# Strongly Monotone Variational Inequalities with Constraints Given by a Separation Oracle

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**Abstract**

This paper considers a case of strongly monotone variational inequality problems defined over a convex set which is given by a "separation oracle". An analytic center based algorithm that uses a mixture of linear and quadratic cuts is devised to solve this problem and its complexity is established.

**Key words:** Variational inequalities, Strong monotonicity, Interior-point methods, Analytic center.

**Résumé**

Nous considérons dans cet article un cas d'inéquations variationnelles forcément monotone définies sur un ensemble convexe donné par un oracle de séparation. Nous développons un algorithme basé sur le centre analytique et utilisont un mélange de coupe linéaires et quadratiques pour résoudre ce problème, et nous établissons la complexité de cet algorithme.

# 1 Introduction

In this paper a strongly monotone variational inequality problem with a feasible set defined implicitly by a " separation oracle" is considered. An analytic center based interior point algorithm for solving this problem will be described. The algorithm is applied to the formulation of the problem as a feasibility problem, with linear constraints corresponding to the feasible set of the original problem and strongly convex quadratic constraints corresponding to the operator. The algorithm starts with a unit ball, that is assumed to contain the feasible set, and adds the constraints as they are needed at the "approximate analytic center" of the current "outer approximation" of the (reformulated) feasible set. The complexity of the algorithm will also be established.

There have been several analytic center based cutting plane (or surface) algorithms for solving variational inequality problems. These include the homogeneous analytic center cutting plane method (HACCPM) of Nesterov and Vial [6], the analytic center cutting plane method of Goffin, Marcotte, and Zhu [1] and the analytic center quadratic cut method of Lüthi and Büeler [3, 4]. In all these methods the feasible set is assumed to be defined by finitely many convex or linear constraints. However, in the problem considered in this paper, the feasible set is implicitly defined by a separation oracle.

The following standard notations will be used throughout the paper. For any generic vector $u$, $u^T$ will denote the vector transpose. The interior of a convex set $\Omega$ will be denoted by $\Omega_{int}$. For two symmetric $m \times m$ matrices $A$ and $B$, $A \succeq B$ means that the matrix $A - B$ is positive semidefinite. The following notation will also be used:

$$\begin{aligned} \|u\|_H &= \langle u, Hu \rangle^{1/2}, \\ \|u\|_H^* &= \langle u, H^{-1}u \rangle^{1/2}, \end{aligned}$$

where $H$ is a given symmetric positive definite matrix and $\langle \cdot, \cdot \rangle$ indicates the standard Euclidean inner product.

# 2 Problem Formulation and Basic Definitions

Consider the following variational inequality problem:

$$VI(\Gamma, T): \quad \text{find } y^* \in \Gamma \;\; \text{such that } \langle T(y^*), y - y^* \rangle \geq 0 \;\; \forall y \in \Gamma,$$

where $\Gamma$ is a closed convex set in $\mathsf{R}^m$, with a nonempty interior, and is contained in the unit ball $\Omega^0 = \{y \in \mathsf{R}^m : \; \|y\| \leq 1\}$. Furthermore, operator $T : \Omega^0 \to \mathsf{R}^m$ is a strongly monotone continuous operator over $\Omega^0$, that is for all $x$, $y \in \Omega^0$ and for some $\alpha > 0$

$$\langle T(x) - T(y), x - y \rangle \geq \alpha \|x - y\|^2.$$

Let $M = \max\{\|T(y)\| \; : \; y \in \Omega^0\}$. It will be assumed, without loss of generality, that $\max\{M, \alpha\} \leq 1$. The set $\Gamma$ is defined implicitly by a "separation oracle" which for every $\bar{y}$ in $\Omega^0$ either answers that $\bar{y}$ belongs to $\Gamma$ or generates a cut $a^T y \leq a^T \bar{y}$ with the property that $\{y \in \mathsf{R}^m : a^T y \leq a^T \bar{y}\} \supset \Gamma$. It will be assumed, without loss of generality, that

$\|a\| = 1$. The set $\Gamma$ can be, for example, defined by a system (finite or infinite) of linear, convex (possibly nonsmooth) or pseudoconvex inequalities or by a system in which some constraints are not explicitly known. The following theorem will be used in the formulation of $VI(\Gamma, T)$ as a feasibility problem.

