)_i^T\zeta) = 0\,,\ \forall\, i\,. \tag{26d}$$

It is therefore clear that, by solving the following optimization problem, one gets a valid candidate for u_k:

$$\begin{aligned}
\underset{x\in\mathcal{X},\zeta\in\mathcal{U},y,\lambda}{\text{maximize}} \quad & \lambda_k \\
\text{subject to} \quad & Ax + By \leq \Psi(x)\zeta \\
& A^T\lambda = d \\
& \lambda \geq 0 \\
& \lambda_i(-a_i^T x - b_i^T y + \Psi(x)_i^T\zeta) = 0\,,\ \forall\, i\,.
\end{aligned}$$

One can finally linearize the complementary slackness conditions by introducing binary variables into this problem, *i.e.,* $v_i \in \{0,1\}$ stating whether λ_i is equal to zero or not. This leads to problem (25). □

Similarly as was the case for the original LRC model, one can uncover an intimate connection between MLRC and conservative approximations that are obtained using AARC. This connection is made explicit in the following proposition.

Proposition 6 *The MLRC (22) is equivalent to applying affine adjustments to the following two-stage problem:*

$$\begin{aligned}
\underset{x\in\mathcal{X},y(\zeta),z(\zeta)}{\text{maximize}} \quad & \min_{\zeta\in\mathcal{U}} c^T x + d^T y(\zeta) + u^T z(\zeta) && (27a)\\
\text{subject to} \quad & Ax + By(\zeta) \leq \Psi(x)\zeta + z(\zeta),\ \forall\, \zeta \in \mathcal{U} && (27b)\\
& z(\zeta) \geq 0\,,\ \forall\, \zeta \in \mathcal{U}\,. && (27c)
\end{aligned}$$

Note that the optimal affine adjustments obtained from solving model (27) might not be implementable, as their only purpose is to identify good first-stage decisions. This means that once x^* is implemented and $\bar{\zeta}$ is observed, it is important to solve the specific recourse problem that is being experienced, *i.e.,* $\text{maximize}_y\, d^T y$, subject to $Ax^* + By \leq \Psi(x^*)\bar{\zeta}$.

Proof. Adding constraints (23a) and (24) to problem (11) leads to the following formulation:

$$\begin{aligned}
g_{MLRC}(x) := \quad \min_{\zeta,\lambda,\Delta} \quad & c^T x + \mathrm{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && (28a)\\
\text{subject to} \quad & B^T\lambda = d && (28b)\\
& P\zeta \leq q && (28c)
\end{aligned}$$

$$
\begin{aligned}
&\Delta B = \zeta d^T && \text{(28d)}\\
&P\Delta \leq q\lambda^T && \text{(28e)}\\
&0 \leq \lambda \leq u && \text{(28f)}\\
&P\Delta \geq q\lambda^T - (q - P\zeta)u^T\,. && \text{(28g)}
\end{aligned}
$$

Similarly to what was described in the proof of Proposition 3, the function $g_{MLRC}(x)$ can be reformulated as

$$
\begin{aligned}
g_{MLRC}(x) = \quad & \max_{Y,y,W,w,S} && \min_{\zeta\in\mathcal{U}} c^T x + d^T(Y\zeta + y) + u^T(w + W^T(q - P\zeta)) && \text{(29a)}\\
& \text{subject to} && P^T S = Y^T B^T - \Psi(x)^T + P^T W && \text{(29b)}\\
&&& Ax + Bw + S^T q \leq W^T q + w && \text{(29c)}\\
&&& S \geq 0 && \text{(29d)}\\
&&& w \geq 0\,,\ W \geq 0\,, && \text{(29e)}
\end{aligned}
$$

where $w \in \mathbb{R}^m$ and $W \in \mathbb{R}^{n_{\mathcal{U}}\times m}$ are respectively the dual variables associated with constraints (28f) and (28g). Again, the constraints (29b)–(29d) can be replaced with

$$
Ax + B(Y\zeta + y) \leq \Psi(x)\zeta + w + W^T(q - P\zeta),\ \forall\,\zeta\in\mathcal{U}\,,
$$

and decision variable S removed from the optimization problem. In this way, we obtain

$$
\begin{aligned}
g_{MLRC}(x) = \quad & \max_{Y,y,W,w} && \min_{\zeta\in\mathcal{U}} c^T x + d^T(Y\zeta + y) + u^T(w + W^T(q - P\zeta)) && \text{(30a)}\\
& \text{subject to} && Ax + B(Y\zeta + y) \leq \Psi(x)\zeta + w + W^T(q - P\zeta),\ \forall\,\zeta\in\mathcal{U} && \text{(30b)}\\
&&& w \geq 0\,,\ W \geq 0\,. && \text{(30c)}
\end{aligned}
$$

We next introduce new variables z and Z as

$$
z := w + W^T q\,, \qquad Z := -W^T P\,,
$$

where $z \in \mathbb{R}^m$ and $Z \in \mathbb{R}^{m\times n_\zeta}$. Therefore, problem (29) can be reformulated as

$$
\begin{aligned}
g_{MLRC}(x) = \quad & \max_{Y,y,W,Z,z} && \min_{\zeta\in\mathcal{U}} c^T x + d^T(Y\zeta + y) + u^T(z + Z\zeta) && \text{(31a)}\\
& \text{subject to} && Ax + B(Y\zeta + y) \leq \Psi(x)\zeta + z + Z\zeta,\ \forall\,\zeta\in\mathcal{U} && \text{(31b)}\\
&&& Z = -W^T P && \text{(31c)}\\
&&& z - W^T q \geq 0 && \text{(31d)}\\
&&& W \geq 0\,. && \text{(31e)}
\end{aligned}
$$

We finally show that constraints (31c)–(31e) are equivalent to the following constraint:

$$
z + Z\zeta \geq 0\,,\ \forall\,\zeta\in\mathcal{U} \qquad \text{(32)}
$$

This is done by considering that, for some fixed j, duality can once again be used to reformulate the constraint that

$$
\begin{aligned}
& \max_{w_{:j}} && z_j - W_{:j}^T q && \geq 0 && \text{(33a)}\\
& \text{subject to} && Z_{j:}^T = -P^T W_{:j} &&&& \text{(33b)}\\
&&& W_{:j} \geq 0 &&&& \text{(33c)}
\end{aligned}
$$

as the constraint that

$$
\begin{aligned}
& \min_{\zeta} && z_j + Z_{j:}\zeta && \geq 0\,. && \text{(34a)}\\
& \text{subject to} && P\zeta \leq q &&&& \text{(34b)}
\end{aligned}
$$

This completes our proof. □

The equivalence between MLRC and AARC can be easily exploited to demonstrate that MLRC is always a feasible approximation model, namely it provides a feasible first-stage solution even in situations where AARC model (2) is an infeasible problem.

Corollary 1 *The MLRC (22) always admits a feasible solution.*

This result follows easily from the fact that MLRC is equivalent to applying affine adjustments to problem (27) which can be shown to have a feasible solution. In particular, let $\bar{x}$ be any member of $\mathcal{X}$, while letting $y(\zeta) := 0$ and for each $z_i(\zeta) := \max(0; \max_{\zeta\in\mathcal{U}} A\bar{x} - \Psi(\bar{x})\zeta)$ which is finite since $\mathcal{U}$ was assumed to be bounded. There must therefore exist a feasible assignment for MLRC (22) otherwise it would not be equivalent to employing affine adjustments in problem (27) which would contradict Proposition 6.

