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Tight-and-cheap conic relaxation for the optimal reactive power dispatch problem

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Abstract: The optimal reactive power dispatch (ORPD) problem is an alternating current optimal power flow (ACOPF) problem where discrete control devices for regulating the reactive power, such as shunt elements and tap changers, are considered. The ORPD problem is modelled as a mixed-integer nonlinear optimization problem and its complexity is increased compared to the ACOPF problem, which is highly nonconvex and generally hard to solve. Recently, convex relaxations of the ACOPF problem have attracted a significant interest since they can lead to global optimality. We propose a tight conic relaxation of the ORPD problem and computational results show that a round-off technique applied with this relaxation instead of the continuous relaxation of the ORPD problem leads to near-global optimal solutions with very small guaranteed optimality gaps on selected MATPOWER test cases with up to 3375 buses.

Keywords: Conic optimization, discrete variables, optimal power flow, power systems, semidefinite programming.

Résumé : Le problème d'écoulement de puissance optimal est un problème d'optimisation non convexe et très difficile à résoudre. Plusieurs relaxations convexes développées récemment, entre autres la relaxation semidéfinie et la relaxation conique du second ordre, ont eu un intérêt significatif dans la communauté scientifique. Parmi toutes ces relaxations, la relaxation semidéfinie est la plus forte et s'est avérée exacte pour plusieurs instances IEEE. Toutefois, sa résolution pour de réseaux de grande taille est trop coûteuse comparativement à celle de la relaxation conique du second ordre. Nous proposons une nouvelle relaxation conique plus forte que la relaxation conique du second ordre et presque aussi forte que la relaxation semidéfinie. Pour des instances ayant jusqu'à 6515 nœuds, la résolution de la nouvelle relaxation est beaucoup plus rapide que celle de la relaxation semidéfinie.

Mots clés : Optimisation conique, écoulement de puissance optimal, réseaux électriques, optimisation semidéfinie

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Nomenclature

Notation	
$\mathbb{R}/\mathbb{C}$	Set of real/complex numbers,
j	Imaginary unit,
a/a	Real/complex number,
$\mathbf{a}/\mathbf{a}$	Real/complex vector,
A/A	Real/complex matrix.
Operators	
$\text{Re}(\cdot)/\text{Im}(\cdot)$	Real/imaginary part operator,
$(\cdot)^*$	Conjugate operator,
$ \cdot $	Magnitude or cardinality set operator,
$\angle(\cdot)$	Phase operator,
$(\cdot)^H$	Conjugate transpose operator,
$\text{rank}(\cdot)$	Rank operator.
Input data	
$\mathcal{P} = (\mathcal{N}, \mathcal{L})$	Power network,
$\mathcal{N}$	Set of buses,
$\mathcal{U} \subseteq \mathcal{N}$	Set of buses k where a shunt element is connected,
$\mathcal{G} = \bigcup_{k \in \mathcal{N}} \mathcal{G}_k$	Set of generators,
$\mathcal{G}_k$	Set of generators connected to bus k ,
$\mathcal{L}$	Set of branches,
$\mathcal{T} \subseteq \mathcal{L}$	Set of branches with tap changers,
p_{Dk}/q_{Dk}	Active/reactive power demand at bus k ,
g'_k/b'_k	Conductance/susceptance of shunt element at bus k ,
$y_\ell^{-1} = r_\ell + jx_\ell$	Series impedance of branch ℓ ,
b'_ℓ	Total shunt susceptance of branch ℓ .
Variables	
p_{Gg}/q_{Gg}	Active/reactive power generation by generator g ,
v_k	Complex (phasor) voltage at bus k ,
u_k	Shunt variable corresponding to the shunt element connected at bus k ,
$p_{f\ell}/q_{f\ell}$	Active/reactive power flow injected along branch ℓ by its <i>from</i> end,
$p_{t\ell}/q_{t\ell}$	Active/reactive power flow injected along branch ℓ by its <i>to</i> end,
t_ℓ	Turns ratio of tap changer ℓ .

1 Introduction

The optimal power flow (OPF) problem, first formulated in [1], consists in finding a network operating point that optimizes an objective function subject to power flow equations and other operational constraints [2, 3, 4, 5]. The continuous version with AC power flow equations, also called AC optimal power flow (ACOPF) problem, is nonconvex and NP-hard [6, 7]. The optimal reactive power dispatch (ORPD) problem, also known as Volt/VAR optimization problem, can be seen as an ACOPF problem with discrete control devices for regulating the reactive power such as shunt elements and tap changers [8, 9]. Because of the presence of discrete variables, the ORPD problem is generally more difficult than the ACOPF.