**Theorem 2.1** *The point $y^* \in \Gamma$ is a solution of $VI(\Gamma, T)$ if and only if*

$$\langle T(y), y^* - y\rangle + \alpha\|y^* - y\|^2 \leq 0 \qquad \forall y \in \Gamma. \tag{1}$$

**Proof.** It follows from strong monotonicity of the operator $T$ that

$$\langle T(y), y^* - y\rangle + \alpha\|y^* - y\|^2 \leq \langle T(y^*), y^* - y\rangle.$$

Hence if $y^*$ is a solution of $VI(\Gamma, T)$, then (1) is satisfied. On the other hand, if $y^*$ satisfies (1) then it also satisfies

$$\langle T(y), y^* - y\rangle \leq 0, \qquad \forall y \in \Gamma$$

which means that $y^*$ is a "weak" solution of $VI(\Gamma, T)$. It then follows from the continuity of $T$ that $y^*$ is a solution of $VI(\Gamma, T)$. □

Therefore, $VI(\Gamma, T)$ can be formulated as the following problem:

$$FP(\Gamma, T): \quad \text{find } y^* \in \Gamma \text{ such that } \langle T(y), y^* - y\rangle + \alpha\|y^* - y\|^2 \leq 0 \;\; \forall y \in \Gamma.$$

Now, let

$$g_p(\Gamma, y) = \max_{x \in \Gamma}\langle T(y), y - x\rangle$$

It is well known that since $T$ is a continuous monotone operator, $y^*$ is a solution of $VI(\Gamma, T)$ if and only if

$$g_p(\Gamma, y^*) = 0.$$

**Definition 2.1** *Suppose $y^*$ is the unique solution of $VI(\Gamma, T)$. A point $\hat{y} \in \Gamma$ is called an $\epsilon$-solution of $VI(\Gamma, T)$ if $\|\hat{y} - y^*\|^2 \leq \epsilon$.*

Suppose that $y^*$ is a solution of $VI(\Gamma, T)$ and $\hat{y} \in \Gamma$. Then

$$\langle T(y^*), \hat{y} - y^*\rangle \geq 0$$

Furthermore, due to strong monotonicity of $T$,

$$\begin{aligned} \alpha\|\hat{y} - y^*\|^2 &\leq \langle T(\hat{y}) - T(y^*), \hat{y} - y^*\rangle \\ &\leq \langle T(\hat{y}), \hat{y} - y^*\rangle \\ &\leq g_p(\Gamma, \hat{y}). \end{aligned}$$

Hence, if $g_p(\Gamma, \hat{y}) \leq \alpha\epsilon$ then $\|\hat{y} - y^*\|^2 \leq \epsilon$. This will be used as the stopping rule in the algorithm.

## 3 Analytic Center

Let $\Omega$ be a bounded set in $\mathsf{R}^m$ defined by $n$ linear or convex quadratic inequalities as follows:

$$\Omega = \{y \in \mathsf{R}^m \ : \ f_j(y) \leq 0, \ \ j = 1, \ldots, n\} \tag{2}$$

For $j = 1, \ldots, n$, the potential function of $\Omega$ at $y$ is

$$\phi(\Omega, y) = -\sum_{j=1}^{n} \ln(-f_j(y))$$

The min potential of $\Omega$ is

$$P(\Omega) = \min_{y \in \Omega_{int}} \phi(\Omega, y)$$

The "analytic center" of $\Omega$ is the unique point at which the above minimum is attained. The gradient and Hessian of $\phi(\Omega, y)$ are given by

$$g(\Omega, y) = \phi'(\Omega, y) = \sum_{j=1}^{n} \frac{f_j'(y)}{-f_j(y)}$$

and

$$H(\Omega, y) = \phi''(\Omega, y) = \sum_{j=1}^{n} \left( \frac{f_j'(y)(f_j'(y))^T}{(f_j(y))^2} + \frac{f_j''(y)}{-f_j(y)} \right)$$