Our second source of improvement for the LRC model comes from considering the following set of quadratic equalities:

$$\begin{bmatrix} \Lambda & \Delta^T \\ \Delta & \Xi \end{bmatrix} = \begin{bmatrix} \lambda \\ \zeta \end{bmatrix} \begin{bmatrix} \lambda^T & \zeta^T \end{bmatrix}, \tag{35}$$

where $\Lambda \in \mathbb{R}^{m\times m}$ and $\Xi \in \mathbb{R}^{n_\zeta\times n_\zeta}$, such that $\Lambda := \lambda\lambda^T$ and $\Xi := \zeta\zeta^T$. It is well known that this system of equations can be relaxed using the following matrix inequality

$$\begin{bmatrix} \Lambda & \Delta^T \\ \Delta & \Xi \end{bmatrix} \succeq \begin{bmatrix} \lambda \\ \zeta \end{bmatrix} \begin{bmatrix} \lambda^T & \zeta^T \end{bmatrix},$$

where $A \succeq B$ indicates that $A - B$ is in the cone of positive semi-definite matrices. This non-linear matrix inequality reduces to a linear matrix inequality after applying Schur's complement

$$\begin{bmatrix} \Lambda & \Delta^T & \lambda \\ \Delta & \Xi & \zeta \\ \lambda^T & \zeta^T & 1 \end{bmatrix} \succeq 0\,.$$

This constraint can be added to problem (11) with additional valid inequalities involving Λ and Ξ to obtain the tighter SDP-LRC model.

Proposition 7 *Given Assumption 4, the following semi-definite programming linearized robust counterpart is a conservative approximation of problem (1):*

$$\begin{aligned}
g_{\text{SDP-LRC}}(x) = \quad & \min_{\zeta,\lambda,\Delta,\Lambda,\Xi} && c^T x + \mathrm{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && \text{(36a)}\\
& \text{subject to} && B^T\lambda = d && \text{(36b)}\\
&&& P\zeta \le q && \text{(36c)}\\
&&& 0 \le \lambda \le u && \text{(36d)}\\
&&& \Delta B = \zeta d^T && \text{(36e)}\\
&&& P\Delta \le q\lambda^T && \text{(36f)}\\
&&& P\Delta \ge q\lambda^T - (q - P\zeta)u^T && \text{(36g)}\\
&&& \begin{bmatrix} \Lambda & \Delta^T & \lambda \\ \Delta & \Xi & \zeta \\ \lambda^T & \zeta^T & 1 \end{bmatrix} \succeq 0 && \text{(36h)}\\
&&& \Lambda B = \lambda d^T && \text{(36i)}\\
&&& P\Xi P^T + qq^T \ge P\zeta q^T + q\zeta^T P^T && \text{(36j)}\\
&&& 0 \le \Lambda \le u\lambda^T\,. && \text{(36k)}
\end{aligned}$$

Moreover, the optimal value of $\max_{x\in\mathcal{X}} g_{\text{SDP-LRC}}(x)$ *is necessarily larger than or equal to the optimal value of MLRC (22) and LRC (10).*

Proof. We simply start by including the new variables Λ and Ξ and constraint (36) in problem (11). One can then realize that constraints (9g) and (9h) can now help tighten the feasible region. Finally, a final tightening step can be achieved by exploiting the fact that $P\zeta \leq q$ implies the following:

$$q - P\zeta \geq 0 \Rightarrow (q - P\zeta)(q - P\zeta)^T \geq 0 \Rightarrow P\zeta\zeta^T P^T + qq^T \geq P\zeta q^T + q\zeta^T P^T$$

and that $\lambda \leq u$ implies that $\lambda\lambda^T \leq u\lambda^T$, which together lead to constraint (36j) after replacing $\Xi := \zeta\zeta^T$ and $\Lambda := \lambda\lambda^T$. □

Note that, although we presented $g_{\text{SDP-LRC}}(x)$ as a minimization problem, a semi-definite programming duality can be employed to obtain a maximization representation of this function that can be integrated with the maximization in x as was done with other LRC models. We however omit the details of this reformulation for aesthetics reasons. Given the connections to AARC that were established regarding the LRC and MLRC models, we suspect that a similar connection could be obtained for the SDP-LRC model. In fact, the authors of Ardestani-Jaafari and Delage (2016) were able to establish such a connection for a special case of the SDP-LRC model. A quick look at their result suggests that the connection that could be established here is highly technical and would provide rather limited new insights.

5 LRC for general uncertainty sets

In this section, we extend our LRC model so that it can accomodate general convex uncertainty sets, *i.e.,* $\mathcal{U}$ is not polyhedral. In this regard, we will instead consider uncertainty sets that can be represented as

$$\mathcal{U}_{general} := \{\zeta \in \mathbb{R}^{n_\zeta} \mid f_l(\zeta) \leq q_l, \forall\, l = 1, \ldots, L\} \tag{37}$$

using a set of convex $f_l(\cdot)$ functions. To establish an extension of LRC, we will need to make use of the notion of perspective functions for a special class of convex functions.

Assumption 5 *The uncertainty set $\mathcal{U}_{general}$ defines a bounded convex set containing a strictly feasible solution $\bar{\zeta}$ such that $f_l(\bar{\zeta}) < q_l$ for all $l = 1, \ldots, L$. Furthermore, for each $l = 1, \ldots, L$, the function $f_l(\cdot)$ is a lower semi-continuous convex function. This implies that, according to the Fenchel-Moreau Theorem, it must be that $f_l(x) = \sup_y x^T y - f_*(y)$, where $f_{l*}(y)$ is the convex conjugate of $f_l(\cdot)$, i.e., $f_{l*}(y) := \sup_z y^T z - f_l(z)$.*

Definition 2 *(Hiriart-Urruty and Lemaréchal, 2001) For each $l = 1, \ldots, L$, let $h_l : \mathbb{R}^{n_\zeta} \times \mathbb{R}_+ \to \mathbb{R}$ be the closure of the perspective function $h_l(z,t) := cl\, t f_l(z/t)$. In particular, given that $f_l(\cdot)$ satisfies Assumption 5, we have that*

$$h_l(z,t) := \sup_y z^T y - t f_{l*}(y) = \begin{cases} t f_l(z/t) & \text{if } t > 0 \\ \lim_{t\to 0^+} t f_l(z/t) & \text{if } t = 0 \\ \infty & \text{otherwise} \end{cases}.$$

Based on this definition, it is clear that $h_l(z,t)$ is jointly convex in z and t.

Under the uncertainty set $\mathcal{U}_{general}$, when Assumption 4 is satisfied, the value of $g(x)$ becomes

$$g(x) = \min_{\zeta,\lambda} \quad c^T x + (\Psi(x)\zeta)^T\lambda - (Ax)^T\lambda \tag{38a}$$

$$\text{subject to} \quad B^T\lambda = d \tag{38b}$$

$$f_l(\zeta) \leq q_l, \forall\, l \tag{38c}$$

$$0 \leq \lambda \leq u \tag{38d}$$

$$\zeta\lambda^T B = \zeta d^T \tag{38e}$$

$$\lambda_i f_l(\zeta) \leq \lambda_i q_l\,, \forall\, i\,, \forall\, l \tag{38f}$$

$$(q_l - f_l(\zeta))(u_i - \lambda_i) \geq 0\,, \forall\, i\,, \forall\, l\,. \tag{38g}$$

