**Les Cahiers du GERAD** ISSN: 0711–2440

# A two-factor structural model for valuing corporate securities

M. Ben-Abdellatif, H. Ben-Ameur,
B. Rémillard

G–2016–119

December 2016



**GERAD** HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

**Tél. : 514 340-6053**
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

# A two-factor structural model for valuing corporate securities

**Malek Ben-Abdellatif** [a]
**Hatem Ben-Ameur** [a]
**Bruno Rémillard** [a,b]

[a] *GERAD & Department of Decision Sciences, HEC Montréal, Montréal (Québec), Canada, H3T 2A7*

[b] *CRM, Montréal (Québec), Canada, H3C 3J7*

malek.ben-abdellatif@hec.ca
hatem.ben-ameur@hec.ca
bruno.remillard@hec.ca

**December 2016**

**Les Cahiers du GERAD**

**G–2016–119**

**Abstract:** We develop a general structural model for valuing risky corporate debts that takes into account both default and interest rate risk. We propose a two-dimensional model in which the state variables are the value of the firm's assets and the short-term interest rate. The former follows a lognormal process and the latter a mean-reverting Gaussian process. Our methodology is based on dynamic programming and finite elements. We use parallel computing to enhance its efficiency. Our model accommodates flexible debt structure, multiple seniority classes, tax benefits, and bankruptcy costs. The results we obtain are consistent with empirical evidence documented in the literature.

**Keywords:** Credit risk, stochastic interest rate, dynamic programming, finite elements, paralell computing

**Résumé :** Nous développons un modèle structurel élargi pour évaluer les dettes corporatives risquées qui prend en compte le risque de défaut et le risque du taux d'intérêt. Nous proposons un modèle bivarié dans lequel les deux facteurs sont la valeur des actifs d'une firme et le taux d'intérêt à court terme. Le premier suit un processus lognormal et le deuxième un processus Gaussien de retour vers la moyenne. Notre méthodologie est basée sur la programmation dynamique couplée aux éléments finis. Nous utilisons le calcul parallèle pour améliorer l'efficacité. Notre modèle permet une structure de dette flexible, plusieurs classes de séniorité, et prend en compte également les économies de taxes et les coûts de faillite. Les résultats obtenus sont consistants avec les effets empiriques documentés dans la littérature.

**Mots clés :** Risque de crédit, taux d'intérêt stochastique, programmation dynamique, éléments finis, calcul parallèle

**Acknowledgments:** Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada, the Fonds pour la formation de chercheurs et l'aide à la recherche du Gouvernement du Québec, and the Institut de Finance Mathématique du Québec.

# 1 Introduction

We propose a structural model for valuing risky debts when the interest rate is stochastic. Our methodology is based on two-dimensional dynamic programming coupled with finite elements. We use parallel computing to enhance our procedure's efficiency. Classical structural models assume a fixed interest rate, but this assumption is too simplistic as interest rates are stochastic in practice, particularly since we observe long maturities for corporate debts. Empirical evidence suggests that the interest rate and credit risk are negatively correlated (Litterman and Scheinkman, 1991, Collin-Dufresne et al., 2001). The activity of the underlying company and its capital structure policy can be sensitive to the term structure of interest rates. The empirical work of Longstaff and Schwartz (1995) shows that bonds with similar credit ratings, but in different industries, have widely varying credit spreads. Theses differences are explained by the manifold correlations with interest rates. Contrary to the traditional approach, which implies that credit spreads depend only on an asset-value factor, Longstaff and Schwartz (1995) show that credit spreads for corporate bonds are driven by an asset-value factor and an interest-rate factor; the dependence between the two factors plays a crucial role in determining credit spreads. It is thus important to include interest rate uncertainty in the credit risk modeling framework.

Structural models are based on the pioneer work of Merton (1974) who considered the firm's assets to follow geometric-Brownian motion. Default occurs if the firm's assets are insufficient to pay the debt at maturity. Considering the debt to be a pure bond, he uses option-pricing theory; the firm's equity is evaluated as a European call option on the firm's assets, with the same maturity as the bond, and a strike price equal to the principal amount. Although simple and unrealistic, this work has generated several developments in credit risk modeling and is the basis for more general models.

Black and Cox (1976) propose a barrier-triggered default which allows for default to happen before maturity of the debt. Several authors consider more complex debt structures or include frictions (Ericsson and Reneby, 1998, Collin-Dufresne and Goldstein, 2001, Hsu et al., 2010, Geske, 1977), endogenous default barriers (Leland, 1994, Anderson and Sundaresan, 1996, Leland and Toft, 1996, Mella-Barral and Perraudin, 1997, François and Morellec, 2004), and jumps in the firm's asset process (Zhou, 2001, Chen and Kou, 2009).

To incorporate interest rate risk in the corporate debt valuation, various articles include a stochastic interest rate in structural models. Shimko et al. (1993) add a stochastic short-term interest rate that evolves according to Vasicek's (1977) model to Merton's (1974) model. In this case, a closed-form solution is available as the problem becomes equivalent to pricing an European call option on a stock under the stochastic interest rate.

Kim et al. (1993) and Longstaff and Schwartz (1995) extend the Black and Cox (1976) model. The former considers a CIR dynamic following Cox et al. (1985) for the short-term interest rate while the latter uses Vasicek's (1977) model; both derive a quasi-closed form for the debt value. Cathcart and El-Jahel (1998) and Briys and De Varenne (1997) propose some corrections to the Longstaff and schwartz's (1995) model. The first adds a CIR process for the short-term rate to avoid having positive probability that the interest rate becomes negative. The second corrects for weaknesses such as bondholders recovering an amount that, in case of default, does not depend on the remaining firm's asset value. All these models consider very simple settings regarding the firm's capital structure and the default mechanism. Allowing endogenous default barriers or more general debt structures requires using a numerical approach to solve the problem.

We extend Altieri and Vargiolu (2001) and Ayadi et al. (2016) by adding a stochastic interest rate to a general structural model which allows for a flexible debt structure with multiple seniority classes, and accounts for bankruptcy costs and tax benefits. We use a mean-reverting Gaussian process for the short-term interest rate as proposed by Vasicek (1977). The proposed methodology is based on a two-dimensional dynamic program coupled with finite elements. As this procedure is time demanding, we use parallel computing to expedite our procedure and improve its efficiency. Our results demonstrate convergence and remain consistent with empirical evidence documented in the literature.

This paper is organized as follows: Section 2 presents our model, Section 3 describes our dynamic program, Section 4 shows our numerical investigation, and Section 5 concludes our paper.

## 2 Model and notations

We propose a structural model for valuing risky debt by allowing for both default risk and interest rate risk. The stochastic short-term interest rate $r_t$ evolves according to a mean-reverting Gaussian process as in Vasicek's (1977) model

$$dr_t = \alpha(\beta - r_t)dt + \sigma_r dZ_t^1, \tag{1}$$

where $\beta$ is the long-term mean level, $\alpha$ is the speed of reversion to this level, and $\sigma_r$ is the instantaneous volatility. The firm's assets value $V_t$ moves according to geometric-Brownian motion

$$\frac{dV_t}{V_t} = (r_t - \delta)dt + \sigma_V(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2), \tag{2}$$

where $\delta$ is the firm's payout rate and $\sigma_v$ is its assets volatility. Both dynamics are under the risk neutral measure $\mathbb{Q}$. $Z_t^1$ and $Z_t^2$ are two independent Brownian motions and $\rho$ represents the correlation between the two processes.

