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**GERAD** HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

**Tél. : 514 340-6053**
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

# Valuing corporate securities when the firm's assets are illiquid

**Hatem Ben-Ameur** [a]
**Tarek Fakhfakh** [b]
**Alexandre Roch** [c]

[a] *GERAD & Department of Decision Sciences, HEC Montréal, Montréal (Québec), Canada*

[b] *FSEG Sfax, Sfax, Tunisia*

[c] *Finance Department, UQAM, Montréal (Québec), Canada*

`hatem.ben-ameur@hec.ca`
`hamadi.fakhfakh@fsegs.rnu.tn`
`roch.alexandre_f@uqam.ca`

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**Abstract:** We use stochastic dynamic programming to design and solve an extended structural setting for which the illiquidity of the firm's assets is interpreted as an intangible corporate security. To assess our construction, we provide a sensitivity analysis of the values of corporate securities with respect to the illiquidity parameter.

**Keywords:** Structural model, corporate securities, illiquidity costs, stochastic dynamic programming

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# 1 Introduction

We further extend the construction of Ben-Ameur et al. (2017) to illiquidity costs. This intangible asset generates a proportional loss on the firm's asset value under liquidation. We use stochastic dynamic programming (SDP) to design and solve an extended structural model that accommodates 1) a large set of Markov state processes, 2) several intangible assets, that is, tax benefits as well as reorganization, illiquidity, and bankruptcy costs, 3) an arbitrary coupon/capital schedule, 4) multiple seniority classes, and 5) a reorganization process. To assess our construction, we provide a sensitivity analysis of the values of corporate securities with respect to the illiquidity parameter.

The relation between bond prices and liquidity is usually studied from market perspectives. Chen et al. (2007) find that illiquid bonds show higher yield spreads. Amihud and Mendelson (1991) study the yield spreads between U.S. Treasury notes and bills with the same characteristics except for liquidity. They give evidence that these spreads decrease in terms of the debt's time to maturity. Chordia et al. (2005) provide evidence that a shock on liquidity spreads in the stock market affects their associated counterparts in the bond market. Furthermore, they find that volatility is an important determinant of liquidity, suggesting the existence of a common factor for liquidity and volatility. Ericsson and Renault (2006) propose a structural model in which investors can be forced to sell their bonds due to an external exogenous liquidity shock. Chevalier et al. (2013) solve the optimal dividend problem in a setting where illiquidity is modeled in terms of fixed costs and increased interest rates associated to debt reorganisation, while Løkka and Zervos (2008) consider the same problem in a setting where the firm pays a proportional transaction cost when new shares of equity are issued or dividends are payed out.

We introduce illiquidity costs as an intangible asset of a public company. We use SDP to design and solve the associated extended structural model. SDP runs backward in time from the debt maturity to the origin and, at each step of the recursion, alternates between value function approximation and value function integration at earlier evaluation/payment date. Despite the flexibility of the proposed structural setting, its resolution, which combines between SDP and finite elements, is extremely efficient as it assumes only a state-space (but not a time) discretization and a numerical (but not a statistical) error.

The rest of this paper is organized as follows. Section 2 presents the model and its corporate securities. Section 3 is a numerical investigation, while Section 4 concludes.

# 2 Model and notation

We consider a public company with a debt portfolio whose payment dates belong to $\mathcal{P} = \{t_0 = 0, \ldots, t_N = T\}$. The firm's balance-sheet equality, which holds for all $t \in [0, T]$ and all $a = A_t > 0$, is

$$a + \mathrm{TB}\,(t, a) - \mathrm{RC}\,(t, a) - \mathrm{IL}\,(t, a) - \mathrm{BC}\,(t, a) = D^s\,(t, a) + D^j\,(t, a) + \mathcal{E}\,(t, a)\,, \tag{1}$$

where $a = A_t$ is the level of the firm's asset value (the state process) at time $t \in [0, T]$. The tangible and intangible corporate securities $A$, TB, RC, IL, BC, $D^s$, $D^j$, $D = D^s + D^j$, and $\mathcal{E}$ represent the value of 1) the firm's assets, 2) tax benefits, 3) reorganization costs, 4) illiquidity costs, 5) bankruptcy costs, 6) the senior debt portfolio, 7) the junior debt portfolio, 8) the overall debt portfolio, and 9) the firm's equity, respectively. Extending Leland (1994), we define the total value of the firm TV as the left-hand side of Equation (1). The state process $A$ can be any Markov process consistent with the no-arbitrage principle as long as European vanilla options on $A$ can be valued in closed form over each time interval $[t_n, t_{n+1}]$.

The firm is committed to paying $d_n^s + d_n^j = d_n$ at $t_n \in \mathcal{P}$ to its creditors, where $d_n^s$ and $d_n^j$ are the outflows generated at $t_n$ by the senior and junior debts, respectively. The total outflow $d_n$ at $t_n$ includes interest payments $C_n = C_n^s + C_n^j$ as well as principal payments $P_n = P_n^s + P_n^j$. Set these payments at zero whenever necessary. The amounts $C_n^s$, $C_n^j$, $P_n^s$, and $P_n^j$ are known to all investors from the very beginning. The last payment dates of the senior and junior debts, both in $\mathcal{P}$, are indicated by $T^s$ and $T^j$, respectively. Several authors consider a senior coupon bond and a junior coupon bond with a longer maturity, that is, $0 \le T^s < T^j = T$. Senior bondholders are therefore assured capital payment before junior bondholders. This realistic case is embedded in our setting.