As was done for polyhedral sets, under Assumption 5 this optimization model can be linearized using perspective functions:

$$
\begin{aligned}
g(x) \geq \; g_{GLRC}(x) = \quad & \min_{\zeta,\lambda,\Delta} && c^T x + \operatorname{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && \text{(39a)}\\
& \text{subject to} && B^T\lambda = d && \text{(39b)}\\
& && f_l(\zeta) \leq q_l, \, \forall\, l && \text{(39c)}\\
& && 0 \leq \lambda \leq u && \text{(39d)}\\
& && \Delta B = \zeta d^T && \text{(39e)}\\
& && h_l(\Delta_{:i}, \lambda_i) \leq q_l \lambda_i \,,\, \forall\, i \,,\, \forall\, l && \text{(39f)}\\
& && h_l(\zeta u_i - \Delta_{:i}, u_i - \lambda_i) \leq q_l(u_i - \lambda_i)\,, && \text{(39g)}
\end{aligned}
$$

where constraint (39f) is derived from

$$
f_l(\zeta) \leq q_l \Rightarrow \lambda_i f_l(\zeta) \leq q_l\lambda_i \Rightarrow \lambda_i(\sup_y \zeta^T y - f_{l*}(y)) \leq q_l\lambda_i \Rightarrow \sup_y (\lambda_i\zeta)^T y - \lambda_i f_{l*}(y) = h(\Delta_{:i}, \lambda_i) \leq q_l\lambda_i
$$

and constraint (39g) is derived from

$$
\begin{aligned}
\big(q_l - f_l(\zeta)\big)(u_i - \lambda_i) \geq 0 &\Rightarrow \sup_y (u_i - \lambda_i)\zeta^T y - (u_i - \lambda_i) f_{l*}(y) \leq q_l(u_i - \lambda_i)\\
&\Rightarrow h_l(\zeta u_i - \Delta_{:i}, u_i - \lambda_i) \leq q_l(u_i - \lambda_i)
\end{aligned}
$$

for all i and for all l, and where the term $\zeta\lambda_i$ is linearized through $\Delta_{:i}$. One might apply duality theory to problem (39) to derive a compact mathematical programming representation of the LRC model under $\mathcal{U}_{general}$. Regarding the relation between this more general LRC model and AARC, Appendix A will demonstrate that the problem maximize$_{x\in\mathcal{X}}\, g_{GLRC}(x)$ always produces a tighter conservative approximation than employing affine adjustments in problem (27). It is worth noting however that, based on the proof presented in Appendix A, it appears legitimate to believe that the two schemes are equivalent in most practical situations.

6 Examples

In this section, we provide some examples to show how to apply LRC in practice. The first example is a robust multi-item newsvendor problem that is an instance of problem (3) for which it is possible to identify conditions under which the LRC is an exact model, *i.e.,* the relaxation gap γ that is described in Proposition 2 is equal to one. Moreover, we describe two logistics applications of ARO wherein LRC can be improved by using the linear valid inequalities presented in Proposition 4. Finally, we describe an application of ARO to a surgery block allocation problem, where LRC can be improved by using the conic valid inequalities presented in Proposition 7. Surprisingly, a straightforward application of affine adjustment will be shown to have the potential to provide strictly better solutions than the model presented in Denton et al. (2010), although the latter model was believed by the authors to provide an optimal robust solution.

We note that, in this section, in order to be more concise in our descriptions, we let $\mathcal{I}$ and $\mathcal{J}$ represent the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively, and consider that $i \in \mathcal{I}$ and $j \in \mathcal{J}$ when the membership for i and j is left unspecified.

Example 1 *(Multi-item newsvendor problem) Consider the following robust multi-item newsvendor problem:*

$$
\max_{x\in\mathcal{X}} \; \min_{\zeta\in\mathcal{U}} \sum_j r_j \min(x_j, \zeta_j) - c_j x_j + s_j \max_j(x_j - \zeta_j, 0) - p_j \max(\zeta_j - x_j, 0)\,, \tag{40}
$$

where r_j, c_j, $s_j \leq r_j$, and p_j denote sale price, ordering cost, salvage price, and shortage cost of a unit of the j-th item, $j \in \mathcal{J}$, respectively, and ζ_j denotes the demand for item j for each j. Problem (40) is a special case of ARO, as

$$
\max_{x\in\mathcal{X}, y(\zeta)} \quad \min_{\zeta\in\mathcal{U}} \sum_j y_j(\zeta)
$$

$$
\begin{aligned}
\text{subject to} \quad & y_j(\zeta) \le (r_j - c_j)x_j - (r_j - s_j)(x_j - \zeta_j)\,,\ \forall\, j \in \mathcal{J}\,, \forall\, \zeta \in \mathcal{U} \\
& y_j(\zeta) \le (r_j - c_j)x_j - p_j(\zeta_j - x_j)\,,\ \forall\, j \in \mathcal{J}\,, \forall\, \zeta \in \mathcal{U}\,.
\end{aligned}
$$

In Ardestani-Jaafari and Delage (2016), the authors show that $g_{LRC}(x)$ is tight in this problem when $\mathcal{U}$ is defined as

$$
\mathcal{U}(\Gamma) = \left\{ \zeta \;\middle|\; \exists\, \delta^+, \delta^- \in \mathbb{R}^n, \; \begin{array}{l} \delta_j^- \ge 0, \delta_j^+ \ge 0 \\ \delta_j^+ + \delta_j^- \le 1\,, \forall\, j \\ \sum_j \delta_j^+ + \delta_j^- = \Gamma \\ \zeta_j = \bar{\zeta}_j + (\delta_j^+ - \delta_j^-)\hat{\zeta}_j \end{array} \right\},
$$

affine adjustments are made with respect to (δ^+, δ^-), and when Γ is an integer value. Furthermore, one could explore for other uncertainty set whether it is possible to get a tighter conservative approximation by employing Proposition 4. Yet, one can show that the dual variables of the recourse problem are implicitly bounded in this multi-item newsvendor problem. Specifically, the dual problem takes the shape of

$$
\begin{aligned}
\min_{\lambda^1, \lambda^2} \quad & \sum_j \lambda_j^1((r_j - c_j)x_j - (r_j - s_j)(x_j - \zeta_j)) + \sum_j \lambda_j^2((r_j - c_j)x_j - p_j(\zeta_j - x_j)) \\
\text{subject to} \quad & \lambda_j^1 + \lambda_j^2 = 1\,, \forall\, j \in \mathcal{J}\,, \\
& \lambda^1 \ge 0\,,\ \lambda^2 \ge 0\,,
\end{aligned}
$$

which already implies that $\lambda^1 \le 1$ and $\lambda^2 \le 1$ at optimum. Hence, MLRC model (22) with $u = 1$ becomes trivially equivalent to LRC model (10) in this case. In order to obtain a tighter conservative approximation, one should instead employ the SDP-LRC approximation model.

Now, let consider an example with $n = 3$, $r = [\ 80 \quad 80 \quad 80\ ]$, $c = [\ 70 \quad 50 \quad 20\ ]$, $s = [\ 20 \quad 15 \quad 10\ ]$, and $p = [\ 60 \quad 60 \quad 50\ ]$. Demand vector ζ is defined in the following uncertainty set $\mathcal{U}$:

$$
\tilde{\mathcal{U}}(\Gamma) = \left\{ \zeta \;\middle|\; \exists\, \delta^+, \delta^- \in \mathbb{R}^n, \quad \begin{array}{l} \delta_j^- \ge 0, \delta_j^+ \ge 0 \\ \delta_j^+ + \delta_j^- \le 1\,, \forall\, j \\ \sum_j \delta_j^+ + \delta_j^- = \Gamma \end{array} \quad \& \quad \begin{array}{l} \zeta_1 = \bar{\zeta}_j + \hat{\zeta}_j(\delta_1^+ + \delta_2^+ - \delta_1^- - \delta_2^-)/2 \\ \zeta_2 = \bar{\zeta}_j + \hat{\zeta}_j(\delta_2^+ + \delta_3^+ - \delta_2^- - \delta_3^-)/2 \\ \zeta_3 = \bar{\zeta}_j + \hat{\zeta}_j(\delta_3^+ + \delta_1^+ - \delta_3^- - \delta_1^-)/2 \end{array} \right\},
$$

We compare, as it is shown in Table 1, the optimal profit and the worst-case profit of LRC, SDP-LRC, and the SDP model, denoted by SDP (A&D) below, proposed by Ardestani-Jaafari and Delage (2016) for this problem. In this example, LRC is not exact but it can be improved using SDP models such that SDP (A&D) and SDP-LRC improve respectively the worst-case profit of LRC by factor 3 and 16, and the sub-optimality gap of LRC is decreased from %95 to %19.5 when we use SDP-LRC. This example implies the impact of using SDP-LRC compared to LRC and the SDP (A&D).