Consider that the firm's capital structure contains a portfolio of senior and junior bonds and a common stock. The firm makes coupon payments to the bondholders which results in collecting tax benefits. The firm also pays bankruptcy costs in case of default. The model assumes that the stockholders determine the time of default by maximizing the firm's total value subject to the limited liability constraint. Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n, \ldots, t_N\}$ be a set of payment dates, and let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a complete probability space, and for each $k \geq n \in \{0, \ldots, N\}$, set $r_n^k = -\int_{t_n}^{t_k} r_s ds$; the discount factor is then $e^{-r_n^k}$. The value functions, in terms of the bankruptcy time $k$, are expressed as follows:

**Bankruptcy costs:** The costs connected to default are equal to $wV_\tau$, where $w \in (0,1)$ is a fixed fraction. The value of bankruptcy costs at time $t_n$ is given by

$$BC_k^{(n)} = \begin{cases} 0, & k < n \text{ or } k = N+1; \\ we^{-r_n^k}V_k, & n \leq k < N+1. \end{cases}$$

**Debt:** At each date $t_n$, the firm is committed to pay $d_n^{(sen)} + d_n^{(jun)} = d_n$ to its creditors, where $d_n^{(sen)}$ and $d_n^{(jun)}$ are the payments due to the senior and junior bondholders, respectively. These payments include interest as well as principal payments. The interest payment is denoted $d_n^{int}$. The last payment dates of the senior and junior debts are indicated by $T^s$ and $T^j$, with $0 \leq T^s \leq T^j = T$. The senior and junior debts are

$$DS_k^{(n)} = \begin{cases} 0, & k < n; \\ e^{-r_n^k} \min\left\{(1-w)V_k, d_k^{(sen)}\right\} + \\ \sum_{j=n}^{k-1} e^{-r_n^j} d_j^{(sen)}, & n \leq k \leq N \\ \sum_{j=n}^{N} e^{-r_n^j} d_j^{(sen)}, & k = N+1. \end{cases}$$

$$DJ_k^{(n)} = \begin{cases} 0, & k < n; \\ r_n^k \max\left\{(1-w)V_k - d_k^{(sen)}, 0\right\} + \\ \sum_{j=n}^{k-1} e^{-r_n^j} d_j^{(jun)}, & n \leq k \leq N \\ \sum_{j=n}^{N} e^{-r_n^j} d_j^{(jun)}, & k = N+1. \end{cases}$$

The total debt at time $t_n$ is then

$$D_k^{(n)} = \begin{cases} 0, & k < n; \\ (1-w)e^{-r_n^k}V_k + \sum_{j=n}^{k-1} e^{-r_n^j} d_j, & n \leq k \leq N \\ \sum_{j=n}^{N} e^{-r_n^j} d_j, & k = N+1. \end{cases}$$

**Tax benefits:** The tax benefits associated with the cost of debt are proportional to the interest payment $d_n^{int}$. Let $r_n^c \in [0,1]$ be the periodic corporate tax rate over $[t_n, t_{n+1}]$ and $tb_n = r_n^c d_n^{int}$. The tax benefits are then

$$TB_k^{(n)} = \begin{cases} 0, & k < n; \\ \sum_{j=n}^{k-1} e^{-r_n^k} tb_j, & n \le k \le N+1. \end{cases}$$

**The total value of the firm:** The total value of the firm represents the assets' value increased by the tax benefits, net of the bankruptcy costs,

$$\begin{aligned} W_k^{(n)} &= V_n + TB_k^{(n)} - BC_k^{(n)} \\ &= \begin{cases} 0, & k < n; \\ V_n + \sum_{j=n}^{k-1} e^{-r_n^j} tb_j - we^{-r_n^k} V_k, & n \le k \le N, \\ V_n + \sum_{j=n}^{N} e^{-r_n^j} tb_j, & k = N+1. \end{cases} \end{aligned}$$

**Equity value:** In case of survival at date $t_n$, the stockholders receive the total value of the firm minus the total debt value

$$\mathcal{E}_k^{(n)} = W_k^{(n)} - D_k^{(n)}.$$

Let $\mathcal{T}$ be the set of stopping times with values in $\{0, \ldots, N+1\}$. As a result, for any stopping time $\tau \in \mathcal{T}$ with $\tau \ge n$, one obtains

$$E\left(\mathcal{E}_\tau^{(n)} | \mathcal{F}_n\right) = \mathcal{B}_n^{(\tau)} \mathbf{1}(\tau > n),$$

where $\mathcal{B}_N^{(\tau \vee N)} = \mathcal{B}_N = V_N + tb_N - d_N$ and

$$\mathcal{B}_n^{(\tau)} = V_n + tb_n - d_n - E\left(e^{-r_n^{n+1}} V_{n+1} | \mathcal{F}_n\right) + E\left(e^{-r_n^{n+1}} \mathcal{E}_{\tau \vee (n+1)}^{(n+1)} | \mathcal{F}_n\right),$$

for all $n \in \{0, \ldots, N-1\}$.

**Definition 1**

$$\mathcal{T}_n = \left\{\tau \in \mathcal{T}; \tau \ge n,\ \{\tau > k\} \subset \left\{E\left(\mathcal{E}_{\tau \vee k}^{(k)} | \mathcal{F}_k\right) > 0\right\},\ \ for\ k \ge n\right\}.$$

Finally, define $J_\tau^{(n)} = TB_\tau^{(n)} - BC_\tau^{(n)}$, and set

$$\bar{J}_n = \sup_{\tau \in \mathcal{T}_n} E\left(J_\tau^{(n)} \middle| \mathcal{F}_n\right),$$

for all $n \in \{0, \ldots, N\}$. Note that $\sup_{\tau \in \mathcal{T}_n} E\left\{W_\tau^{(n)} \middle| \mathcal{F}_n\right\} = V_n + \bar{J}_n$.

The main aim is to find a sequence of stopping times $\tau_n^\star \in \mathcal{T}_n$, corresponding to optimal bankruptcy times, so that the total expected wealth at time $n$ is maximized, that is $V_n + \bar{J}_n = E\left\{W_{\tau_n^\star}^{(n)} | \mathcal{F}_n\right\}$. The solution is provided by Ben-Abdellatif et al. (2016b) in the following theorem.