For the sake of clarity, we first present the model without a reorganization process, then we describe the full setting. Assume that the evaluation problem has been solved backward in time from the maturity $t_N = T$ untill time $t_{n+1}$, for all levels of the state process. No-arbitrage evaluation and Equation (1) give

$$\begin{aligned} & \mathbb{E}^*_{na}\left[\rho_n A_{t_{n+1}}\right] + \mathbb{E}^*_{na}\left[\rho_n \mathrm{TB}\left(t_{n+1}, A_{t_{n+1}}\right)\right] - \\ & \mathbb{E}^*_{na}\left[\rho_n \mathrm{IL}\left(t_{n+1}, A_{t_{n+1}}\right)\right] - \mathbb{E}^*_{na}\left[\rho_n \mathrm{BC}\left(t_{n+1}, A_{t_{n+1}}\right)\right] \\ = \; & \mathbb{E}^*_{na}\left[\rho_n D^s\left(t_{n+1}, A_{t_{n+1}}\right)\right] + \mathbb{E}^*_{na}\left[\rho_n D^j\left(t_{n+1}, A_{t_{n+1}}\right)\right] + \\ & \mathbb{E}^*_{na}\left[\rho_n \mathcal{E}\left(t_{n+1}, A_{t_{n+1}}\right)\right], \end{aligned} \tag{2}$$

where $\rho_n = e^{-r[t_{n+1}-t_n]}$ is the discount factor over $[t_n, t_{n+1}]$, $r$ the risk-free rate (per year), and $\mathbb{E}^*_{na}[.] = \mathbb{E}^*[. \mid A_{t_n} = a]$ the conditional expectation under a risk-neutral probability measure. Any expression of the form $\mathbb{E}^*_{na}\left[\rho_n v\left(t_{n+1}, A_{t_{n+1}}\right)\right]$ is indicated herein by $\overline{v}(t_n, a)$ since it is, indeed, an average value function of $t_n$ and $a = A_{t_n}$, which refers to the present value of a corporate security, based on its future potentialities but not its immediate cash in- or outflow. This legitimizes the interpretation of a corporate security as a financial derivative on the firm's asset value. Alternatively, $\overline{v}(t_n, a)$ can be interpreted as the value function of a corporate security at $t_n^+$, just after $t_n$ when $A_{t_n^+} = A_{t_n} = a$.

In case of survival at $(t_n, a)$, Equation (2) becomes

$$\begin{aligned} & a + \left[\overline{\mathrm{TB}}(t_n, a) + \mathrm{tb}_n\right] - \overline{\mathrm{IL}}(t_n, a) - \overline{\mathrm{BC}}(t_n, a) \\ & \qquad = \left[\overline{D}^s(t_n, a) + d_n^s\right] + \left[\overline{D}^j(t_n, a) + d_n^j\right] + \left[\overline{\mathcal{E}}(t_n, a) - (d_n - \mathrm{tb}_n)\right], \end{aligned} \tag{3}$$

where $\mathrm{tb}_n = C_n \times r^c$ is the tax savings and $r^c$ the periodic corporate tax rate at $t_n$. The survival condition at $(t_n, a)$, that is,

$$\mathcal{E}(t_n, a) = \overline{\mathcal{E}}(t_n, a) - (d_n - \mathrm{tb}_n) > 0, \tag{4}$$

states that the firm's equity value at $t_n^+$ exceeds the due payment on the overall debt net of the immediate tax savings at $t_n$. This condition implicitly assumes that the firm issues new shares of equity at $(t_n, a)$ equivalent to $(d_n - \mathrm{tb}_n) > 0$ (in dollars), as shown by the following equation:

$$\overline{\mathcal{E}}(t_n, a) = \mathcal{E}(t_n, a) + (d_n - \mathrm{tb}_n) > \mathcal{E}(t_n, a).$$

This rule can be seen as a protection covenant for the firm's bondholders. Since the firm's equity value at time $t_n$ is a continuous and increasing function of the state variable, there exists a default (liquidation) barrier $b_n$ under which the firm is liquidated, that is,

$$\mathcal{E}(t_n, a) = 0, \quad \text{for } a \le b_n,$$

while the strict priority rule applies, which highlights the compound-call-option nature of the residual asset

$$\mathcal{E}(t_n, a) = \max\left[0, \overline{\mathcal{E}}(t_n, a) - (d_n - \mathrm{tb}_n)\right],$$

as claimed by Geske (1977).

Under liquidation, we observe a first loss on the firm's asset value at the rate $v$, due to illiquidity costs, followed by a second loss at the rate $w$, due to bankruptcy costs. The remaining firm's asset value $\overline{w}\overline{v}a = (1-w)(1-v)a$ is used to (partially) pay bondholders by their seniority class. It can happen that, under liquidation at time $t_n$, senior bondholders are paid partially, while junior bondholders are not paid at all, in which case, there exists a threshold $0 < b_n^s < b_n$ such that

$$D^j(t_n, a) = 0, \quad \text{for } a \le b_n^s.$$

The barrier $b_n^s$ can be interpreted as a loss barrier for senior bondholders. This version of the model includes the seminal works of Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994). The gain in flexibility comes with a minor loss of efficiency. Their explicit approach is exchanged here for a numerical approach based on SDP and finite elements. Ayadi et al. (2014) discuss and use SDP and finite differences to evaluate options on futures on stock indices. Their methodology applies for all derivative products.

**Table 1: Value functions before maturity without reorganization.**

| Balance-sheet equality | Liquidation $a \leq b_n$ | Survival $a > b_n$ |
|---|---|---|
| $+a = A_{t_n}$ | $a$ | $a$ |
| $+\mathrm{TB}(t_n, a)$ | $0$ | $\overline{\mathrm{TB}}\,(t_n, a) + \mathrm{tb}_n$ |
| $-\mathrm{IL}(t_n, a)$ | $-va$ | $-\overline{\mathrm{IL}}\,(t_n, a)$ |
| $-\mathrm{BC}(t_n, a)$ | $-w\overline{v}a$ | $-\overline{\mathrm{BC}}\,(t_n, a)$ |
| $=$ | $=$ | $=$ |
| $+D^s\,(t_n, a)$ | $\min\left[\overline{wv}a, \overline{D}^s\,(t_n, a) + d_n^s\right]$ | $\overline{D}^s\,(t_n, a) + d_n^s$ |
| $+D^j\,(t_n, a)$ | $\max\left[0, \overline{wv}a - D^s\,(t_n, a)\right]$ | $\overline{D}^j\,(t_n, a) + d_n^j$ |
| $+\mathcal{E}\,(t_n, a)$ | $0$ | $\overline{\mathcal{E}}\,(t_n, a) - (d_n - \mathrm{tb}_n)$ |