Table 1: Comparison of optimal bound and worst-case profit associated to solutions obtained from approximation models in a newsvendor problem

	LRC	SDP (A&D)	SDP-LRC	Exact model
Optimal bound on worst-case profit	41.83	113.01	411.08	825.83
Worst-case profit of solution	41.83	150.94	664.76	825.83

Example 2 *(Location-transportation problem) the robust location-transportation problem (RLTP) can be formulated as*

$$
\begin{aligned}
\underset{x, y(\zeta), v}{\text{maximize}} \quad & \min_{\zeta \in \mathcal{U}} \; -\sum_i c_i x_i - k_i v_i + \sum_i \sum_j \eta_{ij} y_{ij}(\zeta) && \text{(41a)} \\
\text{subject to} \quad & \sum_i y_{ij}(\zeta) \le \zeta_j\,,\ \forall\, j \in \mathcal{J}\,,\ \forall\, \zeta \in \mathcal{U} && \text{(41b)} \\
& \sum_j y_{ij}(\zeta) \le x_i\,,\ \forall\, i \in \mathcal{I}\,,\ \forall\, \zeta \in \mathcal{U} && \text{(41c)} \\
& y(\zeta) \ge 0\,,\ \forall\, \zeta \in \mathcal{U} && \text{(41d)}
\end{aligned}
$$

$$0 \le x_i \le M v_i \,,\ \forall\, i \in \mathcal{I} \tag{41e}$$
$$v_i \in \{0,1\} \,,\ \forall\, i \in \mathcal{I} \,. \tag{41f}$$

In this problem, variable v_i indicates that, if there is an open facility in location i for each $i \in \mathcal{I}$, variable x_i denotes the production capacity of the facility i, and variable y_{ij} denotes how many goods are shipped from facility i to customers at location j, with $j \in \mathcal{J}$. The demand for location j is characterized by ζ_j. Parameter $\eta_{ij} > 0$ denotes the unit revenue of goods shipped from facility i to customer j, while c_i and k_i denote variable and fixed capacity cost for facility i respectively.

Here, the dual formulation of the recourse function takes the form

$$\underset{\lambda^1,\lambda^2,\lambda^3}{\text{minimize}} \quad \sum_j \zeta_j \lambda_j^1 + \sum_i x_i \lambda_i^2 \tag{42a}$$
$$\text{subject to} \quad \lambda_j^1 + \lambda_i^2 - \lambda_{ij}^3 = \eta_{ij} \,,\ \forall\, i \in \mathcal{I} \,,\ \forall\, j \in \mathcal{J} \tag{42b}$$
$$\lambda^1 \ge 0 \,,\ \lambda^2 \ge 0 \,,\ \lambda^3 \ge 0 \,, \tag{42c}$$

where λ^1, λ^2, and $\lambda^3 \in \mathbb{R}^{m\times n}$ are dual variables associated with constraints (41b)–(41d) respectively. Since the objective function of problem (42) is non-decreasing in λ^1 and λ^2, one can conclude that, at optimum, each term of λ^{1*} and λ^{2*} will be such that it will be involved in at least one active constraint among the set of constraints

$$\lambda_j^1 + \lambda_i^2 \ge \eta_{ij} \,,\ \forall\, i \in \mathcal{I} \,,\ \forall\, j \in \mathcal{J} \,.$$

It must therefore be that

$$\lambda_j^{1*} \le \max_i \eta_{ij} - \lambda_i^{2*} \le \max_i \eta_{ij} = u_j^1 \,,\ \forall\, j \in \mathcal{J} \,,$$

and that

$$\lambda_i^2 \le \max_j \eta_{ij} - \lambda_j^{1*} \le \max_j \eta_{ij} = u_j^2 \,,\ \forall\, i \in \mathcal{I} \,.$$

Finally, since $\lambda_{ij}^{3*} = \lambda_j^{1*} + \lambda_i^{2*} - \eta_{ij}$, one could conclude that $\lambda_{ij}^3 \le u_{ij}^3 := \max_i \eta_{ij} + \max_j \eta_{ij} - \eta_{ij}$ but this constraint would be redundant in problem (42) after imposing the upper bounds on λ^1 and λ^2, so that one can leave $u^3 = \infty$. Based on Propositions 4 and 6, we know that it is possible to obtain a tighter conservative approximation to problem (41) by employing affine adjustments in the following augmented model:

$$\begin{aligned}
\underset{x,y(\zeta),z^1(\zeta),z^2(\zeta),v}{\text{maximize}} \quad & \min_{\zeta\in\mathcal{U}} \ -\sum_i c_i x_i - K_i I_i + \sum_i \sum_j \eta_{ij} y_{ij}(\zeta) + {u^1}^T z^1(\zeta) + {u^2}^T z^2(\zeta) \\
\text{subject to} \quad & \sum_i y_{ij}(\zeta) \le \zeta_j + z_j^1(\zeta) \,,\ \forall\, j \in \mathcal{J} \,,\ \forall\, \zeta \in \mathcal{U} \\
& \sum_j y_{ij}(\zeta) \le x_i + z_i^2(\zeta) \,,\ \forall\, i \in \mathcal{I} \,,\ \forall\, \zeta \in \mathcal{U} \\
& y(\zeta) \ge 0 \,,\ \forall\, \zeta \in \mathcal{U} \\
& z^1(\zeta) \ge 0 \,,\ z^2(\zeta) \ge 0 \,,\ \forall\, \zeta \in \mathcal{U} \\
& 0 \le x_i \le M v_i \,,\ \forall\, i \in \mathcal{I} \\
& v_i \in \{0,1\} \,,\ \forall\, i \in \mathcal{I} \,.
\end{aligned}$$

where $z^1 : \mathbb{R}^n \to \mathbb{R}^n$ and $z^2 : \mathbb{R}^n \to \mathbb{R}^m$ can be interpreted as violation adjustments for constraints (41b) and (41c). In Ardestani-Jaafari and Delage (2014), the authors employed a special case of such a conservative approximation where $z^2(\zeta) := 0$ while $z_j^1(\zeta) := z_j^1 \zeta_j$, and showed that there exist instances of the location-transportation problem for which this conservative approximation is strictly tighter than employing affine adjustments directly on model (41).

Let us now consider an example with 2 facilities and 3 customers where the nominal demand, $\bar{\zeta}_j$, is equal to 20,000 for all j, the parameters c and K are respectively equal to 0.6 per unit and 100,000, the matrix η is defined as $\eta = \begin{bmatrix} 5.9 & 5.6 & 4.9 \\ 5.6 & 5.9 & 4.9 \end{bmatrix}$, and ζ is defined in the following budgeted uncertainty set $\mathcal{U}$:

$$\mathcal{U} = \{\zeta | \exists\, \delta \in [0,1]^n,\ \zeta_j = \bar{\zeta}_j - \hat{\zeta}_j \delta_j \,,\ \forall\, j \,,\ \sum_j \delta_j \le \Gamma\} \,,$$

where $\Gamma := 2$ denotes the budget of uncertainty, $\hat{\zeta} := 18000$ denotes the maximum deviation from nominal demand. Table 2 compares the performance of AARC proposed by Ben-Tal et al. (2004), ELAARC proposed by Ardestani-Jaafari and Delage (2014) and our proposed MLRC in this problem instance. One can observe from the table that while AARC exhibits "over-conservatism" by refusing to open any of the facilities, ELAARC instead provides facility location plan that that achieves 70% of the best worst-case profit achievable. The optimality gap is further closed down to zero when using MLRC. It is worth remind the reader that AARC (and implicitely ELAARC and MLRC) was shown in Ardestani-Jaafari and Delage (2014) to provide an exact solution to this model when $\Gamma = 1$ or $\Gamma = 3$, however this example confirms that some improvement is possible for other sizes of budgets and that MLRC is a valuable candidate to do so.