The point $y$ is called a "$\theta$-approximate analytic center" or "$\theta$-center" of $\Omega$, if

$$\|g(\Omega, y)\|^*_{H(\Omega, y)} \leq \theta.$$

## 4 Algorithm and Its Convergence

The algorithm devised to solve $FP(\Gamma, T)$ and its convergence will be studied in this section. The notation used in the algorithm is as follows: In the $k$-th iteration, the set $\Gamma^k$ indicates the outer approximation of $\Gamma$, which is updated by feasibility cuts only; the set $\Omega^k$ indicates the current set of localization which is updated both by operator and feasibility cuts; $k_q$ indicates the total number of strongly convex quadratic cuts (operator cuts) added and $k_l$ indicates the total number of linear cuts (feasibility cuts) added.

**Algorithm**

**0.** (Initialization) Let $\Omega^0 = \{y \in \mathsf{R}^m : \|y\| \leq 1\}$.
Set $y^0 = 0$; $\Gamma^0 = \Omega^0$; $k = k_l = k_q = 0$; determine $0 < \theta < (\sqrt{2} - 1)^2$.

**1.** ($k$-th iteration) Find the $\theta$-center of $\Omega^k$, $y^k$.

**2.** If $y^k \in \Gamma$ then
   If $g_p(\Gamma^k, y^k) \leq \alpha\epsilon$ stop
   Else (add an operator cut through $y^k$ as follows:)
      Update $\Omega^{k+1} = \Omega^k \cap \{y : \langle T(y^k), y - y^k \rangle + \alpha\|y^k - y\|^2 \leq 0\}$;

$\Gamma^{k+1} = \Gamma^k$;
$k_q = k_q + 1$; $k = k + 1$ and go to Step 1.
Else (add a feasibility cut through $y^k$ as follows:)
Update $\Gamma^{k+1} = \Gamma^k \cap \{y : a_{k+1}^T y \leq a_{k+1}^T y^k\}$;
$\Omega^{k+1} = \Omega^k \cap \{y : a_{k+1}^T y \leq a_{k+1}^T y^k\}$;
$k_l = k_l + 1$; $k = k + 1$ and go to Step 1.

Note that since $\Gamma \subset \Gamma^k$, then $g_p(\Gamma^k, y^k) \leq \alpha\epsilon$ implies that $g_p(\Gamma, y^k) \leq \alpha\epsilon$ which means that $y^k$ is an $\epsilon$-solution of $VI(\Gamma, T)$.

Before studying the overall convergence, one first needs to consider an updating direction to move from a $\theta$-center of $\Omega^k$ towards an interior point of $\Omega^{k+1}$ and from there to recenter towards a $\theta$-center of $\Omega^{k+1}$.

In case a quadratic cut is added, the same direction as given by Sharifi-Mokhtarian and Goffin [8] is used. This direction is given in the following theorem which is a restatement of [8, Theorem 3.1, Theorem 3.2].

**Theorem 4.1** *Suppose $\Omega$, as defined by (2), is changed to $\Omega^+$ as follows:*

$$\Omega^+ = \{f_j(y) \leq 0, \;\; j = 1, \dots, n, \quad h_{n+1}(y) \leq h_{n+1}(y^a)\}$$

*where $y^a$ is a $\theta$-center of $\Omega$ with $0 < \theta < 1$ and*

$$h_{n+1}(y) = -c_{n+1} + \langle a_{n+1}, y\rangle + \frac{1}{2}\langle y, Q_{n+1} y\rangle$$

*with $Q_{n+1} \succeq 0$. Let (Updating step)*

$$y^u = y^a - \frac{\beta}{\bar{r}} \bar{H}^{-1} a^*$$

*where $0 < \beta < 1$ and*

$$\begin{aligned} a^* &= a_{n+1} + Q_{n+1} y^a \\ \bar{H} &= H(\Omega, y^a) + \frac{Q_{n+1}}{\bar{r}} \\ \chi(\bar{r}) &= \langle a^*, \bar{H}^{-1} a^* \rangle^{\frac{1}{2}} \end{aligned}$$