**Table 2: Value functions at maturity without reorganization.**

| Balance-sheet equality | Liquidation $a \leq b_N$ | Survival $a > b_N$ |
|---|---|---|
| $+a = A_{t_N}$ | $a$ | $a$ |
| $+\mathrm{TB}(t_N, a)$ | $0$ | $\mathrm{tb}_N$ |
| $-\mathrm{IL}(t_N, a)$ | $-va$ | $0$ |
| $-\mathrm{BC}(t_N, a)$ | $-w\overline{v}a$ | $0$ |
| $=$ | $=$ | $=$ |
| $+D^s\,(t_N, a)$ | $\min\left(\overline{wv}a, d_N^s\right)$ | $d_N^s$ |
| $+D^j\,(t_N, a)$ | $\max\left(0, \overline{wv}a - d_N^s\right)$ | $d_N^j$ |
| $+\mathcal{E}\,(t_N, a)$ | $0$ | $a - (d_N - \mathrm{tb}_N)$ |

Table 1 exhibits the value functions of corporate securities at $t_n$, for $n = 0, \ldots, N-1$, while Table 2 points out their limit conditions at $t_N = T$. At maturity, the default barrier is $b_N = (d_N - \mathrm{tb}_N)$ and the loss barrier for senior bondholders is $b_N^s = \max\left(\overline{wv}a - d_N^s, 0\right)$.

We now introduce a reorganization process for which bondholders assume a moderate immediate loss under default in exchange for a substantial future gain. We propose a design where bondholders consent to reduce their promized cash-inflow payment at $t_n$ at a grace rate $\eta \in [0, 1]$, as long as the firm is under default and the number of grace periods asked for by the firm before $t_n$ doesn't exceed $\overline{g} \in \mathbb{N}$. The variables $\eta$ and $\overline{g}$ can be seen as the design parameters of the proposed reorganization process. For example, let $\overline{g} = 3$, $\eta = 10\%$, $g_n = 2$, while the firm is under default at $(t_n, a)$. Thus, the firm is better off asking for a third and last reorganization period, which reduces its due payment $d_n$ by $\eta = 10\%$ and expands its non-bankrupt event. Broadie and Kaya (2007) show that reorganization processes based on partial forgiveness can augment the total value of the firm, and that the benefits drastically decrease with $\overline{g}$. In sum, short-time reorganization processes are recommended. Our numerical investigation confirms their findings. Tables 3–4 exhibit the SDP value functions in the full setting with a reorganization process.

**Table 3: Value functions before maturity with reorganization.**

| BSE | Survival: $a > b_n\,(g)$ |
|---|---|
| $+a = A_{t_n}$ | $a$ |
| $+\mathrm{TB}(t_n, a, g)$ | $\overline{\mathrm{TB}}\,(t_n, a, g) + \mathrm{tb}_n$ |
| $-\mathrm{RC}(t_n, a, g)$ | $-\overline{\mathrm{RC}}\,(t_n, a, g)$ |
| $-\mathrm{IL}(t_n, a, g)$ | $-\overline{\mathrm{IL}}\,(t_n, a, g)$ |
| $-\mathrm{BC}(t_n, a, g)$ | $-\overline{\mathrm{BC}}\,(t_n, a, g)$ |
| $=$ | $=$ |
| $+D^s\,(t_n, a, g)$ | $\overline{D}^s\,(t_n, a, g) + d_n^s$ |
| $+D^j\,(t_n, a, g)$ | $\overline{D}^j\,(t_n, a, g) + d_n^j$ |
| $+\mathcal{E}\,(t_n, a, g)$ | $\overline{\mathcal{E}}\,(t_n, a, g) - (d_n - \mathrm{tb}_n)$ |

| BSE | Reorganization: $a \in \left[b_n^l\left(g\right), b_n\left(g\right)\right]$ |
|---|---|
| $+a = A_{t_n}$ | $a$ |
| $+\mathrm{TB}(t_n, a, g)$ | $\overline{\mathrm{TB}}\left(t_n, a, g+1\right) + \overline{\eta}\times\mathrm{tb}_n$ |
| $-\mathrm{RC}(t_n, a, g)$ | $-\overline{\mathrm{RC}}\left(t_n, a, g+1\right) - ua$ |
| $-\mathrm{IL}(t_n, a, g)$ | $-\overline{\mathrm{IL}}\left(t_n, a, g+1\right)$ |
| $-\mathrm{BC}(t_n, a, g)$ | $-\overline{\mathrm{BC}}\left(t_n, a, g+1\right)$ |
| $=$ | $=$ |
| $+D^s\left(t_n, a, g\right)$ | $\overline{D}^s\left(t_n, a, g+1\right) + \overline{\eta} d_n^s$ |
| $+D^j\left(t_n, a, g\right)$ | $\overline{D}^j\left(t_n, a, g+1\right) + \overline{\eta} d_n^j$ |
| $+\mathcal{E}\left(t_n, a, g\right)$ | $\overline{\mathcal{E}}\left(t_n, a, g+1\right) - \left[\overline{\eta}\left(d_n - \mathrm{tb}_n\right) + ua\right]$ |
| BSE | Liquidation: $a < b_n^l\left(g\right)$ |
| $+a = A_{t_n}$ | $a$ |
| $+\mathrm{TB}(t_n, a, g)$ | $0$ |
| $-\mathrm{RC}(t_n, a, g)$ | $0$ |
| $-\mathrm{IL}(t_n, a, g)$ | $-va$ |
| $-\mathrm{BC}(t_n, a, g)$ | $-w\overline{v}a$ |
| $=$ | $=$ |
| $+D^s\left(t_n, a, g\right)$ | $\min\left[\overline{wv}a, \overline{D}^s\left(t_n, a, g\right) + d_n^s\right]$ |
| $+D^j\left(t_n, a, g\right)$ | $\max\left[0, \overline{wv}a - D^s\left(t_n, a, g\right)\right]$ |
| $+\mathcal{E}\left(t_n, a, g\right)$ | $0$ |