**Theorem 1** *Set $\mathcal{E}_N = \max(V_N + tb_N - d_N, 0)$. For any $k \in \{0, \ldots, N-1\}$, set*

$$\mathcal{E}_k = \max\left\{V_k + tb_k - d_k - E\left(e^{-r_k^{k+1}} V_{k+1} | \mathcal{F}_k\right) + E\left(e^{-r_k^{k+1}} \mathcal{E}_{k+1} | \mathcal{F}_k\right), 0\right\}.$$

*Next, define*

$$\tau_k^\star = \begin{cases} N+1, & \text{if } \mathcal{E}_j > 0 \text{ for all } j \in \{k, \ldots, N\}, \\ \min\{k \le j \le N; \mathcal{E}_j = 0\}, & \text{otherwise.} \end{cases}$$

*Then*

$$\bar{J}_N = E\left(J_{\tau_N^\star}^{(N)} | \mathcal{F}_N\right) = -\alpha V_N \mathbf{1}(\mathcal{E}_N = 0) + b_N \mathbf{1}(\mathcal{E}_N > 0),$$

*and for all* $k \in \{0, \ldots, N-1\}$,

$$
\begin{aligned}
\bar{J}_k &= E\left(J^{(k)}_{\tau^*_k} | \mathcal{F}_k\right) \\
&= -\alpha V_k \mathbf{1}(\mathcal{E}_k = 0) + \left\{ tb_k + E\left(e^{-r_k^{k+1}} \bar{J}_{k+1} | \mathcal{F}_k\right)\right\} \mathbf{1}(\mathcal{E}_k > 0).
\end{aligned}
$$

The proof of Theorem 1 is given in Ben-Abdellatif et al. (2016b). Now suppose that $V_n = V(t_n)$. Further, set $\mathcal{F}_n = \sigma\{r(u), V(u); 0 \le u \le t_n\}$.

Note that our model satisfies the **Markovian hypothesis**, meaning that there is an expectation operator $T_n$ so that for any integrable function $\Psi$ on $\mathbb{R} \times [0, \infty)$,

$$
E\left[e^{-\int_{t_n}^{t_{n+1}} r(u)du} \Psi\{r(t_{n+1}), V(t_{n+1})\} | \mathcal{F}_n\right] = T_n \Psi\{r(t_{n+1}), V(t_n + 1)\}. \tag{3}
$$

In our setting, this expectation operator is calculated as follows

$$
T_n \Psi\{r(t_{n+1}), V(t_{n+1})\} = B(t_n, t_{n+1}) E^* \left[\Psi\{r(t_{n+1}), V(t_{n+1})\} | \mathcal{F}_n\right],
$$

where $E^*$ is the expectation under the forward measure, and $B(t_n, t_{n+1})$ is the price of a zero-coupon bond with maturity $t_{n+1}$ at time $t_n$. The change of measure using the forward measure is done according to Jamshidian (1989) and is described in Appendix A. For this setting, the following proposition from Ben-Abdellatif et al. (2016b) gives the expression of the value functions.

**Proposition 1** *Set* $r = r(t_n)$ *and* $v = V(t_n)$. *Under the Markovian hypothesis, for* $k = N$, *one has*

$$
\begin{aligned}
\mathcal{E}_N(r, v) &= \max(v + tb_N - d_N, 0), & (4)\\
D_N(r, v) &= (1 - w)v \mathbf{1}\{\mathcal{E}_N(r, v) = 0\} + d_N \mathbf{1}\{\mathcal{E}_N(r, v) > 0\}, \\
DS_N(r, v) &= \min\left\{(1 - w)v, d_N^{(sen)}\right\} \mathbf{1}\{\mathcal{E}_N(r, v) = 0\} + d_N^{(sen)} \mathbf{1}\{\mathcal{E}_N(r, v) > 0\}, & (5)\\
DJ_N(r, v) &= \max\left\{(1 - w)v - d_N^{(sen)}, 0\right\} \mathbf{1}\{\mathcal{E}_N(r, v) = 0\} + d_N^{(jun)} \mathbf{1}\{\mathcal{E}_N(r, v) > 0\}, & (6)\\
TB_N(r, v) &= tb_N \mathbf{1}\{\mathcal{E}_N(r, v) > 0\}, & (7)\\
BC_N(r, v) &= wv \mathbf{1}\{\mathcal{E}_N(r, v) = 0\}, & (8)
\end{aligned}
$$

*and for any* $k \in \{0, \ldots, N-1\}$

$$
\begin{aligned}
\mathcal{E}_k(r, v) &= \max\left\{b_k - d_k + T_k \mathcal{E}_{k+1}(r, v), 0\right\}, & (9)\\
D_k(r, v) &= (1 - w)v \mathbf{1}\{\mathcal{E}_k(r, v) = 0\} + \{d_k + T_k D_{k+1}(r, v)\} \mathbf{1}\{\mathcal{E}_k(r, v) > 0\}, \\
DS_k(r, v) &= \min\left\{(1 - w)v, d_k^{(sen)}\right\} \mathbf{1}\{\mathcal{E}_k(r, v) = 0\} + \left\{d_k^{(sen)} + T_k DS_{k+1}(r, v)\right\} \mathbf{1}\{\mathcal{E}_k(r, v) > 0\}, & (10)\\
DJ_k(r, v) &= \max\left\{(1 - w)v - d_k^{(sen)}, 0\right\} \mathbf{1}\{\mathcal{E}_k(r, v) = 0\} + \\
&\quad \left\{d_k^{(jun)} + T_k DJ_{k+1}(r, v)\right\} \mathbf{1}\{\mathcal{E}_k(r, v) > 0\}, & (11)\\
TB_k(r, v) &= \{tb_n + T_k TB_{k+1}\} \mathbf{1}\{\mathcal{E}_k(r, v) > 0\}, & (12)\\
BC_k(r, v) &= \alpha V_n \mathbf{1}\{\mathcal{E}_k(r, v) = 0\} + T_k BC_{k+1} \mathbf{1}\{\mathcal{E}_k(r, v) > 0\}. & (13)
\end{aligned}
$$

# 3 Dynamic programming

The implementation of the optimal stopping time problem presented in Section 2 is done by using dynamic programming coupled with finite elements and bilinear interpolations. Parallel computing is used to accelerate the execution time of our program and enhance its efficiency.

Let $\mathcal{G}$ be a set of grid points $\{(a_1, b_1), (a_1, b_2), \ldots, (a_p, b_q)\}$ such that $\max(\Delta a_k, \Delta b_l) \to 0$ and $\mathbb{Q}[(V_t, r_t) \in [a_p, \infty) \times \mathbb{R}^*_+ \cup \mathbb{R}^*_+ \times [b_q, \infty)] \to 0$, when $p$ and $q \to 0$. Let $a_0 = b_0 = 0$ and $a_{p+1} = b_{q+1} = \infty$. The rectangle $[a_i, a_{i+1}) \times [b_j, b_{j+1})$ is designated by $R_{ij}$.

Dynamic programming acts as follows.

1. At date $t_N = T$, the value functions are known in closed form and are computed following Equation (4), (5), (6), (7) and (8).