*and $0 < \bar{r} < 1$ satisfies*

$$\frac{1}{\sqrt{2}} \bar{r} \leq \chi(\bar{r}) \leq \bar{r} \; .$$

*Then $y^u \in \Omega^+_{int}$ ,*

$$P(\Omega^+) \leq \phi(\Omega^+, y^u) \leq \phi(\Omega, y^a) - \ln \bar{r} + \alpha_1(\beta, \theta)$$

*where $\alpha_1(\beta, \theta) = -\ln(1-\beta) - \ln\left[\beta\left(1 - \frac{\beta}{2}\right)\right] - (1-\theta)\beta$, and*

$$P(\Omega^+) \geq \phi(\Omega, y^a) - \ln \bar{r} + \alpha_2(\theta) \tag{3}$$

*where*

$$\alpha_2(\theta) = \min_{v>0}\{(1-\theta)v - \ln(1+v) - \ln(1+\sqrt{1+v^2})\}. \tag{4}$$

□

In case of the problem studied in this paper, the strongly convex quadratic cut generated by the algorithm at, say the $n$-th iteration, is given by $Q_{n+1} = 2\alpha I$, $a_{n+1} = T(y^a)$ and $\Omega = \Omega^n$.

Furthermore, as is discussed in [8, Theorem 6.1], $\chi(\bar{r})$ satisfies

$$(\chi(\bar{r}))^2 = \sum_{i=1}^{m} \frac{\tilde{a}_i^2 \bar{r}}{\bar{r} + \lambda_i}$$

where $\tilde{a}_i$ is the $i$-th component of $\tilde{a} = L^{-1}a^*$, with $L$ being a lower triangular matrix such that $H(\Omega, y^a) = LL^T$, and $\lambda_i$ is the $i$-th eigenvalue of $L^{-1}Q_{n+1}L^{-T}$, with $i = 1, 2, \ldots, m$. It can be shown (see [8, Corollary 6.1] that finding $\bar{r}$ is like finding a root of

$$\psi(\bar{r}) = \sum_{i=1}^{m} \frac{\tilde{a}_i^2}{\bar{r} + \lambda_i} - \frac{3\bar{r}}{4}$$

in $(0, 1)$ with accuracy $\frac{\bar{r}}{4}$. The hybrid algorithm of Ye [9] can be used here to find a suitable $\bar{r}$. This hybrid algorithm combines Newton's method and a binary search.

**Remark.** It can be seen using the above (see [8, Theorem 5.1, Corollary 6.1]) that the number of arithmetic operations required to find a suitable $\bar{r}$, in the $k$-th iteration of the algorithm, is at most

$$O\left(\log_2\left(\log_2\left(\frac{(3k+1-3\theta)^2 + 4(1-3\theta)^2}{\epsilon^4(1-3\theta)^2}\right)\right)\right)$$

and the arithmetic complexity of centering towards a $\theta$-center of $\Omega^{k+1}$, in the $k$-th iteration, is at most $O(1)$.

□

The following result will be used in the proof of the overall convergence of the algorithm.

**Lemma 4.1** *Suppose that $k_q$ and $k_l$ indicate the number of the quadratic and linear cuts (respectively) added by the $k$-th iteration of the algorithm. Also, suppose that at the $k$-th iteration of the algorithm, a quadratic cut is added. Let*

$$(\chi(\bar{r}_{k_q-1}))^2 = (a_{k_q}^*)^T (\bar{H}^k)^{-1} a_{k_q}^*$$

*where $\bar{H}^k = H^k + \frac{2\alpha I}{\bar{r}_{k_q-1}}$ with $H^k = H^k(\Omega^k, y^k)$, $a_{k_q}^* = a_{k_q} + 2\alpha y^k$, and $0 < \bar{r}_{k_q-1} < 1$ satisfies*

$$\frac{1}{\sqrt{2}} \bar{r}_{k_q-1} \leq \chi(\bar{r}_{k_q-1}) \leq \bar{r}_{k_q-1}\,. \tag{5}$$