The value function of a corporate security becomes a function of time, the level of the firm's asset value, and the number of reorganizations requested by the firm before that time. For $g \in \{0, \ldots, \overline{g}\}$ and $\eta \in [0, 1]$, the balance-sheet equality in Equation (3) is exchanged for

$$\begin{aligned} & a + \left[\overline{\mathrm{TB}}\left(t_n, a, g\right) + \mathrm{tb}_n\right] - \overline{\mathrm{RC}}\left(t_n, a, g\right) - \\ & \overline{\mathrm{IL}}\left(t_n, a, g\right) - \overline{\mathrm{BC}}\left(t_n, a, g\right) \\ = \; & \left[\overline{D}^s\left(t_n, a, g\right) + d_n^s\right] + \left[\overline{D}^j\left(t_n, a, g\right) + d_n^j\right] + \\ & \left[\overline{\mathcal{E}}\left(t_n, a, g\right) - \left(d_n - \mathrm{tb}_n\right)\right], \end{aligned} \tag{5}$$

under survival and

$$\begin{aligned} & a + \left[\overline{\mathrm{TB}}\left(t_n, a, g+1\right) + \overline{\eta}\mathrm{tb}_n\right] - \left[\overline{\mathrm{RC}}\left(t_n, a, g+1\right) + ua\right] - \\ & \overline{\mathrm{IL}}\left(t_n, a, g+1\right) - \overline{\mathrm{BC}}\left(t_n, a, g+1\right) \\ = \; & \left[\overline{D}^s\left(t_n, a, g+1\right) + \overline{\eta} d_n^s\right] + \left[\overline{D}^j\left(t_n, a, g+1\right) + \overline{\eta} d_n^j\right] + \\ & \left[\overline{\mathcal{E}}\left(t_n, a, g+1\right) - \left(\overline{\eta}\left(d_n - \mathrm{tb}_n\right) + ua\right)\right], \end{aligned} \tag{6}$$

under reorganization, where $u \in [0, 1]$ is the rate of the reorganization costs and $\overline{u} = 1 - u$. This results in a survival event characterized by

$$\overline{\mathcal{E}}\left(t_n, a, g\right) - \left(d_n - \mathrm{tb}_n\right) > 0, \quad \text{while } g \le \overline{g},$$

and a reorganization event characterized by

$$\overline{\mathcal{E}}\left(t_n, a, g\right) - \left(d_n - \mathrm{tb}_n\right) \le 0$$

and

$$\overline{\mathcal{E}}\left(t_n, a, g+1\right) - \left[\overline{\eta}\left(d_n - \mathrm{tb}_n\right) + ua\right] > 0,$$

while $g < \overline{g}$.

For example, $\mathrm{TB}(t_n, a, g) = \overline{\mathrm{TB}}\left(t_n, a, g+1\right) + \overline{\eta}\times\mathrm{tb}_n$ in Table 3 indicates that the firm asks for a grace period at $t_n$ since the firm has asked for $g$ grace periods before $t_n$ and $g + 1$ before $t_n^+$. A reorganization can take place over $[t_n, t_{n+1}]$ and be effective if $\overline{\eta}$ and $u$ are low, which suggests a liquidation barrier $b_n^l\left(g\right)$

Table 4: Value functions at maturity with reorganization.

| BSE | Liquidation | Reorganization | Survival |
|---|---|---|---|
| $+a = A_{t_N}$ | $a$ | $a$ | $a$ |
| $+\mathrm{TB}(t_N, a, g)$ | $0$ | $\overline{\eta}\mathrm{tb}_N$ | $\mathrm{tb}_N$ |
| $-\mathrm{RC}(t_N, a, g)$ | $0$ | $-ua$ | $0$ |
| $-\mathrm{IL}(t_N, a, g)$ | $-va$ | $0$ | $0$ |
| $-\mathrm{BC}(t_N, a, g)$ | $-w\overline{v}a$ | $0$ | $0$ |
| $=$ | $=$ | $=$ | $=$ |
| $+D^s\left(t_N, a, g\right)$ | $\min\left[\overline{wv}a, d_N^s\right]$ | $\overline{\eta}d_N^s$ | $d_N^s$ |
| $+D^j\left(t_N, a, g\right)$ | $\max\left[0, \overline{wv}a - D^s\left(t_N, a, g\right)\right]$ | $\overline{\eta}d_N^j$ | $d_N^j$ |
| $+\mathcal{E}\left(t_N, a, g\right)$ | $0$ | $\overline{u}a - \overline{\eta}\left(d_N - \mathrm{tb}_N\right)$ | $a - \left(d_N - \mathrm{tb}_N\right)$ |

below the default (reorganization) barrier $b_n\left(g\right)$. In sum, at each step of the recursion, SDP computes $\overline{g}+1$ value functions per corporate security as well as $\overline{g}+1$ default (reorganization) and liquidation barriers. Since the firm's equity value cannot be deteriorated with a reorganization process based on partial forgiveness, we define the optimal reorganization design as the solution of the following optimization problem:

$$(\overline{g}, \eta) = \quad \arg \quad \begin{array}{l} \sup D\left(t_0, A_{t_0}, g_0\right) \\ \overline{g} \in \{0, \ldots, N\} \\ \eta \in [0,1] \end{array} \quad ,$$

where $g_0 \leq \overline{g}$ is the number of grace periods called for by the firm before $t_0$ with the convention that $g_0 = 0$ when the count starts at $t_0$. This optimization problem is solved on a grid mesh of $(\overline{g}, \eta)$. The proposed reorganization process is case sensitive in the sense that the number of allowed grace periods and the grace rate depend on the firm's characteristics.

## 3 Numerical investigation

We start our numerical investigation with a debt portfolio reduced to a coupon bond. We perform a sensitivity analysis of SDP value functions with respect to the bond's coupon rate and the illiquidity rate. Then, we consider a bond portfolio made of a senior and a junior coupon bonds with the same maturity, and re-perform the same sensitivity analysis with respect to the debt's time to maturity and the illiquidity rate. We choose to present our numerical investigation under the geometric-Brownian-motion assumption.