2. At each date $t_n$, suppose that an approximation of each value function is available at a future decision date $t_{n+1}$ on $\mathcal{G}$, indicated by $\widetilde{\Psi}_{n+1}(a_k, b_l)$, for $k = 1, \ldots, p$ and $l = 1, \ldots, q$, where $\Psi_n$ represents $TB_n$, $BC_n$, $DS_n$, $DJ_n$, or $\mathcal{E}_n$. Use a bilinear piecewise polynomial, and interpolate each value function $\widetilde{\Psi}_{n+1}$ from $\mathcal{G}$ to the overall state space $[0, \infty)^2$ by setting

$$\widehat{\Psi}_{n+1}(x, e^y) = \sum_{i=0}^{p} \sum_{j=0}^{q} \left(\alpha_{ij}^{n+1} + \beta_{ij}^{n+1} x + \gamma_{ij}^{n+1} e^y + \delta_{ij}^{n+1} x e^y\right) \times \mathbb{I}\left((x, y) \in R_{ij}\right).$$

   The local coefficients of each value function $f_{n+1}$, $\alpha_{ij}^{n+1}$, $\beta_{ij}^{n+1}$, $\gamma_{ij}^{n+1}$, and $\delta_{ij}^{n+1}$, for $i = 0, \ldots, p$ and $j = 0, \ldots, q$, are those of the bilinear interpolation.

3. Approximate every expected discounted value function at $t_n$ on $\mathcal{G}$

$$\begin{aligned} & E\left[e^{-\int_{t_n}^{t_{n+1}} r_s ds}\, \widehat{\Psi}_{n+1}(V_{t_{n+1}}, r_{t_{n+1}}) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\ &= B(t_n, t_{n+1}) E^*\left[\widehat{\Psi}_{n+1}(V_{t_{n+1}}, r_{t_{n+1}}) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\ &= B(t_n, t_{n+1}) \sum_{i,j} \left(\alpha_{ij}^{n+1} T_{klij}^{00} + \beta_{ij}^{n+1} T_{klij}^{10} + \gamma_{ij}^{n+1} T_{klij}^{01} + \delta_{ij}^{n+1} T_{klij}^{11}\right), \end{aligned} \tag{14}$$

   where the transition tables $T^{00}, T^{10}, T^{01}$, and $T^{11}$ are defined by

$$\begin{aligned} T_{klij}^{\nu\mu} \quad = \quad & \mathbb{E}^*\left[(V_{t_{n+1}})^{\nu} (e^{r_{t_{n+1}}})^{\mu} \mathbb{I}\left((V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij}\right) \mid \right. \\ & \left.(V_{t_n}, r_{t_n}) = (a_k, b_l)\right], \quad \text{for } \nu \text{ and } \mu \in \{0, 1\}. \end{aligned}$$

   For example, $T_{klij}^{00}$ represents the transition probability that the Markov process $(V, r)$ moves from $(a_k, b_l)$ at $t_n$ and visits the rectangle $R_{ij}$ at $t_{n+1}$. Closed-form solutions for the transition parameters are given in Appendix B.

4. Compute the value functions at $t_n$ on $\mathcal{G}$ following Equation (9), Equation (11), Equation (10), Equation (12) and Equation (13), using Equation (14).

5. Go to Step 2 and repeat until $n = 0$.

# 4 Numerical investigation

Parallel computing uses multiple central processing units (CPUs) simultaneously to accelerate complex computations. The Message Passing Interface (MPI) library allows the computing process to exchange information between the running CPUs in order to achieve a given job. We parallelize our dynamic program by submitting the computation tasks associated to a given number of grid points to each available CPU. The algorithm used to parallelize our dynamic program is described in detail in Ben-Abdellatif et al. (2016a). This approach allows us to drastically reduce computation times to a reasonable level.

We use the supercomputer Briare managed by Calcul Qubec and Compute Canada.[1] The code lines are written in C and compiled with GCC. We use the MPI library to access parallel computing.

We consider similar parameters to those in Longstaff and Schwartz (1995) for the interest rate dynamic as plausible parameter values. Figure 1 presents the term structure of credit spreads when the firm's leverage

[1] The operation of this supercomputer is funded by the Canada Foundation for Innovation (CFI), ministre de l'conomie, de la Science et de l'Innovation du Qubec (MESI) and the Fonds de recherche du Qubec - Nature et technologies (FRQ-NT).

ratio (debt principle/firm's assets) is changed, but without tax benefits and bankruptcy costs. Credit spreads are greater for a higher leverage ratio, which corresponds to more risky debt. We also observe that credit spreads increase with maturity. Figure 2 considers the case with bankruptcy costs and corporate taxes. The term structure of credit spreads is monotone increasing for firms with a low leverage ratio associated with good rated bonds. Conversely, the credit spreads' term structure is hump shaped for firms with higher leverage ratios, thus corresponding to bonds with low ratings. This is consistent with empirical evidence, as explained by Sarig and Warga (1989) and Kim et al. (1993).

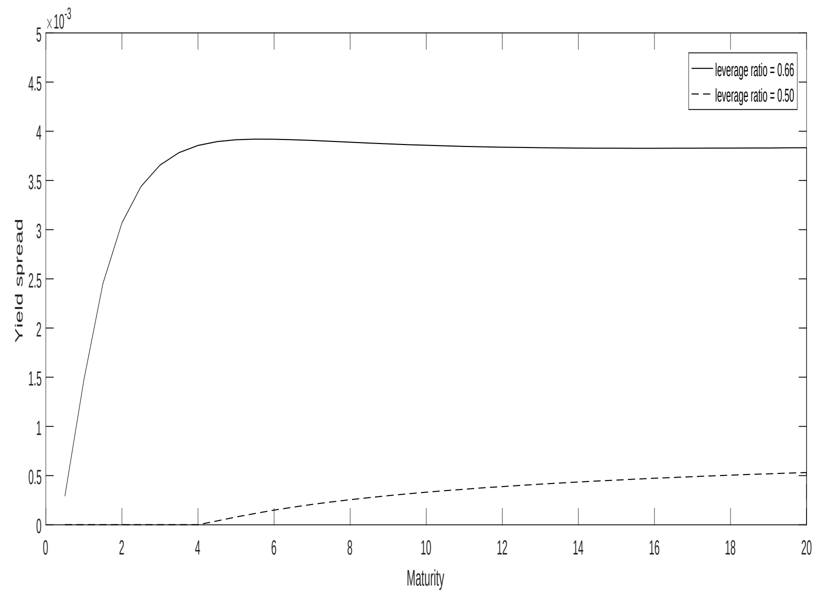


**Figure 1: Credit spreads for an $8\%$ bond for different leverage ratios. The parameters used are $r = 0.04, \alpha = 1, \beta = 0.06, \sigma_r = 0.03, \rho = -0.25, \sigma_v = 0.2, w = 0$, and $r^c = 0$.**