Table 2: Comparison of optimal bound on worst-case profit and worst-case profit associated to solutions of each model in an instance of RLTP

	AARC	ELAARC	MLRC	Exact model
Optimal bound on worst-case profit	0	4024	6600	6600
Worst-case profit of solution	0	4622	6600	6600

Example 3 *(Multi-product assembly problem) In this problem, a manufacturer produces n products using m different types of parts. It is a two-stage problem wherein, the manufacturer pre-orders x_i units for part $i \in \mathcal{I}$ with a cost of c_i per unit in the first stage; and when demand is realized, it must be determined how many products to make, y_j for each product $j \in \mathcal{J}$. The robust multi-product assembly problem can be formulated as follows:*

$$\begin{aligned}
\underset{x,y(\zeta)}{\text{maximize}} \quad & \min_{\zeta\in\mathcal{U}} \; -c^T x + (q-l)^T y(\zeta) + s^T(x - Ay(\zeta)) && \text{(43a)}\\
\text{subject to} \quad & y(\zeta) \le \zeta \,, \forall\, \zeta \in \mathcal{U} && \text{(43b)}\\
& Ay(\zeta) \le x \,, \forall\, \zeta \in \mathcal{U} && \text{(43c)}\\
& y(\zeta) \ge 0 \,, \forall\, \zeta \in \mathcal{U} && \text{(43d)}\\
& 0 \le x \le M \,, && \text{(43e)}
\end{aligned}$$

where $\zeta \in \mathbb{R}^n$ is the uncertain demand for each product and where parameters q and l denote, respectively, the selling price and production cost per unit of the products, while s denotes the salvage unit value of unused parts. Finally A_{ij} denotes the number of units of part i that is required to assemble product j.

As was done for the previous example, one can hope to identify a tighter conservative approximation than with AARC by applying an affine adjustment on the following augmented model:

$$\begin{aligned}
\underset{x,y(\zeta),z^1(\zeta),z^2(\zeta)}{\text{maximize}} \quad & \min_{\zeta\in\mathcal{U}} \; -c^T x + (q-l)^T y(\zeta) + s^T(x - Ay(\zeta)) + {u^1}^T z^1(\zeta) + {u^2}^T z^2(\zeta) && \text{(44a)}\\
\text{subject to} \quad & y(\zeta) \le \zeta + z^1(\zeta) \,, \forall\, \zeta \in \mathcal{U} && \text{(44b)}\\
& Ay(\zeta) \le x + z^2(\zeta) \,, \forall\, \zeta \in \mathcal{U} && \text{(44c)}\\
& y(\zeta) \ge 0 \,, \forall\, \zeta \in \mathcal{U} && \text{(44d)}\\
& z^1(\zeta) \ge 0 \,, \forall\, \zeta \in \mathcal{U} && \text{(44e)}\\
& z^2(\zeta) \ge 0 \,, \forall\, \zeta \in \mathcal{U} && \text{(44f)}\\
& 0 \le x \le M \,, && \text{(44g)}
\end{aligned}$$

where $z^1 : \mathbb{R}^n \to \mathbb{R}^n$ and $z^2 : \mathbb{R}^n \to \mathbb{R}^m$ can be interpreted as violation adjustments for constraints (43b) and (43c). Yet, in this case, the u bounds are obtained from the dual problem:

$$\begin{aligned}
\underset{\lambda^1,\lambda^2}{\text{minimize}} \quad & \zeta^T \lambda^1 + x^T \lambda^2 && \text{(45a)}\\
\text{subject to} \quad & \lambda^1_j + \sum_i A_{ij}\lambda^2_i \ge q_j - l_j + A_{:j}^T s \,, \; \forall\, j \in \mathcal{J} && \text{(45b)}\\
& \lambda^1 \ge 0 \,, \; \lambda^2 \ge 0 \,, && \text{(45c)}
\end{aligned}$$

where $\lambda^1 \in \mathbb{R}^n$ and $\lambda^2 \in \mathbb{R}^n$ are the dual variables associated to constraints (43b) and (43c). Here again, the objective function is non-decreasing in λ^1 and λ^2 so that, at optimum, each term of these two vectors must be involved in at least one active constraint among the following set:

$$\lambda_j^1 + \sum_i A_{ij}\lambda_i^2 \geq q_j - l_j + A_{:j}^T s\,, \forall\, j \in \mathcal{J}\,.$$

This indicates to us that

$$\lambda_j^{1*} \leq q_j - l_j + A_{:j}^T s - \sum_i A_{ij}\lambda_i^{2*} \leq q_j - l_j + A_{:j}^T s = u_j^1\,,$$

and that

$$\lambda_i^{2*} \leq \max_{j\in\mathcal{J}_i} \frac{1}{A_{ij}}(q_j - l_j + A_{:j}^T s - \lambda_j^1 - \sum_{i'\neq i} A_{ij}\lambda_i^{2*}) \leq \max_{j\in\mathcal{J}_i} \frac{1}{A_{ij}}(q_j - l_j + A_{:j}^T s) = u_i^2\,,$$

where the set of indices $\mathcal{J}_i := \{j \,|\, A_{ij} \neq 0\}$.

We conclude this example with a description of the specific context in which exploiting the information about the bound u on λ^* leads to a strictly tighter conservative approximation. In particular, consider a multi-product assembly problem with three products and two different types of parts. The pre-order variable x is bounded by 100,000, the cost of parts A and B are, respectively, \$25 per unit and \$3 per unit, while the salvage value is \$4 per unit and \$1 per unit. Furthermore, the difference between the selling price and the unit production cost of each product is: \$380/unit, \$800/unit, and \$1200/unit respectively for products #1 to #3. Next, we have that product #1 requires 9 units of both parts, product #2 requires 5 units of part B, and #3 requires 9 units of A and 4 units of B. Finally, for products #1 to #3, the nominal demand is respectively of 9000, 10,000, and 8000 units while the worst-case demand for each is 1000, 2000, and 0 units respectively. In this specific context, one can set the bound vectors as $u^1 := [\ 425 \quad 805 \quad 1240\ ]^T$ and $u^2 := [\ 140 \quad 310\ ]^T$.

As it is shown in Table 3, when the budget of uncertainty is set to $\Gamma = 2$, a direct application of affine adjustments on problem (43) will lead to a worst-case profit estimated at 2.474 million dollars; meanwhile applying affine adjustments on the equivalent formulation that allows penalized violations, i.e., problem (44), achieves a worst-case profit estimated at 2.722 million dollars (namely a 10% increase in profit). This confirms that the MLRC model can provide a strictly tighter conservative approximation for this type of problem.