Let $D$ and $D'$ be two identical bonds except that $D$ is more liquid than $D'$. We define their liquidity spread by the difference in their yield spreads, that is,

$$\mathrm{LS} = \mathrm{YS}' - \mathrm{YS} \geq 0, \tag{7}$$

where LS stands for the liquidity spread and YS for the yield spread. The expected sign of LS is positive since, ceteris paribus, more liquid bonds are more attractive for investors. In our context, illiquidity costs, introduced as an intangible asset, represent the loss on the firm's asset value under liquidation. We assume a proportional loss that applies at the (illiquidity) rate $v \in [0, 1]$. The illiquidity assumption on the firm's assets propagate naturally to the corporate debt portfolio, which results in the following: $D$ is more liquid than $D'$ means that $D$ and $D'$ are identical corporate debts except for their associated illiquidity rates $v$ and $v'$ with $v \leq v'$. One has

$$\mathrm{LS} = \mathrm{YS}\left(v'\right) - \mathrm{YS}\left(v\right) \geq 0.$$

The higher is $v \in [0, 1]$ and the larger is the loss of value for bondholders under liquidation.

As documented in Figure 1, the firm's equity value is a decreasing function of the coupon rate since a larger coupon results in a larger default probability. Moreover, the firm's equity value is insensitive to the illiquidity rate since it's null under liquidation, where illiquidity costs are effective. A higher coupon rate results in two conflicting effects on the debt value: 1) a higher promised cash-flow stream, which has a positive impact on the debt value, and 2) a higher default probability, which has a negative impact on the debt value. Figure 2 locates the firm's maximum debt capacity when the coupon changes. The cash-flow effect dominates for investment-grade bonds and the default-probability effect dominates for junk bonds. As expected, Figure 3 exhibits larger yield spreads for higher coupon levels. Figures 2–3 show that the debt value (yield spread) is a decreasing (increasing) function of the illiquidity rate. The liquidity spread, LS, remains positive, which is consistent with Chen et al. (2007), and becomes larger for high coupon rates as if illiquidity adverse effects were accentuated for high-yield bonds. From Figure 4, the total value of the firm TV is sensitive to the debt–equity ratio and shows a global maximum when the coupon changes. Moreover, TV is a decreasing function of the illiquidity rate since illiquidity costs tend to lower the total value of the firm. Tax benefits, TB, as shown in Figure 5, are represented by an increasing (a decreasing) function of the coupon rate for high- (low-) quality bonds. For low coupon rates, the cash-flow effect dominates, while, for high-coupon rates, the default-probability effect dominates. TB is insensitive to the illiquidity rate as the former is null under liquidation. Bankruptcy costs, BC, as shown in Figure 6, are represented by an increasing function of the bond's coupon. The BC function shows convergence to $w\overline{v}A_0$ when the coupon increases, which characterizes an immediate liquidation. By construction, BC is a decreasing function of the illiquidity rate. Figure 7 represents illiquidity costs, IL, as an increasing function of the bond's coupon and of the illiquidity rate. The IL function shows convergence to $v \times A_0$ when the coupon increases, which characterizes an immediate liquidation. The total-default-probability function in Figure 8 is an increasing function of the coupon, but insensitive to the illiquidity rate. Illiquidy costs have no effect on the survival event.

We now consider a debt portfolio made of a senior and a junior bond with the same maturity. Figure 9 shows that more liquid senior bonds show lower yield spreads, which is consistent with Chen et al. (2007). Meanwhile, Figure 10 shows that senior liquidity spreads decrease with the bond's time to maturity, which is consistent with Amihud and Mendelson (1991). Long-time horizons tend to attenuate the adverse effects of illiquidity. Figure 11 investigates further these issues. By contrast, the term structure of junior yield spreads is practically insensitive to the illiquidity rate. This holds true when junior bondholders are not paid at all under liquidation.

The proposed reorganization process is still valuable in the presence of illiquidity costs. Figures 12–14 point out that a larger illiquidity rate results in a different reorganization plan with a larger grace rate and grace periods.