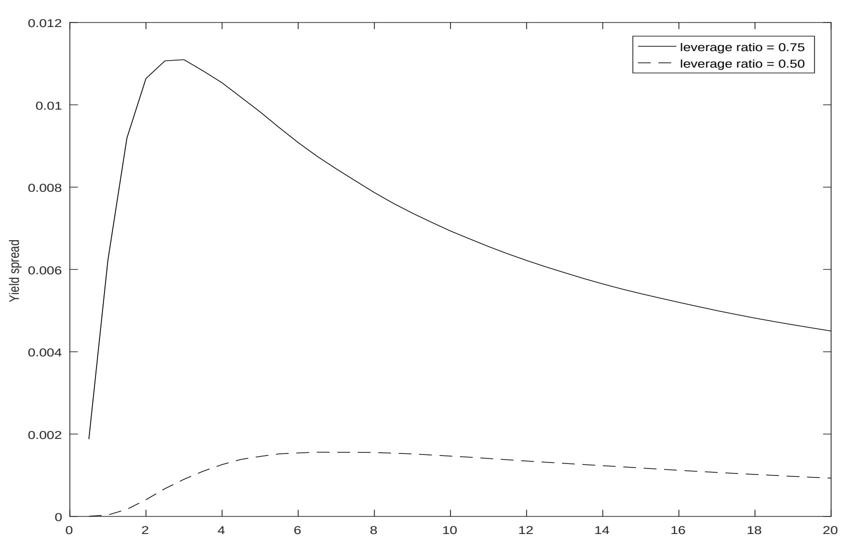


**Figure 2: Credit spreads for an $8\%$ bond for different leverage ratios. The parameters used are $r = 0.04, \alpha = 1, \beta = 0.06, \sigma_r = 0.03, \rho = -0.25, \sigma_v = 0.2, w = 0.3$, and $r^c = 0.35$.**

Figure 3 plots the term structure of the credit spread for various levels of the current interest rate $r$, and shows a negative relation between credit spreads and the level of the short-term interest rate. An increase in $r$ tends to reduce the default probability as it affects the drift on the firm's assets dynamic, reducing the yield spread. However, the magnitude of decrease in the credit spread depends on the correlation between asset returns and changes in the interest rate. As shown in Figure 4, the credit spread increases when the correlation increases. As explained by Longstaff and Schwartz (1995), differences in the duration of bonds across industries is related to the differences in correlation with the interest rate level.

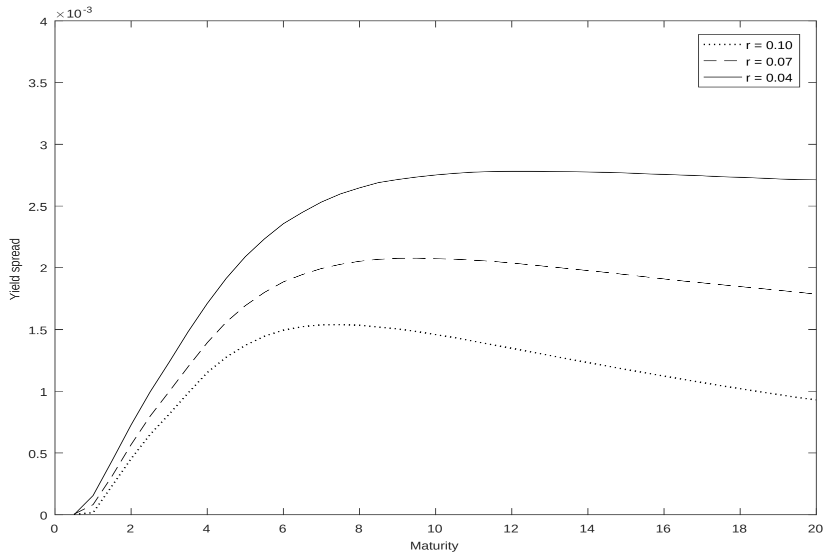


**Figure 3: Credit spreads for an** $8\%$ **bond for different values of** $r$**. The parameters used are** $\alpha = 1, \beta = 0.06, \sigma_r = 0.03, \rho = -0.25, \sigma_v = 0.2, w = 0.3, r^c = 0.35$**, and leverage ratio** $= 0.5$**.**

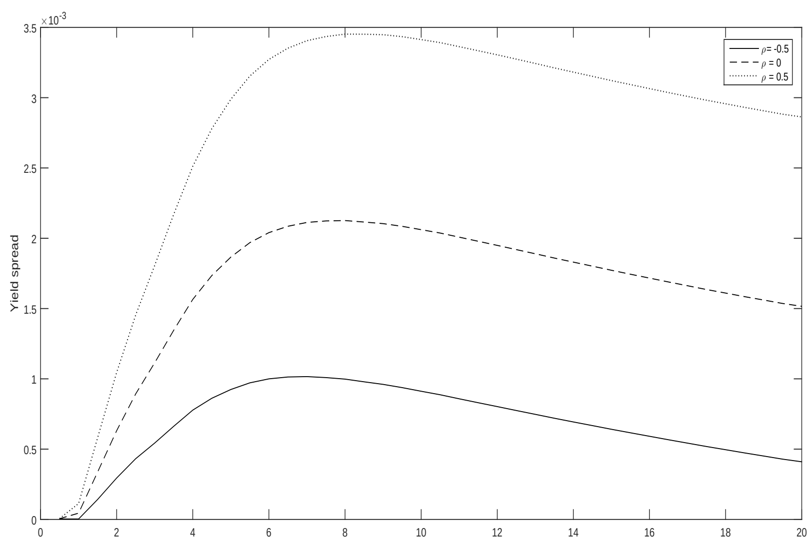


**Figure 4: Credit spreads for an** $8\%$ **bond for different values of** $\rho$**. The parameters used are** $r = 0.04, \alpha = 1, \beta = 0.06, \sigma_r = 0.03, \sigma_v = 0.2, w = 0.3, r^c = 0.35$**, and leverage ratio** $= 0.5$**.**

Figure 5 plots the credit spread for different values of volatility for the firm's assets $\sigma_r$. As the latter increases, the credit spread increases. The term structure of credit spreads is monotone increasing for firms with low risk activities, while it is hump shaped for more risky firms.

Our paper does not address the estimation problem but it is interesting to notice that under the structural credit model, it remains an issue. The main difficulty of the estimation problem is that the firm's assets value cannot be directly observed. This is further complicated by the fact that the data samples only comprise of

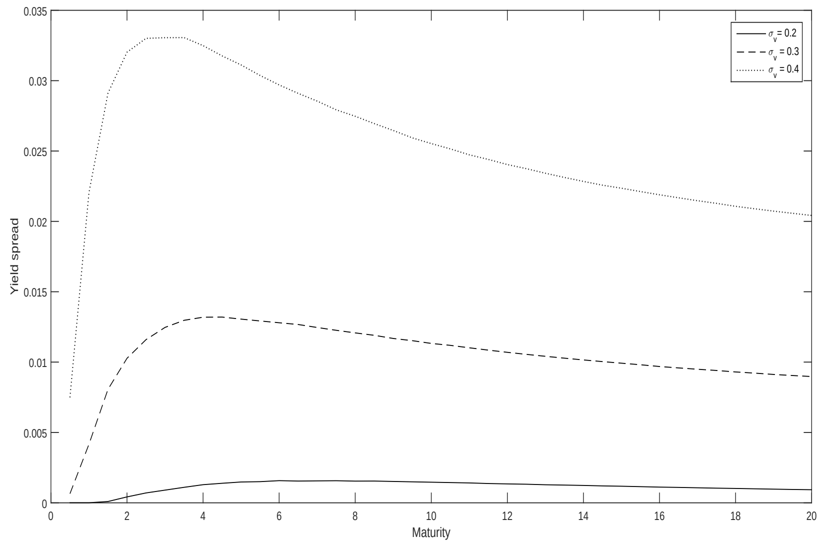


**Figure 5: Credit spreads for an $8\%$ bond for different values of $\sigma_v$. The parameters used are $r = 0.04, \alpha = 1, \beta = 0.06, \sigma_r = 0.03, \rho = -0.25, w = 0.3, r^c = 0.35$, and leverage ratio $= 0.5$.**