Table 3: Optimal solution of AARC and MLRC in the instance of multi-product assembly problem

Model	# of parts type A	# of parts type B	Optimal bound on worst-case profit	Worst-case profit of solution
AARC	92,793	91,000	\$2.474 million	\$2.474 million
MLRC	81,000	91,000	\$2.722 million	\$2.722 million

Example 4 *(Surgery block allocation problem) Consider the following surgery block allocation problem:*

$$\begin{aligned}
\underset{x,Z,y(\zeta)}{\text{minimize}} \quad & \max_{\zeta\in\mathcal{U}} \; c\sum_i x_i + d\sum_i y_i(\zeta) && \text{(46a)}\\
\text{subject to} \quad & y_i(\zeta) \geq \sum_j \zeta_j Z_{ij} - w x_i\,, \forall\, i \in \mathcal{I}\,, \forall\, \zeta \in \mathcal{U} && \text{(46b)}\\
& y(\zeta) \geq 0\,, \forall\, \zeta \in \mathcal{U} && \text{(46c)}\\
& \sum_i Z_{ij} = 1\,, \forall\, j \in \mathcal{J} && \text{(46d)}\\
& Z_{ij} \leq x_i\,, \forall\, i \in \mathcal{I} && \text{(46e)}\\
& x \in \{0,1\}^m\,,\; Z \in \{0,1\}^{m\times n}\,, && \text{(46f)}
\end{aligned}$$

where for each $i = 1, 2, \ldots, m$, variable x_i denotes whether we will open Operating Room (OR) i or not, while, for each $j = 1, 2, \ldots, n$, the variable $Z_{ij} \in \{0,1\}$ decides whether surgery block j will be allocated to

OR i. Each ζ_j captures the duration of surgery block j, which is a priori not known exactly. As the surgeries are performed, if the total amount of time needed in OR i exceeds the planned session length w, then one has to schedule some overtime y_i. The cost model includes a fixed cost c for opening any OR and a variable overtime cost d. Note that constraint (46d) captures the fact that a surgery block needs to be assigned to exactly one OR, while constraint (46e) captures the fact that surgery blocks can be assigned to an OR only if it is opened.

Proposition 8 *When $\mathcal{U} := \{\zeta \in \mathbb{R}^n \,|\, \exists\, \delta \in [0,1]^n, \zeta = \bar{\zeta} + \hat{\zeta}\delta,\ \sum_j \delta_j \leq \Gamma\}$, applying affine adjustments to the surgery block allocation problem provides a conservative approximation that is at least as tight as the reformulation proposed in Denton et al. (2010) (see model (40) in their paper).*

Proof. Proof. Indeed, the model presented in Denton et al. (2010) can be rewritten as

$$\begin{aligned} \underset{x,Z}{\text{minimize}} \quad & g_{Denton}(x, Z) \\ \text{subject to} \quad & \sum_i Z_{ij} = 1\,,\ \forall\, j \in \mathcal{J} \\ & Z_{ij} \leq x_i\,,\ \forall\, i \in \mathcal{I} \\ & x \in \{0,1\}^m\ ,\ Z \in \{0,1\}^{m\times n}\,, \end{aligned}$$

where

$$\begin{aligned} g_{Denton}(x, Z) := \quad \min_{\alpha,\gamma,\kappa} \quad & \sum_i cx_i + \sum_i \gamma_i + \Gamma\alpha \\ \text{subject to} \quad & \alpha \geq d\hat{\zeta}_j Z_{ij} - \kappa_{ij}\,,\ \forall\, i \in \mathcal{I}\,,\ \forall\, j \in \mathcal{J} \\ & \gamma_i \geq \sum_j \kappa_{ij} - d(wx_i - \sum_j \bar{\zeta}_j Z_{ij}) \\ & \alpha \geq 0\,,\ \gamma \geq 0\,,\ \kappa \geq 0\,, \end{aligned}$$

where $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, and $\kappa \in \mathbb{R}^{m\times n}$. By duality, we can also represent this function as

$$\begin{aligned} g_{Denton}(x, Z) := \quad \max_{\lambda,\Delta} \quad & \sum_i cx_i + \sum_{ij} d\hat{\zeta}_j Z_{ij}\Delta_{ij} - \sum_i d(wx_i - \sum_j \bar{\zeta}_j Z_{ij})\lambda_i \\ \text{subject to} \quad & 0 \leq \lambda_i \leq 1\,,\ \forall\, i \in \mathcal{I} \\ & 0 \leq \Delta_{ij} \leq \lambda_i\,,\ \forall\, i \in \mathcal{I}\,,\ \forall\, j \in \mathcal{J} \\ & \sum_{ij} \Delta_{ij} \leq \Gamma\,, \end{aligned}$$

where $\lambda \in \mathbb{R}^m$ and $\Delta \in \mathbb{R}^{m\times n}$.

Based on Proposition 3 and the details presented in the proof of Proposition 10, we now know that applying affine adjustments on problem (46) is equivalent to optimizing

$$\begin{aligned} \underset{x,Z}{\text{minimize}} \quad & g_{LRC}(x, Z) \\ \text{subject to} \quad & (46d), (46e), (46f)\,, \end{aligned}$$

where

$$\begin{aligned} g_{LRC}(x, Z) := \quad \max_{\lambda,\delta,\Delta} \quad & \sum_i cx_i + \sum_{ij} d\hat{\zeta}_j Z_{ij}\Delta_{ij} - \sum_i d(wx_i - \sum_j \bar{\zeta}_j Z_{ij})\lambda_i \\ \text{subject to} \quad & 0 \leq \lambda_i \leq 1\,,\ \forall\, i \in \mathcal{I} \\ & 0 \leq \delta_j \leq 1\,,\ \forall\, j \in \mathcal{J} \\ & \sum_j \delta_j \leq \Gamma \end{aligned}$$

$$\begin{aligned}
&0 \leq \Delta_{ij} \leq \delta_j \,,\, \forall i \in \mathcal{I} \,,\, \forall j \in \mathcal{J} \\
&\Delta_{ij} \leq \lambda_i \,,\, \forall i \in \mathcal{I} \,,\, \forall j \in \mathcal{J} \\
&\sum_j \Delta_{ij} \leq \Gamma z_i \,,\, \forall i \in \mathcal{I} \\
&1 - \delta_j - \lambda_i + \Delta_{ij} \geq 0 \,,\, \forall i \in \mathcal{I} \,,\, \forall j \in \mathcal{J} \\
&\sum_j \delta_j - \sum_j \Delta_{ij} \leq \Gamma(1 - \lambda_i) \,,\, \forall i \in \mathcal{I} \,.
\end{aligned}$$

We will now exploit the fact that we can add the constraint $\sum_{ij} \Delta_{ij} \leq \Gamma$ to the problem associated to $g_{LRC}(x, Z)$ without affecting the optimal value that it will return. This is because, for any optimal solution $(\lambda^*, \delta^*, \Delta^*)$, one can simply replace Δ^* with Δ' such that $\Delta'_{ij} := Z_{ij}\Delta_{ij}$ satisfies all constraints and achieves the same objective. Indeed we have that

$$\begin{aligned}
&0 \leq \Delta_{ij} \leq \delta_j \Rightarrow 0 \leq \Delta_{ij} Z_{ij} \leq \delta_j \Rightarrow 0 \leq \Delta'_{ij} \leq \delta_j \\
&\Delta_{ij} \leq z_i \Rightarrow \Delta_{ij} Z_{ij} \leq z_i \Rightarrow \Delta'_{ij} \leq z_i \\
&\sum_j \Delta_{ij} \leq \Gamma z_i \Rightarrow \sum_j \Delta_{ij} Z_{ij} \leq \Gamma z_i \Rightarrow \sum_j \Delta'_{ij} \leq \Gamma z_i \\
&\sum_{ij} \Delta'_{ij} = \sum_{ij} \Delta_{ij} Z_{ij} \leq \sum_{ij} \delta_j Z_{ij} = \leq \sum_j \delta_j \sum_i Z_{ij} = \sum_j \delta_j \leq \Gamma \,.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
g_{LRC}(x, Z) = \quad & \max_{\lambda,\delta,\Delta} & & \sum_i c x_i + \sum_{ij} d\hat{\zeta}_j Z_{ij}\Delta_{ij} - \sum_i d(w x_i - \sum_j \hat{\zeta}_j Z_{ij}) z_i \\
& \text{subject to} & & 0 \leq z_i \leq 1 \,,\, \forall i \in \mathcal{I} \\
& & & 0 \leq \delta_j \leq 1 \,,\, \forall j \in \mathcal{J} \\
& & & \sum_j \delta_j \leq \Gamma \\
& & & 0 \leq \Delta_{ij} \leq \delta_j \,,\, \forall i \in \mathcal{I} \,,\, \forall j \in \mathcal{J} \\
& & & \Delta_{ij} \leq z_i \,,\, \forall i \,,\, \forall j \in \mathcal{J} \\
& & & \sum_j \Delta_{ij} \leq \Gamma z_i \,,\, \forall i \in \mathcal{I} \\
& & & 1 - \delta_j - \lambda_i + \Delta_{ij} \geq 0 \,,\, \forall i \in \mathcal{I} \,,\, \forall j \in \mathcal{J} \\
& & & \sum_j \delta_j - \sum_j \Delta_{ij} \leq \Gamma(1 - \lambda_i) \,,\, \forall i \in \mathcal{I} \\
& & & \sum_{ij} \Delta_{ij} \leq \Gamma \,,
\end{aligned}$$