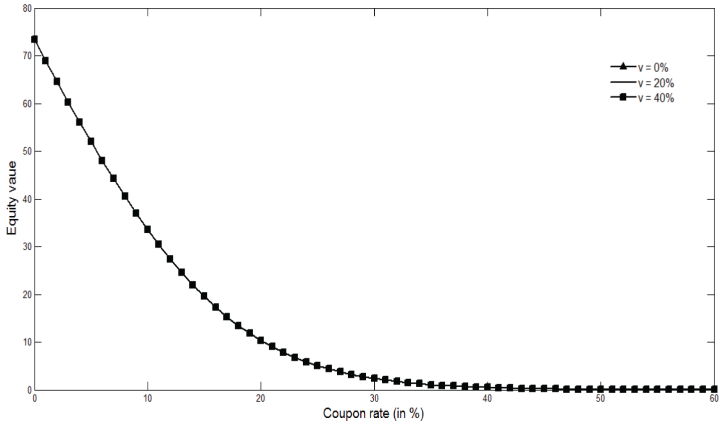


**Figure 1: Equity value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

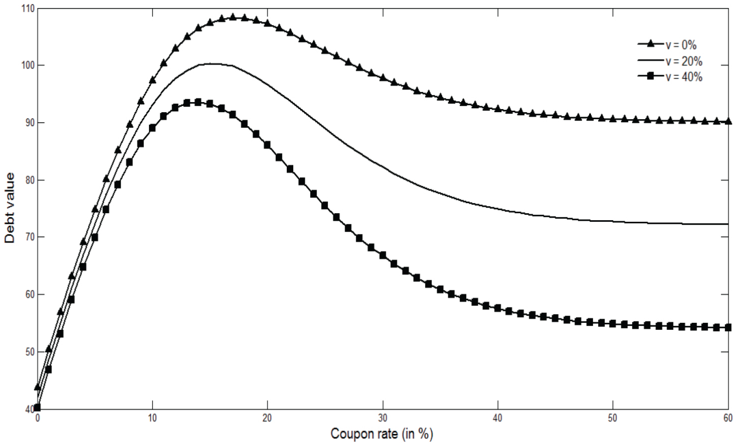


**Figure 2: Debt value as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

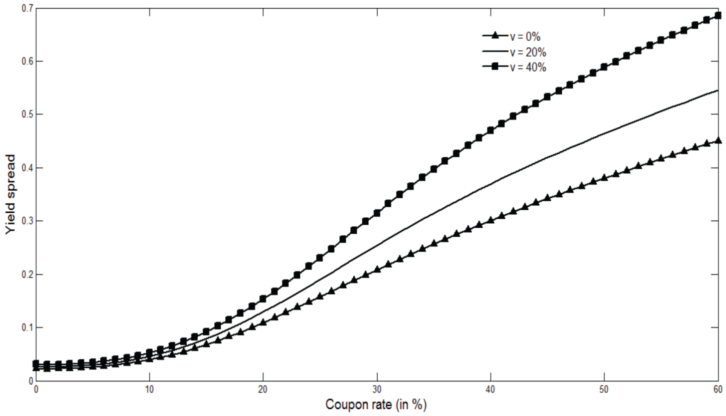


**Figure 3: Yield spread as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

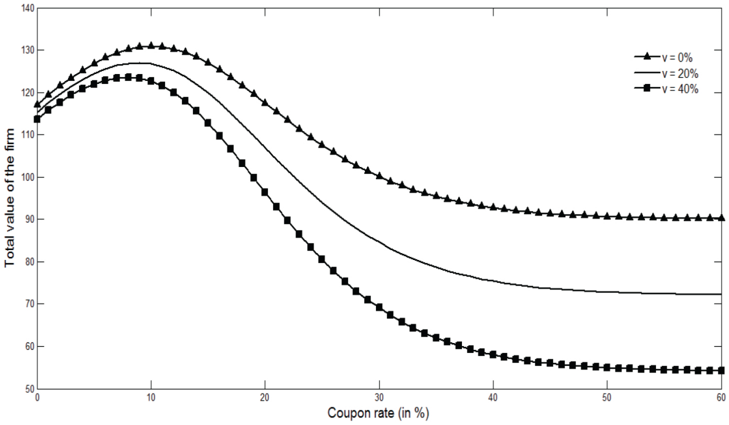


**Figure 4: Total value of the firm as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

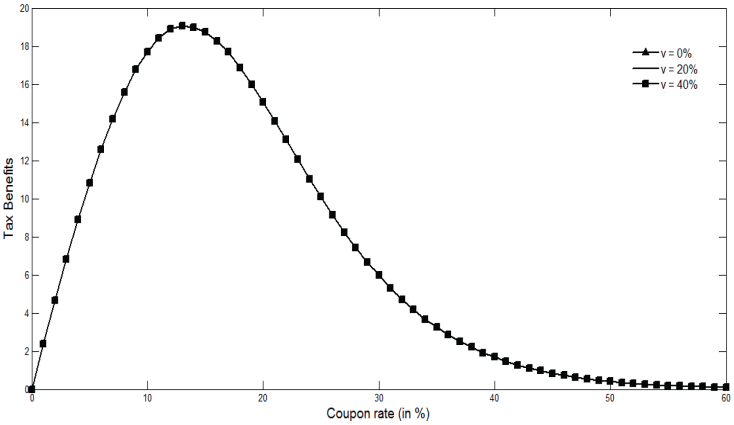


**Figure 5: Tax benefits as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

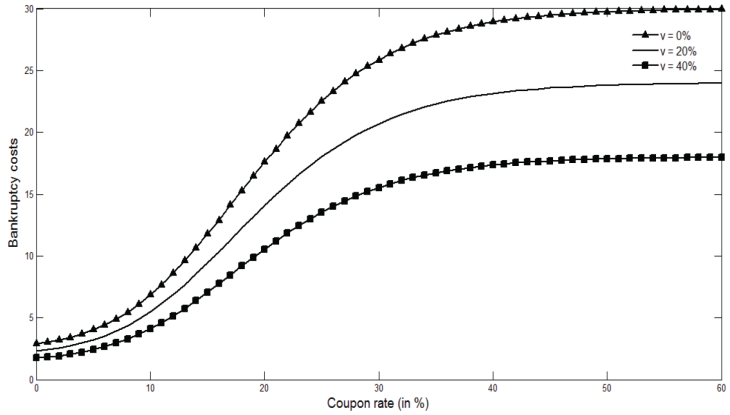


**Figure 6: Bankruptcy costs as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

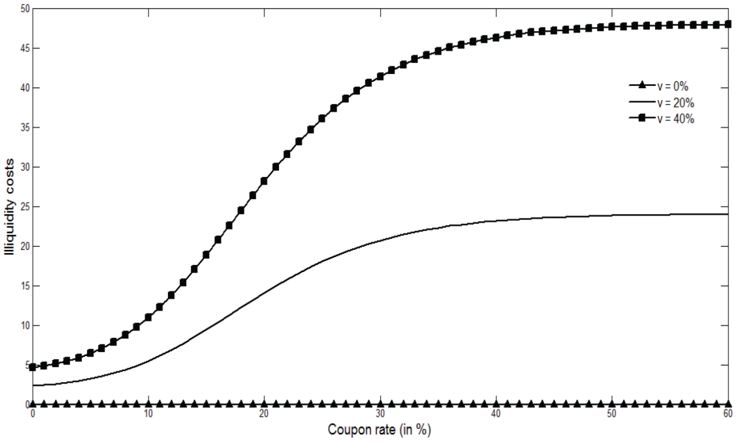


**Figure 7: Illiquidity costs as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

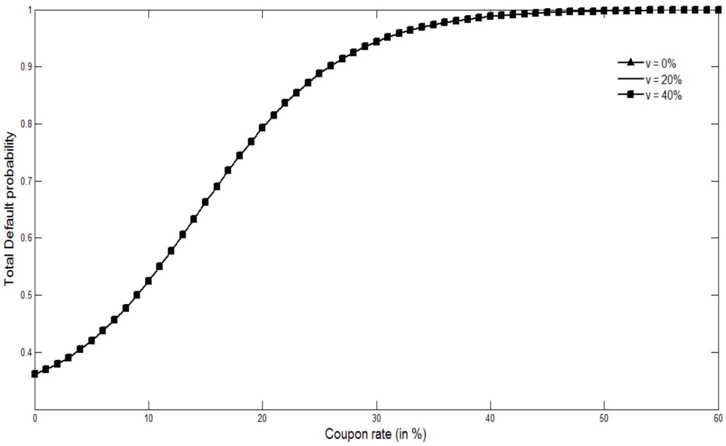


**Figure 8: Total default probability as a function of the bond's coupon.** The debt is a coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