surviving firms. Several approaches were proposed to tackle the estimation problem. We briefly discuss the main two methodologies; the first one is based on a transformed-data-maximum likelihood method. Duan (1994, 2000) was the pioneer and proposes a likelihood function based on the observed equity prices. He views them as a sample of transformed data using the equity pricing equation. Later, and under the same spirit, the transformed-data MLE method was also applied in credit risk analysis by Ericsson and Reneby (2004), Wong and Choi (2004) and Duan et al. (2004). The latter derive maximum likelihood estimators for parameters under deterministic and stochastic interest rates. Under Longstaff and Schwartz's (1995) model they propose a two-stage estimation procedure that first analyzes a reduced version of the model by setting the interest rate to a constant. Finally, they address the full version of Longstaff and Schwartz's (1995) structural model. The second estimation methodology KMV, as is called in the financial industry, is based on an iterated algorithm. Interestingly, Duan et al. (2005) proved that the KMV method is somewhat equivalent to the transformed-data MLE method that he proposed in earlier research (Duan, 1994, 2000). As a future research avenue, we will conduct an empirical analysis based on our valuation algorithm and oppose it to real data.

# 5 Conclusion

We propose a general model for valuing risky corporate debt that incorporates both default risk and interest-rate risk. Our methodology is based on a dynamic program coupled with piecewise bilinear approximations where we use parallel computing to enhance efficiency. The proposed model allows for any debt structure with different seniority classes and takes into account tax benefits and bankruptcy costs. Our methodology is flexible and general, and can easily be used to perform realistic empirical credit-risk studies.

We examine the theoretical effect of interest rate uncertainty on the valuation of corporate debt by incorporating a mean-reverting process to model the short-term interest rate. As expected, our results are consistent with empirical evidence documented in the literature. In fact, the interest-rate risk affects the credit spreads' level, and both are negatively correlated. In addition, the correlation between the interest rate and the firms' economic activities explains the observed different credit spreads for bonds with the same rating but in various industries.

Future research avenues include considering a reorganization process for this framework, and the valuation of options embedded in corporate bonds, such as exchangeable convertible bonds. Moreover, one can extend this two-dimensional dynamic program to higher dimensions by including an additional factor to the valuation framework. For example, one could consider a corporate debt for which coupon payments are due in a foreign currency; then, the exchange rate thus becomes the third factor of the model. The extension is challenging but feasible as we can rely on parallel computing to control the computing times, and we can combine the dynamic program with quasi-Monte Carlo simulations instead of finite elements.

## Appendix A Forward measure

The forward measure $\mathbb{P}^{T_F}$ for any date $T_F$ is the measure associated with taking the bond $B(t,T_F)$ as a numeraire asset. Under the forward measure, the ratio $B(t,T)/B(t,T_F)$ is a martingale for $T \leq T_F$. From Girsanov's Theorem, it follows that the process $W^{T_F}$ defined by

$$dW_t^{T_F} = dZ_t^1 + \frac{\sigma_r}{\alpha}(1 - e^{-\alpha(T_F - t)}),$$

is standard Brownian motion under $\mathbb{P}_{T_F}$. Thus, the dynamic of the interest rate becomes

$$dr_t = \left(\theta - \alpha r_t - \frac{\sigma_r^2}{\alpha}(1 - e^{-\alpha(T_F - t)})\right) dt + \sigma_r dW_t^{T_F},$$

with $\theta = \alpha\beta$ and the dynamic of $X_t = \ln(V_t)$ is

$$\begin{aligned} dX_t = &\left(r_t - \delta - \frac{\sigma_V^2}{2} - \frac{\rho\sigma_V\sigma_r}{\alpha}(1 - e^{-\alpha(T_F - t)})\right) dt + \\ &\sigma_V\left(\rho dW_t^{T_F} + \sqrt{1-\rho^2}dZ_t^2\right). \end{aligned}$$

The solutions are given by

$$\begin{aligned} r_t &= r_u e^{-\alpha(t-u)} + \left(\frac{\theta}{\alpha} - \frac{\sigma_r^2}{\alpha^2}\right)\left(1 - e^{-\alpha(t-u)}\right) + \\ &\quad \frac{\sigma_r^2}{2\alpha^2}\left(e^{-\alpha(T_F - t)} - e^{-\alpha(T_F + t - 2u)}\right) + \sigma_r \int_u^t e^{-\alpha(t-s)} dW_s^{T_F}, \\ X_t &= X_u + \beta(u,t) - \left(\frac{\sigma_V^2}{2} + \frac{\rho\sigma_V\sigma_r}{\alpha}\right)(t-u) + \\ &\quad \frac{\rho\sigma_V\sigma_r}{\alpha^2}\left(e^{-\alpha(T_F - u)} - e^{-\alpha(T_F - u)}\right) + \\ &\quad \int_u^t \left(\rho\sigma_V + \frac{\sigma_r}{\alpha}\left(1 - e^{-\alpha(t-s)}\right)\right) dW_s^{T_F}, \quad \text{for } 0 \leq u \leq t, \end{aligned}$$

with

$$\begin{aligned} \beta(u,t) &= \frac{r_u}{\alpha}\left(1 - e^{-\alpha(t-u)}\right) + \left(\frac{\theta}{\alpha} - \frac{\sigma_r^2}{\alpha^2}\right)\left(-\frac{1 - e^{-\alpha(t-u)}}{\alpha} + t - u\right) \\ &\quad + \frac{\sigma_r^2}{2\alpha^3}\left(e^{-\alpha(T_F - t)} - 2e^{-\alpha(T_F - u)} + e^{-\alpha(T_F + t - 2u)}\right). \end{aligned}$$