meaning that, for any feasible x and Z, it must be that $g_{LRC}(x, Z) \leq g_{Denton}(x, Z)$, since the latter involves an optimization model that is exactly the same as the former except that it imposes fewer constraints. We conclude that exploiting affine adjustments must lead to a tighter conservative approximation. □

Proposition 9 *There exists an instance of the surgery block allocation problem for which applying affine adjustment provides a conservative approximation that is strictly tighter than the reformulation proposed in Denton et al. (2010).*

Proof. Proof. Consider a particular problem instance in which there are three surgery blocks and 2 operating rooms that can run for 8 hours. The cost of opening a room is \$39,000, and the overtime cost is \$100 per minute. The duration of each of the three surgery blocks is planned to be equal to 0 min, 240 min, and 320 min, but could last up to 160 min, 352 min, and 512 min respectively. We set the budget to $\Gamma = 2$.

In this context, one can show that the model proposed by Denton will suggest opening only one OR, where all blocks will be scheduled for an estimated worst-case total cost of \$822,000. On the other hand, one can verify that opening both ORs and scheduling the biggest block in one OR and the two smaller ones in the second OR leads to a worst-case total cost of \$812,000. Note that the worst-case total cost of this solution is estimated at \$828,000 by the Denton model. One can further confirm that the exact optimal solution is the one that is returned by the LRC model. Table 4 summarizes the conservative approximation obtained for the two types of solutions. □

Table 4: Comparison of the worst-case cost for different solution methods to the surgery block allocation problem

Candidate solution	Worst-cast total cost	Denton's bound on worst-case cost	LRC's bound on worst-case cost
Open one OR	\$822,000	\$822,000	\$822,000
Open two ORs	\$812,000	\$828,000	\$812,000

The fact that applying affine adjustments can provide better solutions than the model in Denton et al. (2010) may come as a surprise, as it is stated in Denton et al. (2010) that their proposed reformulation provides an exact solution to the robust surgery block allocation problem. The issue in the argument presented by the authors of that paper is found in their Proposition 6, which states that a certain polyhedron only has integer extreme points. Appendix B provides arguments that disprove this proposition.

7 Conclusions

In this paper, we extended the linearization scheme presented in Ardestani-Jaafari and Delage (2016) so that it can be used to construct tractable conservative approximation models for two-stage adjustable robust optimization problems with right-hand side uncertainty. We showed that, as in Ardestani-Jaafari and Delage (2016), this scheme provides an alternate interpretation of models obtained through the use of AARC. Yet, by considering the adversarial problem as a bilinear optimization problem that needs to be linearized, it become very natural to identify modifications based on linear and conic valid inequalities that will improve LRC and consequently provide tightening procedures for AARC. Based on these results, it is clear that the LRC model can help us to better understand the quality of solutions obtained from AARC and offers a perspective that might help design better approximation methods for two-stage adjustable robust optimization models. We finally surveyed the types of improvement that our proposed models might offer in four different applications of operations management problems. These applications included a surgery block allocation problem, for which we were able to shed some light on the misconception that the model proposed by Denton et al. (2010) provides an exact robust solution.

Appendix

A Relation to AARC for General Uncertainty Sets

For simplicity, we present the connection between GLRC and AARC for a convex uncertainty set described as $\mathcal{U}_{general} := \{\zeta \in \mathbb{R}^{n_\zeta} \mid f(\zeta) \leq q\}$ and when no bounds are known for the dual variables $\lambda \in \mathbb{R}^m$.

Proposition 10 *Given that $f(\cdot)$ satisfies Assumption 5, the GLRC model presented below provides a tighter conservative approximation than the AARC model presented in (2) when the uncertainty set is described as $\mathcal{U}_{general}$:*

$$\underset{x\in\mathcal{X}}{\text{maximize}} \quad g_{GLRC}(x)\,,$$

where

$$\begin{aligned}
g_{GLRC}(x) := \quad & \min_{\zeta,\lambda,\Delta} && c^T x + \text{tr}(\Psi(x)\Delta) - (Ax)^T\lambda && \text{(47a)}\\
& \text{subject to} && B^T\lambda = d && \text{(47b)}\\
& && f(\zeta) \leq q && \text{(47c)}\\
& && \lambda \geq 0 && \text{(47d)}\\
& && \Delta B = \zeta d^T && \text{(47e)}\\
& && h(\Delta_{:i}, \lambda_i) \leq q\lambda_i\,,\ \forall\, i\,. && \text{(47f)}
\end{aligned}$$

Proof. Proof. Based on Definition 2, constraint (47f) can be explicitely described as

$$\sup_z \Delta_{:i}^T z - \lambda_i f_*(z) \leq q\lambda_i\,,\ \forall\, i\,.$$

One can then construct the Lagrangian function of problem (47) using the following form :

$$\begin{aligned}
\mathcal{L}(\zeta,\lambda,\Delta,y,Y,s) &:= c^T x + \text{tr}(\Psi(x)\Delta) - (Ax)^T\lambda + y^T(d - B^T\lambda) + \text{tr}(Y(\zeta d^T - \Delta B))\\
&\qquad + \sum_i s_i(\sup_z \Delta_{:i}^T z - \lambda_i f_*(z) - q\lambda_i)\\
&= \sup_{z_1,\ldots,z_m} c^T x + \text{tr}(\Psi(x)\Delta) - (Ax)^T\lambda + y^T(d - B^T\lambda) + \text{tr}(Y(\zeta d^T - \Delta B))\\
&\qquad + \sum_i s_i\Delta_{:i}^T z_i - \lambda_i s_i f_*(z_i) - q s_i\lambda_i)\,,
\end{aligned}$$

where $y \in \mathbb{R}^{n_y}$, $Y \in \mathbb{R}^{n_y \times n_\zeta}$, and $s \in \mathbb{R}^m$ are respectively the dual variables associated to constraints (47b), (47e), and (47f). Now letting $\mathcal{L}(\zeta,\lambda,\Delta,y,Y,s,\{z_i\}_{i=1}^m)$ denote the expression on the right of the $\sup_{z_1,\ldots,z_m}$ operator, Langrangian weak duality theory states that

$$g_{GLRC}(x) \geq \inf_{\zeta: f(\zeta)\leq q}\ \sup_{y,Y,s\geq 0,\{z_i\}_{i=1}^m}\ \inf_{\lambda\geq 0,\Delta} \mathcal{L}(\zeta,\lambda,\Delta,y,Y,s,\{z_i\}_{i=1}^m)\,, \tag{48}$$

with equality being met when strong duality conditions are satisfied. One can analytically resolve the optimum in terms of λ and Δ as

$$\begin{aligned}
g_{GLRC}(x) \geq \min_{\zeta: f(\zeta)\leq q}\quad & \max_{y,Y,s\geq 0,\{z_i\}_{i=1}^m} && c^T x + d^T(y + Y\zeta)\\
& \text{subject to} && (\Psi(x))_{i:}^T - Y^T B_{i:}^T + s_i z_i = 0\,,\ \forall\, i\\
& && A_{i:}x + B_{i:}y + s_i f_*(z_i) + q s_i \leq 0\,,\ \forall\, i\,.
\end{aligned}$$