*Then*

$$\sum_{j=0}^{k_q-1} \bar{r}_j^2 \leq \frac{18}{\alpha}(1 + \ln k_q)\,. \tag{6}$$

**Proof.** Note that

$$\begin{aligned}
\bar{H}^k &\succeq H^k \\
&= H^0(\Omega^0, y^k) + \sum_{j=1}^{k_q} \left[ \frac{(T(y^j) + 2\alpha y^k)(T(y^j) + 2\alpha y^k)^T}{(s_j^q(y^k))^2} + \frac{2\alpha I}{s_j^q(y^k)} \right] + \sum_{j=0}^{k_l - 1} \frac{a_{j+1} a_{j+1}^T}{(s_j^l(y^k))^2} \\
&\succeq 2I + \sum_{j=1}^{k_q} \frac{2\alpha I}{s_j^q(y^k)} \qquad (\text{Note } H^0 \succeq 2I)\,,
\end{aligned}$$

where $s_j^q(y^k)$ and $s_j^l(y^k)$ are the slacks of the $j$-th quadratic and $j$-th linear cuts respectively. Furthermore,

$$\begin{aligned}
s_j^q(y^k) &= -\langle T(y^j), y^k - y^j \rangle - \alpha \|y^k - y^j\|^2 \\
&\leq \|T(y^j)\| \|y^k - y^j\| \\
&\leq \|y^k - y^j\| \leq 2
\end{aligned}$$

Thus, since $\alpha \leq 1$,

$$\bar{H}^k \succeq H^k \succeq 2I + k_q \alpha I \succeq \alpha(k_q + 1) I\,.$$

Now in the $J$-th iteration ($1 \leq J \leq k$) let $j+1$ (with $0 \leq j \leq k_q$) indicate the number of quadratic cuts added. Also note that $\|y^j\| \leq 1$, $\|a_{j+1}\| \leq 1$. This means that $\|a_{j+1}^*\| \leq 3$. Hence

$$(\bar{\chi}(\bar{r}_j))^2 = (a_{j+1}^*)^T (\bar{H}^J)^{-1} a_{j+1}^* \leq \frac{\|a_{j+1}^*\|^2}{\alpha(j+1)} \leq \frac{9}{\alpha(j+1)}$$

On the other hand, it follows from the definition of $\bar{r}_j$ that $\bar{r}_j^2 \leq 2(\chi(\bar{r}_j))^2$. Hence

$$\bar{r}_j^2 \leq \frac{18}{\alpha(j+1)}$$

Finally,

$$\begin{aligned}
\sum_{j=0}^{k_q - 1} r_j^2 &\leq \sum_{j=0}^{k_q - 1} \frac{18}{\alpha(j+1)} \\
&\leq \frac{18}{\alpha} \sum_{j=0}^{k_q - 1} \frac{1}{j+1} \\
&\leq \frac{18}{\alpha}(1 + \ln k_q)
\end{aligned}$$

where the last inequality follows from the fact that $\sum_{j=0}^{k_q-1} \frac{1}{j+1} < 1 + \ln k_q$ .

□

In case a linear cut is added the following result due to Nesterov [5, Lemma 2.1] will be used.

**Theorem 4.2** *Let $y^a$ be a $\theta$-center of $\Omega$, with $0 \le \theta < (\sqrt{2}-1)^2$. Consider*

$$\Omega^+ = \{y : f_i(y) \le 0,\ i = 1, \ldots, n,\ (a_{n+1})^T y \le (a_{n+1})^T y^a\}$$

*then (updating step)*

$$y^+ = y^a - \frac{[H(\Omega, y^a)]^{-1} a_{n+1}}{3\|a_{n+1}\|^*_{H(\Omega,y^a)}} \in (\Omega^+)_{int}\,;$$

$$\begin{aligned} \phi(\Omega, y^a) - \ln \|a_{n+1}\|^*_{H(\Omega,y^a)} + c(\theta) &\le P(\Omega^+) \le \phi(\Omega^+, y^+) \\ &\le \phi(\Omega, y^a) - \ln \|a_{n+1}\|^*_{H(\Omega,y^a)} + \bar{c}(\theta) \end{aligned}$$