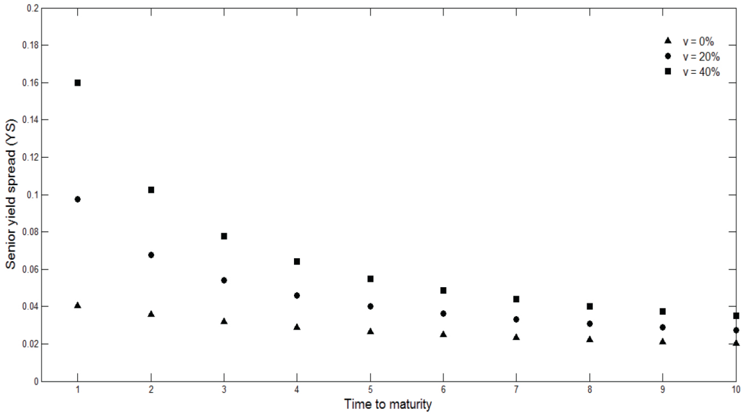


**Figure 9: Term structure of senior yield spreads.** The senior debt is acoupon bond with a maturity of 10 years, a principal amount of \$70; and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30; and an annual coupon rate of 10%. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

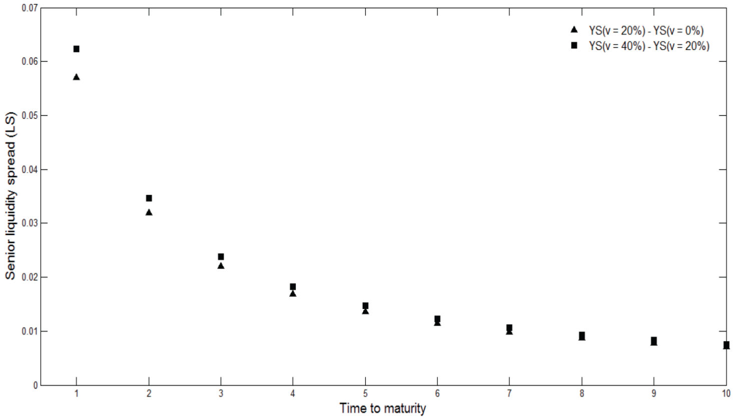


**Figure 10: Term structure of senior liquidity spreads.** The senior debt is a coupon bond with a maturity of 10 years, a principal amount of \$70; and an annual coupon rate of 7%. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30; and an annual coupon rate of 10%. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0, \upsilon \in \{0\%, 20\%, 40\%\},$ and $\omega = 25\%$.

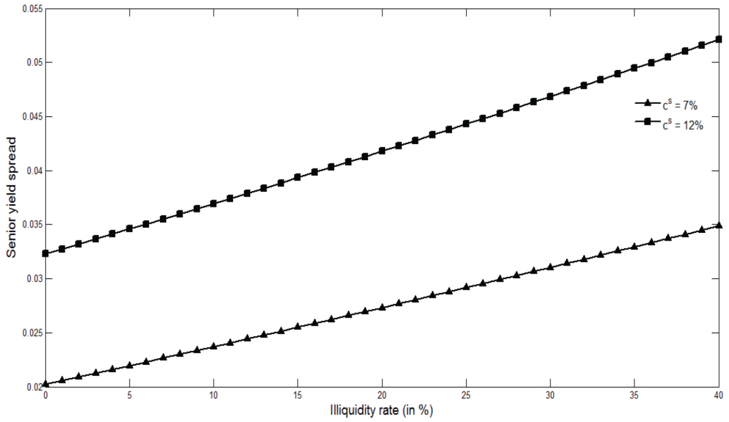
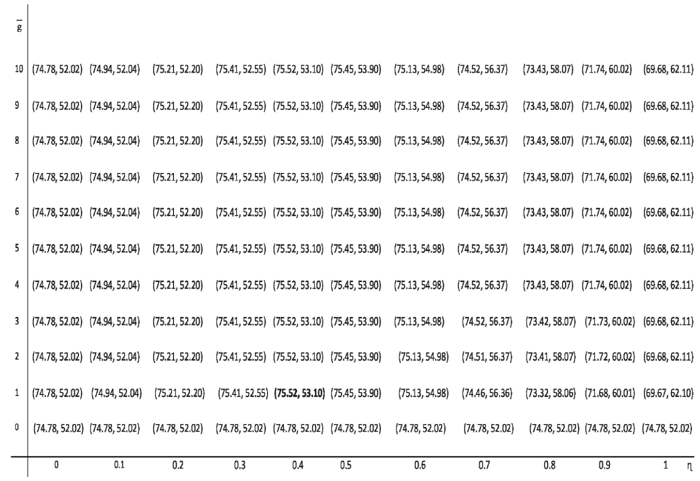
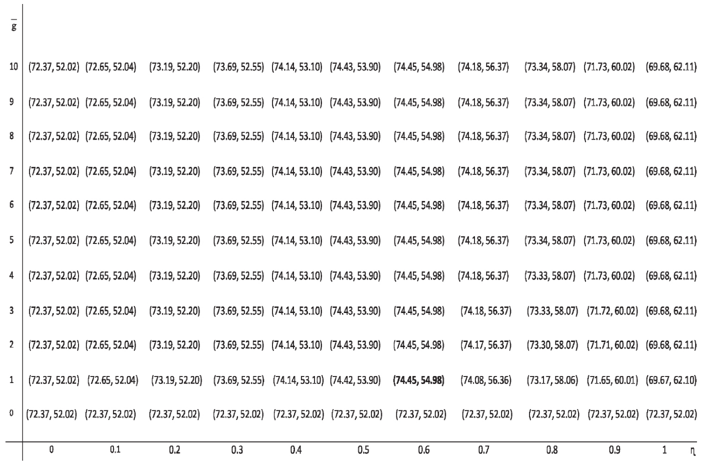