The equality constraint can further be used in conjunction with the fact that $s_i f_*(z_i) = h_*(s_i z_i, s_i) := \sup_y s_i z_i^T y - s_i f(y)$, to obtain

$$g_{GLRC}(x) \geq \min_{\zeta: f(\zeta)\leq q} \quad \max_{y,Y,s\geq 0} \quad c^T x + d^T(y + Y\zeta)$$

$$\text{subject to} \qquad A_{i:}x + B_{i:}y + h_*(Y^T B_{i:}^T - (\Psi(x))_{i:}^T, s_i) + qs_i \leq 0\,, \forall\, i\,.$$

After applying Sion's minimax theorem as was done in the proof of Proposition 3, one obtains

$$\begin{aligned} g_{GLRC}(x) \geq \quad & \max_{y,Y,s\geq 0} \quad \min_{\zeta:f(\zeta)\leq q} c^T x + d^T(y + Y\zeta) \\ & \text{subject to} \quad A_{i:}x + B_{i:}y + h_*(Y^T B_{i:}^T - (\Psi(x))_{i:}^T, s_i) + qs_i \leq 0\,, \forall\, i\,, \end{aligned}$$

where the last constraint can be reformulated as

$$A_{i:}x + B_{i:}y + \inf_{s_i\geq 0} \sup_{\zeta} B_{i:}Y\zeta - \Psi(x)_{i:}\zeta - s_i f(\zeta) + qs_i \leq 0$$

since s_i is not involved in the objective function. Given that there exists a point $\bar{\zeta}$ such that $f(\bar{\zeta}) < q$, strong duality theory will apply here and allow one to reformulate this constraint as

$$A_{i:}x + B_{i:}y + \sup_{\zeta} \inf_{s_i\geq 0} B_{i:}Y\zeta - \Psi(x)_{i:}\zeta - s_i f(\zeta) + qs_i \leq 0\,,$$

and finally

$$A_{i:}x + B_{i:}y + \sup_{\zeta:f(\zeta)\leq q} B_{i:}Y\zeta - \Psi(x)_{i:}\zeta \leq 0\,.$$

These steps allow us to reach our conclusion:

$$\begin{aligned} g_{GLRC}(x) \geq \quad & \max_{y,Y} \quad \min_{\zeta:f(\zeta)\leq q} c^T x + d^T(y + Y\zeta) \\ & \text{subject to} \quad Ax + B(y + Y\zeta) \leq \Psi(x)\zeta \leq 0\,, \forall\, \zeta : f(\zeta) \leq q\,, \end{aligned}$$

which is the conservative approximation of the worst-case performance for x when employing affine adjustments. Note that equality is met in this expression as long as strong duality applies in deriving equation (48). In this case, LRC becomes equivalent to AARC. □

B Counterargument to Proposition 6 in Denton et al. (2010)

We first recall the proposition that can be found in Denton et al. (2010).

Proposition 11 (Proposition 6 from Denton et al. (2010)) *The polyhedron defined by the following constraints has integer extreme points when τ is an integer*

$$\sum_{ij} \Delta_{ij} \leq \tau \tag{49a}$$

$$0 \leq \Delta_{ij} \leq Y_{ij} z_j\,, \forall\, i \in \mathcal{I}, \forall\, j \in \mathcal{J} \tag{49b}$$

$$0 \leq z_j \leq 1\,, \forall\, j \in \mathcal{J}\,, \tag{49c}$$

where $\Delta \in \mathbb{R}^{n\times m}$ and $z \in \mathbb{R}^m$ and where the parameter $Y \in \{0,1\}^{n\times m}$ satisfies the property that $\sum_j Y_{ij} = 1$ for all $i \in \mathcal{I}$.

Here is a counter-proposition.

Proposition 12 *Let $n = 3$, $m = 2$, $Y_{12} = Y_{21} = Y_{31} = 1$, and $\Gamma = 2$, then the polyhedron defined by equations (49a), (49b), and (49c) has the following extreme point:*

$$\bar{\Delta} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \\ 0.5 & 0 \end{bmatrix} \qquad \bar{z} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}\,.$$

Proof. Proof. This can easily be shown by verifying that this solution is feasible and that it satisfies exactly a set of 8 linearly independent constraints. The eight constraints are

$$\begin{array}{llll} \sum_{ij} \Delta_{ij} \leq 2 & -\Delta_{11} \leq 0 & \Delta_{12} \leq z_2 & \Delta_{21} \leq z_1 \\ -\Delta_{22} \leq 0 & \Delta_{31} \leq z_1 & -\Delta_{32} \leq 0 & z_2 \leq 1\,. \end{array}$$

Putting all these together we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_{11} \\ \Delta_{12} \\ \Delta_{21} \\ \Delta_{22} \\ \Delta_{31} \\ \Delta_{32} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since this matrix is invertible and the pair $(\bar{\Delta}, \bar{z})$ satisfies this system of equations, we have confirmed that this assignment describes an extreme point of the polyhedron. □

References

Ardestani-Jaafari, A., E. Delage. 2014. The value of flexibility in robust location-transportation problem. working paper .

Ardestani-Jaafari, A., E. Delage. 2016. Robust optimization of sums of piecewise linear functions with application to inventory problems. Operations Research 64(2) 474–494.

Babonneau, F, J-P Vial, R Apparigliato. 2010. Robust optimization for environmental and energy planning. Uncertainty and Environmental Decision Making 79–126.

Ben-Tal, A., A. Goryashko, E. Guslitzer, A. Nemirovski. 2004. Adjustable robust solutions of uncertain linear programs. Mathematical Programming 99(2) 351–376.

Ben-Tal, Aharon, Byung Do Chung, Supreet Reddy Mandala, Tao Yao. 2011. Robust optimization for emergency logistics planning: Risk mitigation in humanitarian relief supply chains. Transportation research part B: methodological 45(8) 1177–1189.

Ben-Tal, Aharon, Boaz Golany, Arkadi Nemirovski, Jean-Philippe Vial. 2005. Retailer-supplier flexible commitments contracts: a robust optimization approach. Manufacturing & Service Operations Management 7(3) 248–271.

Denton, B. T., A. J. Miller, H. J . Balasubramanian, T. R. Huschka. 2010. Optimal allocation of surgery blocks to operating rooms under uncertainty. Operations research 58(4-part-1) 802–816.

Hiriart-Urruty, J.-B., C. Lemaréchal. 2001. Fundamentals of Convex Analysis. Grundlehren Text Editions, Springer-Verlag Berlin Heidelberg.

Jabr, Rabih A, Sami Karaki, Joe Akl Korbane. 2015. Robust multi-period opf with storage and renewables. Power Systems, IEEE Transactions on 30(5) 2790–2799.

Melamed, Michal, Aharon Ben-Tal, Boaz Golany. 2016. On the average performance of the adjustable ro and its use as an offline tool for multi-period production planning under uncertainty. Computational Management Science 1–23.

Ouorou, Adam. 2013. Tractable approximations to a robust capacity assignment model in telecommunications under demand uncertainty. Computers & Operations Research 40(1) 318–327.

Sherali, H. D., A. Alameddine. 1992. A new reformulation-linearization technique for bilinear programming problems. Journal of Global optimization 2(4) 379–410.