*where*

$$c(\theta) = 2 - \sqrt{2} + \frac{\ln 2}{2} + \ln((\sqrt{2}-1)^2 - \theta) \tag{7}$$

$$\bar{c}(\theta) = 2\ln 3 - \ln 2 - \frac{1-\theta}{3}\,.$$

**Remark.** It follows from the above that the total number of (damped) Newton steps to update to a new $\theta$-center after addition of a linear cut, is at most $O(1)$ (depending on $\theta$ only). Furthermore,

$$P(\Omega^+) \ge P(\Omega) - \ln \|a_{n+1}\|^*_{H(\Omega,y^c)} + c(\theta) \tag{8}$$

□

The following lemma adapts a result of Nesterov [5, Theorem 3.1] to the problem studied in this paper and will be used in the proof of the overall convergence of the algorithm.

**Lemma 4.2** *Suppose that $k_l$ and $k_q$ indicate the number of the linear and quadratic cuts (respectively) added by the $k$-th iteration of the algorithm. Let $r_{k_{l-1}} = \|a_{k_l}\|^*_{H(\Omega^k,y^k)}$. Then*

$$\sum_{j=0}^{k_l-1} r_j^2 \le \frac{m}{2\ln(9/8)} \ln\left(1 + \frac{k_l}{8m}\right)\,. \tag{9}$$

**Proof.** Let $B^0 = 2I$; $B^{j+1} = B^j + \frac{1}{4} a_{j+1} a_{j+1}^T$ for $0 \le j \le k_l - 1$. Note that since $\Gamma$ is contained in the unit ball, the $j$-th slack corresponding to a feasibility cut satisfies :

$$s_j(y) = a_j^T (y - y^j) \le \|a_j\| \|y - y^j\| \le 2$$

This implies that

$$\begin{aligned} H(\Omega^k, y^k) &= H^0(\Omega^0, y^k) + \sum_{j=1}^{k_q} \left( \frac{f_j'(y)(f_j'(y))^T}{(f_j(y))^2} + \frac{f_j''(y)}{-f_j(y)} \right) + \sum_{j=0}^{k_l-1} \frac{a_{j+1} a_{j+1}^T}{(s_j(y^k))^2} \\ &\quad \text{(with } f_j(y) \le 0 \text{ indicating the } j\text{-th quadratic cut)} \\ &\succeq 2I + \sum_{j=0}^{k_l-1} \frac{a_{j+1} a_{j+1}^T}{4} \qquad (\text{since } H^0 \succeq 2I) \\ &= B^{k_l} \end{aligned}$$

Now in the $J$-th iteration $(1 \leq J \leq k)$ let $j+1$ (with $0 \leq j \leq k_l$) indicate the number of linear cuts added. Also, for $0 \leq j \leq k_l - 1$, let

$$w_j = a_{j+1}^T (B^j)^{-1} a_{j+1}$$

Then $r_j^2 \leq w_j$ and since $B^j \succeq 2I$, $w_j \leq 1/2$. Also

$$\det B^{j+1} = (\det B^j)(1 + \frac{w_j}{4})$$

Therefore,

$$\begin{aligned} \ln(\det B^{j+1}) &= \ln(\det B^j) + \ln(1 + \frac{w_j}{4}) \\ &\geq \ln(\det B^j) + 2w_j \ln(9/8) \end{aligned}$$

(the last inequality follows from the inequality $\ln(1 + \alpha_1 \alpha_2) \geq \alpha_1 \ln(1 + \alpha_2)$ valid for $\alpha_1 \in [0,1]$ and $\alpha_2 \geq 0$. In this case $\alpha_1 = 2w_j$ and $\alpha_2 = \frac{1}{8}$). Finally,

$$[\det B^{j+1}]^{1/m} \leq \frac{\text{Trace } B^{j+1}}{m} \leq 2 + \frac{j+1}{4m}.$$

Hence

$$\begin{aligned} 2\ln(9/8) \sum_{j=0}^{k_l - 1} w_j &\leq m \ln\left(2 + \frac{k_l}{4m}\right) - \ln(\det B^0) \\ &\leq m \ln\left(2 + \frac{k_l}{4m}\right) - m \ln 2 \\ &= m \ln\left(1 + \frac{k_l}{8m}\right). \end{aligned}$$

□

The convergence of the algorithm can now be established.