Under the forward measure, the pair $(X_t, r_t)$ follows a bivariate normal distribution with

$$E[X_t|X_u] = X_u + \beta(u,t) - \left(\frac{\sigma_V^2}{2} + \frac{\rho\sigma_V\sigma_r}{\alpha}\right)(t-u) + \frac{\rho\sigma_V\sigma_r}{\alpha^2}\left(e^{-\alpha(T_F - t)} - e^{-\alpha(T_F - u)}\right),$$

$$Var[X_t|X_u] = \left(\sigma_V^2 + \frac{2\rho\sigma_V\sigma_r}{\alpha} + \frac{\sigma_r^2}{\alpha^2}\right)(t-u) - \frac{2\rho\sigma_V\sigma_r}{\alpha^2} \times \left(1 - e^{-\alpha(t-u)}\right) - \frac{\sigma_r^2}{2\alpha^3}\left(3 - 4e^{-\alpha(t-u)} + e^{-2\alpha(t-u)}\right),$$

$$E[r_t|r_u] = r_u e^{-\alpha(t-u)} + \frac{\theta}{\alpha}\left(1 - e^{-\alpha(t-u)}\right) - \frac{\sigma_r^2}{\alpha^2} \times \left(1 - e^{-\alpha(t-u)}\right) + \frac{\sigma_r^2}{2\alpha^2}\left(e^{-\alpha(T_F - t)} - e^{-\alpha(T_F + t - 2u)}\right),$$

$$Var[r_t|r_u] = \frac{\sigma_r^2}{2\alpha}\left(1 - e^{-2\alpha(t-u)}\right), \text{ and}$$

$$Cov[X_t, r_t|X_u, r_u] = \left(\frac{\rho\sigma_V\sigma_r}{\alpha} + \frac{\sigma_r^2}{\alpha^2}\right)\left(1 - e^{-\alpha(t-u)}\right) - \frac{\sigma_r^2}{2\alpha^2} \times \left(1 - e^{-2\alpha(t-u)}\right).$$

# Appendix B Transitions parameters

The transition parameters $T_{klij}^{\nu\mu}$ for $\nu$ and $\mu \in \{0,1\}$, $k \in \{1,\ldots,p\}$, $l \in \{1,\ldots,q\}$, $i \in \{0,\ldots,p\}$, and $j \in \{0,\ldots,q\}$ are calculated as follows:

$$
\begin{aligned}
T_{klij}^{00} &= \mathbb{E}^* \left[\mathbb{I}\left((V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\
&= \mathbb{Q}^* \left[(V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij} \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\
&= \int_{x_{k,i}}^{x_{k,i+1}} \int_{y_{l,j}}^{y_{l,j+1}} \phi(z_1, z_2, \rho) dz_1 dz_2 \\
&= \Phi(x_{k,i+1}, y_{l,j+1}, \rho) - \Phi(x_{k,i}, y_{l,j+1}, \rho) - \Phi(x_{k,i+1}, y_{l,j}, \rho) + \Phi(x_{k,i}, y_{l,j}, \rho),
\end{aligned}
$$

where

$$
\begin{aligned}
x_{k,i} &= \left(\log(a_i/a_k) - \eta_1\right)/\sqrt{\delta_1} \\
y_{l,j} &= (b_j - \eta_2)/\sqrt{\delta_2}, \\
\eta_1 &= \beta_l - \left(\frac{\sigma_V^2}{2} + \frac{\rho\sigma_V\sigma_r}{\alpha}\right)\Delta t + \frac{\rho\sigma_V\sigma_r}{\alpha^2}\left(1 - e^{-\alpha\Delta t}\right), \\
\delta_1 &= \left(\sigma_V^2 + \frac{2\rho\sigma_V\sigma_r}{\alpha} + \frac{\sigma_r^2}{\alpha^2}\right)\Delta t - \frac{2\rho\sigma_V\sigma_r}{\alpha^2}\left(1 - e^{-\alpha\Delta t}\right) - \frac{\sigma_r^2}{2\alpha^3}\left(3 - 4e^{-\alpha\Delta t} + e^{-2\alpha\Delta t}\right), \\
\eta_2 &= b_l e^{-\alpha\Delta t} + \frac{\theta}{\alpha}\left(1 - e^{-\alpha\Delta t}\right) - \frac{\sigma_r^2}{\alpha^2}\left(1 - e^{-\alpha\Delta t}\right) + \frac{\sigma_r^2}{2\alpha^2}\left(1 - e^{-2\alpha\Delta t}\right), \\
\delta_2 &= \frac{\sigma_r^2}{2\alpha}\left(1 - e^{-2\alpha\Delta t}\right), \\
\beta_l &= \frac{r_l}{\alpha}\left(1 - e^{-\alpha\Delta t}\right) + \left(\frac{\theta}{\alpha} - \frac{\sigma_r^2}{\alpha^2}\right)\left(-\frac{1 - e^{-\alpha\Delta t}}{\alpha} + \Delta t\right) + \frac{\sigma_r^2}{2\alpha^3}\left(1 - 2e^{-\alpha\Delta t} + e^{-2\alpha\Delta t}\right).
\end{aligned}
$$

$\mathbb{E}^*$ is the expectation under the forward measure to the time $t_{n+1}$. The functions $\phi(\cdot,\cdot,\rho)$ and $\Phi(\cdot,\cdot,\rho)$ are the density and cumulative density functions, respectively, of the bivariate standard normal distribution with correlation coefficient $\rho$. The function $\Phi(\cdot,\cdot,\rho)$ is computed according to Genz (2004).

$$
\begin{aligned}
T_{klij}^{10} &= \mathbb{E}^* \left[V_{t_{n+1}} \mathbb{I}\left((V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\
&= w_k^1 \int_{x_{k,i}-\sqrt{\delta_1}}^{x_{k,i+1}-\sqrt{\delta_1}} \int_{y_{l,j}-\rho\sqrt{\delta_1}}^{y_{l,j+1}-\rho\sqrt{\delta_1}} \phi(u_1, u_2, \rho) du_1 du_2 \\
&= w_k^1 \Big[\Phi(x_{k,i+1} - \sqrt{\delta_1}, y_{l,j+1} - \rho\sqrt{\delta_1}, \rho) \\
&\quad - \Phi(x_{k,i} - \sqrt{\delta_1}, y_{l,j+1} - \rho\sqrt{\delta_1}, \rho) \\
&\quad - \Phi(x_{k,i+1} - \sqrt{\delta_1}, y_{l,j} - \rho\sqrt{\delta_1}, \rho) \\
&\quad + \Phi(x_{k,i} - \sqrt{\delta_1}, y_{l,j} - \rho\sqrt{\delta_1}, \rho)\Big],
\end{aligned}
$$

where $w_k^1 = a_k \exp(\eta_1 + \delta_1/2)$.

$$
\begin{aligned}
T_{klij}^{01} &= \mathbb{E}^* \left[e^{r_{t_{n+1}}} \mathbb{I}\left((V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\
&= w_l^2 \int_{x_{k,i}-\rho\sigma_2\Delta t}^{x_{k,i+1}-\rho\sigma_2\Delta t} \int_{y_{l,j}-\sigma_2\Delta t}^{y_{l,j+1}-\sigma_2\Delta t} \phi(u_1, u_2, \rho) du_1 du_2
\end{aligned}
$$

$$
\begin{aligned}
&= w_l^2\Big[\Phi(x_{k,i+1}-\rho\sqrt{\delta_2}, y_{l,j+1}-\sqrt{\delta_2},\rho) \\
&\quad - \Phi(x_{k,i}-\rho\sqrt{\delta_2}, y_{l,j+1}-\sqrt{\delta_2},\rho) \\
&\quad - \Phi(x_{k,i+1}-\rho\sqrt{\delta_2}, y_{l,j}-\sqrt{\delta_2},\rho) \\
&\quad + \Phi(x_{k,i}-\rho\sqrt{\delta_2}, y_{l,j}-\sqrt{\delta_2},\rho)\Big],
\end{aligned}
$$

where $w_l^2 = \exp(\eta_2 + \delta_2/2)$.