**Figure 11: Senior yield spread as a function of the illiquidity rate.** The senior debt is a coupon bond with a maturity of 10 years, a principal amount of \$70; and an annual coupon rate $c^s \in 7\%, 12\%$. The junior debt is a coupon bond with a maturity of 10 years, a principal amount of \$30; and an annual coupon rate of 10%. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, \eta = 0,$ and $\omega = 25\%$.



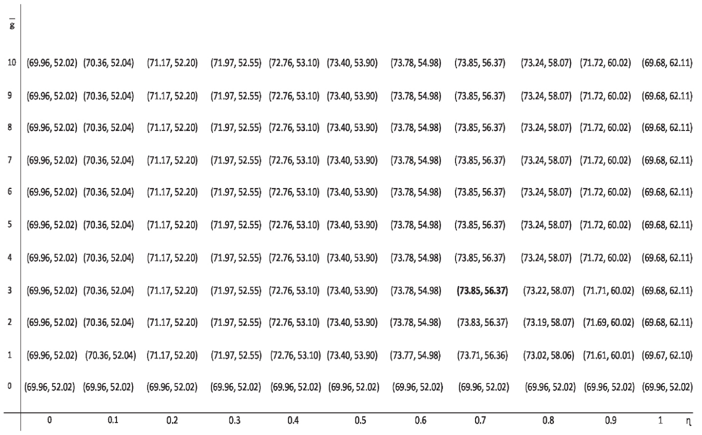
| $\overline{g}$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 9 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 8 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 7 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 6 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 5 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 4 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.43, 58.07) | (71.74, 60.02) | (69.68, 62.11) |
| 3 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.52, 56.37) | (73.42, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 2 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | (75.52, 53.10) | (75.45, 53.90) | (75.13, 54.98) | (74.51, 56.37) | (73.41, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 1 | (74.78, 52.02) | (74.94, 52.04) | (75.21, 52.20) | (75.41, 52.55) | **(75.52, 53.10)** | (75.45, 53.90) | (75.13, 54.98) | (74.46, 56.36) | (73.32, 58.06) | (71.68, 60.01) | (69.67, 62.10) |
| 0 | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) | (74.78, 52.02) |
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 $\eta$ |

**Figure 12: Debt and equity as functions of $\eta$ and $\overline{g}$.** The debt is a 5% coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, u = 5\%, \upsilon = 0\%$ and $\omega = 25\%$.



| $\overline{g}$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 9 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 8 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 7 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 6 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 5 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.34, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 4 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.33, 58.07) | (71.73, 60.02) | (69.68, 62.11) |
| 3 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.18, 56.37) | (73.33, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 2 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.43, 53.90) | (74.45, 54.98) | (74.17, 56.37) | (73.30, 58.07) | (71.71, 60.02) | (69.68, 62.11) |
| 1 | (72.37, 52.02) | (72.65, 52.04) | (73.19, 52.20) | (73.69, 52.55) | (74.14, 53.10) | (74.42, 53.90) | **(74.45, 54.98)** | (74.08, 56.36) | (73.17, 58.06) | (71.65, 60.01) | (69.67, 62.10) |
| 0 | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) | (72.37, 52.02) |
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 $\eta$ |

**Figure 13: Debt and equity as functions of $\eta$ and $\overline{g}$.** The debt is a 5% coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, u = 5\%, \upsilon = 20\%$ and $\omega = 25\%$.



| $\bar{g}$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 9 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 8 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 7 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 6 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 5 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 4 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.85, 56.37) | (73.24, 58.07) | (71.72, 60.02) | (69.68, 62.11) |
| 3 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | **(73.85, 56.37)** | (73.22, 58.07) | (71.71, 60.02) | (69.68, 62.11) |
| 2 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.78, 54.98) | (73.83, 56.37) | (73.19, 58.07) | (71.69, 60.02) | (69.68, 62.11) |
| 1 | (69.96, 52.02) | (70.36, 52.04) | (71.17, 52.20) | (71.97, 52.55) | (72.76, 53.10) | (73.40, 53.90) | (73.77, 54.98) | (73.71, 56.36) | (73.02, 58.06) | (71.61, 60.01) | (69.67, 62.10) |
| 0 | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) | (69.96, 52.02) |
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 η |

**Figure 14: Debt and equity as functions of $\eta$ and $\overline{g}$.** The debt is a 5% coupon bond with a maturity of 10 years and a principal amount of \$100. Set $A_0 = \$120, \sigma = 30\%, r = 6\%, r^c = 35\%, u = 5\%, \upsilon = 40\%$ and $\omega = 25\%$.

# 4 Conclusion

The structural model of Merton (1974) can be extended to accommodate several intangible assets, including illiquidity costs. These are modeled in a natural way as a proportional reduction in the firm's asset value under liquidation. This study is the first to propose a theoretical setting of a firm's assets illiquidity.

Our numerical investigation highlights the sensitivity of the values of corporate securities with respect to the bond's coupon rate, time to maturity, and the illiquidity rate, under reasonable scenarios. Consistently to the literature, we find that more liquid bonds show lower yield spreads (Figure 3 and Figure 9) and that longer bonds' horizons display lower liquidity spreads (Figures 10–11). We also show that higher-yield bonds exhibit larger liquidity spreads (Figure 3) and that riskier junior bonds are less sensitive to illiquidity. Finally, we report that reorganization is still effective in the presence of illiquidity costs (Figures 12–14).

A future research avenue consists of estimating this extended structural model and performing numerical investigations on selected public companies.

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