**Theorem 4.3** *The algorithm finds an $\epsilon$-solution as soon as the following inequality is violated:*

$$\rho^4 \leq 4 \left[\frac{18(1 + \ln k_q)}{\alpha k_q}\right]^{\frac{k_q}{k+1}} \left[\frac{\frac{m}{\ln(9/8)} \ln(1 + \frac{k_l}{8m})}{k_l}\right]^{\frac{k_l}{k+1}} e^{-2\zeta(\theta)\frac{k}{k+1}} \tag{10}$$

*Here*

$$\zeta(\theta) = \min\{\alpha_2(\theta), c(\theta)\}$$

*with $\alpha_2$ and $c$ as given by (4), (7), and $\rho$ is the radius of the largest ball inscribed in $\Gamma \cap B(y^*, \sqrt{\epsilon})$ (with $B(y^*, \sqrt{\epsilon})$ being the ball centered at $y^*$ with radius $\sqrt{\epsilon}$). Furthermore, $k_l$ and $k_q$ indicate the number of the linear and quadratic cuts (respectively) added by the $k$-th iteration of the algorithm.*

**Proof.** Since a ball of radius $0 < \rho < 1$ can be inscribed in $\Gamma$, it follows that (see [2, Lemma 5.1])

$$(k+1)\ln\left(\frac{2}{\rho^2}\right) \geq P(\Omega^k)\,.$$

Hence, by (3), (8),

$$(k+1)\ln\left(\frac{2}{\rho^2}\right) \quad \geq P(\Omega^k \geq \quad P(\Omega^0) - \frac{1}{2}\left(\sum_{j=0}^{k_q-1}\ln(\bar{r}_j^2) + \sum_{j=0}^{k_l-1}\ln(r_j^2)\right) + k\zeta(\theta)$$

Note that $P(\Omega^0) = 0$. So,

$$\begin{aligned}
\ln\left(\frac{\rho^2}{2}\right) + \frac{k}{k+1}\zeta(\theta) \quad &\leq \quad \frac{1}{2(k+1)}\left(\sum_{j=0}^{k_q-1}\ln(\bar{r}_j^2) + \sum_{j=0}^{k_l-1}\ln(r_j^2)\right) \\
&\leq \quad \frac{1}{2(k+1)}\left(k_q\ln\left(\frac{\sum_{j=0}^{k_q-1}\bar{r}_j^2}{k_q}\right) + k_l\ln\left(\frac{\sum_{j=0}^{k_l-1}r_j^2}{k_l}\right)\right) \\
&\qquad \text{due to concavity of the ln function} \\
&\leq \quad \frac{k_q}{2(k+1)}\ln\left[\frac{18(1+\ln k_q)}{\alpha k_q}\right] + \frac{k_l}{2(k+1)}\ln\left[\frac{\frac{m}{\ln(9/8)}\ln(1+\frac{k_l}{8m})}{k_l}\right]
\end{aligned}$$

where the last inequality follows from Lemma 4.1(6) and Lemma 4.2(9). This implies that

$$\frac{\rho^4}{4}e^{2\zeta(\theta)\frac{k}{k+1}} \leq \left[\frac{18(1+\ln k_q)}{\alpha k_q}\right]^{\frac{k_q}{k+1}}\left[\frac{\frac{m}{\ln(9/8)}\ln(1+\frac{k_l}{8m})}{k_l}\right]^{\frac{k_l}{k+1}}$$

□

**Remark.** Note that if $k_l$ or $k_q$ is large then the above inequality (or inequality (10)) is violated.

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