$$
\begin{aligned}
T^{11}_{klij} &= \mathbb{E}^*\left[V_{t_{n+1}} e^{r_{t_{n+1}}} \mathbb{I}\left((V_{t_{n+1}}, r_{t_{n+1}}) \in R_{ij}\right) \mid (V_{t_n}, r_{t_n}) = (a_k, b_l)\right] \\
&= w_{1,k} w_l^2 \exp\left(\rho\sqrt{\delta_1\delta_2}\right) \times \int_{x_{k,i}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}}^{x_{k,i+1}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}} \int_{y_{l,j}-\rho\sqrt{\delta_1}-\sqrt{\delta_2}}^{y_{l,j+1}-\rho\sqrt{\delta_1}-\sqrt{\delta_2}} \phi(u_1,u_2,\rho)du_1du_2 \\
&= w_k^1 w_l^2 \exp\left(\rho\sqrt{\delta_1\delta_2}\right) \times \\
&\quad \Big[\Phi(x_{k,i+1}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}, y_{l,j+1}-\rho\sqrt{\delta_1}-\sqrt{\delta_2},\rho) - \\
&\quad \Phi(x_{k,i}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}, y_{l,j+1}-\rho\sqrt{\delta_1}-\sqrt{\delta_2},\rho) - \\
&\quad \Phi(x_{k,i+1}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}, y_{l,j}-\rho\sqrt{\delta_1}-\sqrt{\delta_2},\rho) + \\
&\quad \Phi(x_{k,i}-\sqrt{\delta_1}-\rho\sqrt{\delta_2}, y_{l,j}- -\rho\sqrt{\delta_1}-\sqrt{\delta_2},\rho)\Big].
\end{aligned}
$$

## References


Altieri, A. and Vargiolu, T. (2001). Optimal default boundary in discrete time models. Rendiconti per gli Studi Economici Quantitativi, 2(1):1–20.

Anderson, R. and Sundaresan, S. (1996). Design and valuation of debt contracts. Review of Financial Studies, 9(1):37–68.

Ayadi, M. A., Ben-Ameur, H., and Fakhfakh, T. (2016). A dynamic program for valuing corporate securities. European Journal of Operational Research, 249(2):751–770.

Ben-Abdellatif, M., Ben-Ameur, H., and Rémillard, B. (2016a). Dynamic programming and parallel computing for valuing two-dimensional American-style options. Working paper G-2016-48, HEC Montréal.

Ben-Abdellatif, M., Rémillard, B., and Chérif, R. (2016b). Valuing corporate securities. Working paper, HEC Montréal.

Black, F. and Cox, J. C. (1976). Valuing corporate securities: some effects of bond indenture provisions. The Journal of Finance, 31(2):351–367.

Briys, E. and De Varenne, F. (1997). Valuing risky fixed rate debt: an extension. Journal of Financial and Quantitative Analysis, 32(02):239–248.

Cathcart, L. and El-Jahel, L. (1998). Valuation of defaultable bonds. The Journal of Fixed Income, 8(1):65–78.

Chen, N. and Kou, S. G. (2009). Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. Mathematical Finance, 19(3):343–378.

Collin-Dufresne, P. and Goldstein, R. S. (2001). Do credit spreads reflect stationary leverage ratios? The Journal of Finance, 56(5):1929–1957.

Collin-Dufresne, P., Goldstein, R. S., and Martin, J. S. (2001). The determinants of credit spread changes. The Journal of Finance, 56(6):2177–2207.

Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. Econometrica: Journal of the Econometric Society, 53:385–407.

Duan, J.-C. (1994). Maximum likelihood estimation using price data of the derivative contract. Mathematical Finance, 4(2):155–167.

Duan, J.-C. (2000). Correction: Maximum likelihood estimation using price data of the derivative contract (mathematical finance 1994, 4/2, 155–167). Mathematical Finance, 10(4):461–462.

Duan, J.-C., Gauthier, G., and Simonato, J.-G. (2005). On the equivalence of the KMV and maximum likelihood methods for structural credit risk models. Working paper G-2005-06, Les Cahiers du GERAD.

Duan, J.-C., Simonato, J.-G., Gauthier, G., and Zaanoun, S. (2004). Estimating merton's model by maximum likelihood with survivorship consideration. In EFA 2004 Maastricht Meetings Paper, number 4190.

Ericsson, J. and Reneby, J. (1998). A framework for valuing corporate securities. Applied Mathematical Finance, 5(3-4):143–163.

Ericsson, J. and Reneby, J. (2004). An empirical study of structural credit risk models using stock and bond prices. The Journal of Fixed Income, 13(4):38–49.

François, P. and Morellec, E. (2004). Capital structure and asset prices: some effects of bankruptcy procedures. The Journal of Business, 77(2):387–411.

Genz, A. (2004). Numerical computation of rectangular bivariate and trivariate normal and t probabilities. Statistics and Computing, 14(3):251–260.

Geske, R. (1977). The valuation of corporate liabilities as compound options. The Journal of Financial and Quantitative Analysis, 12(4):541–552.

Hsu, J., Saá-Requejo, J., and Santa-Clara, P. (2010). A structural model of default risk. Journal of Fixed Income, 19(3):77–94.

Jamshidian, F. (1989). An exact bond option formula. The journal of Finance, 44(1):205–209.

Kim, I. J., Ramaswamy, K., and Sundaresan, S. (1993). Does default risk in coupons affect the valuation of corporate bonds? A contingent claims model. Financial Management, 22(1):117–131.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. The Journal of Finance, 49(4):1213–1252.

Leland, H. E. and Toft, K. B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. The Journal of Finance, 51(3):987–1019.

Litterman, R. B. and Scheinkman, J. (1991). Common factors affecting bond returns. The Journal of Fixed Income, 1(1):54–61.

Longstaff, F. A. and Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. The Journal of Finance, 50(3):789–819.

Mella-Barral, P. and Perraudin, W. (1997). Strategic debt service. The Journal of Finance, 52(2):531–556.

Merton, R. C. (1974). On the pricing of corporate debt: the risk structure of interest rates. The Journal of Finance, 29(2):449–470.

Sarig, O. and Warga, A. (1989). Some empirical estimates of the risk structure of interest rates. The Journal of Finance, 44(5):1351–1360.

Shimko, D. C., Tejima, N., and Van Deventer, D. R. (1993). The pricing of risky debt when interest rates are stochastic. The Journal of Fixed Income, 3(2):58–65.

Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of Financial Economics, 5(2):177–188.

Wong, H. and Choi, T. (2004). The impact of default barrier on the market value of firm's asset. Working paper, Chinese University of Hong Kong.

Zhou, C. (2001). The term structure of credit spreads with jump risk. Journal of Banking and Finance, 25(11):2015–2040.