A mixed-integer nonlinear program (MINLP) is an optimization problem which involves both continuous and integer variables and whose objective function and feasible set are described by nonlinear functions [10]. A MINLP is said to be *convex* if its continuous relaxation, i.e. the problem obtained by dropping the integrity constraints, is a convex optimization problem; otherwise, it is said to be *nonconvex*. MINLPs inherit difficulties from nonlinear programs (NLPs) and mixed-integer linear programs (MILPs) since they are a generalization of both classes. In fact, there exist simple cases of nonconvex MINLPs which are not only NP-hard, but even undecidable. See more details in [11, 12].

Recently, convex relaxations of the ACOPF problem, in particular second-order cone programming (SOCP) [13], semidefinite programming (SDP) [14], and quadratic convex (QC) [15] relaxations, have attracted a significant interest for several reasons. First, they can lead to global optimality; second, because they are relaxations, they provide a bound on the global optimal value of the ACOPF problem; and third, if one of these relaxations is infeasible, then the ACOPF problem is infeasible. Mixed-integer QC relaxations were proposed in [16] and mixed-integer SOCP relaxations in [17, 18, 19] for MINLPs in power systems.

However, [5] emphasized that convex relaxations of the OPF problem are aimed at complementing nonconvex (local) solvers with the valuable information about the quality of the solution obtained, rather than at replacing them.

On the other hand, some heuristic techniques discussed in [20] have been proposed for handling discrete variables in the OPF problem. One approach works as follows: first, solve the continuous relaxation of the OPF problem, i.e. treat the discrete variables as continuous; second, round-off solutions corresponding to discrete variables to their nearest discrete values; and third, fix discrete variables to these values and then solve the corresponding OPF subproblem. This approach is called the *round-off technique* and remains the simplest to deal with discrete variables in the ORPD problem, although it may lead to suboptimal or infeasible solutions. Some deficiencies were already pointed out in [21]: for instance, since the OPF problem is in general highly nonconvex, solving its continuous relaxation with a local optimizer in the first step may lead to a poor local solution which, after round-off in the second step, may lead to a very poor solution in the third step.

Our contribution is in two parts. First, we propose SDP-based relaxations of the ORPD problem with a new tight convex model of tap changer; second, a modified round-off technique where, in the first step, we solve a SDP-based relaxation instead of the continuous relaxation of the ORPD problem. More details about semidefinite optimization can be found in [22]. Computational results show that the SDP-based relaxations of the ORPD problem are tight, and furthermore that applying the round-off technique with these relaxations may lead to near-global solutions, even for large-scale instances.

The remainder of this paper is organized as follows. In Section 2, we state the mathematical model of the ORPD problem. In Section 3, we describe two SDP-based relaxations of the ORPD problem: a simple one already proposed in [23] and a new tighter one. In Section 4, we present our round-off technique, and we report computational results in Section 5. Section 6 concludes the paper.

2 ORPD: Formulation

Consider a typical power network $\mathcal{P} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N} = \{1, 2, \dots, n\}$ and $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ denote respectively the set of buses and the set of branches (transmission lines and tap changers). We denote by $\mathcal{U} \subseteq \mathcal{N}$ the set of buses k where a shunt element is connected, and $\mathcal{T} \subseteq \mathcal{L}$ the set of branches with tap changers. Each branch $\ell \in \mathcal{L}$ has a *from* end k (on the *tap side*) and a *to* end m as modeled in [24]. We denote $\ell = (k, m)$. The ORPD problem is given as:

$$\min \sum_{g \in \mathcal{G}} c_{g2} p_{Gg}^2 + c_{g1} p_{Gg} + c_{g0} \quad (1a)$$

over variables $\mathbf{u} \in \{0, 1\}^{|\mathcal{U}|}$, $\mathbf{t} \in \mathbb{R}^{|\mathcal{T}|}$, $\mathbf{p}_G, \mathbf{q}_G \in \mathbb{R}^{|\mathcal{G}|}$, $\mathbf{p}_f, \mathbf{q}_f, \mathbf{p}_t, \mathbf{q}_t \in \mathbb{R}^{|\mathcal{L}|}$, and $\mathbf{v} \in \mathbb{C}^{|\mathcal{N}|}$, subject to

- Power balance equations:

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k u_k |\mathbf{v}_k|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{U}, \quad (1b)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k u_k |\mathbf{v}_k|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{U}, \quad (1c)$$

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{N} \setminus \mathcal{U}, \quad (1d)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{N} \setminus \mathcal{U}, \quad (1e)$$

- Branch flow equations:

$$p_{f\ell} + j q_{f\ell} = \frac{\mathbf{v}_k}{t_\ell} \left[\left(j \frac{b'_\ell}{2} + y_\ell \right) \frac{\mathbf{v}_k}{t_\ell} - y_\ell \mathbf{v}_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (1f)$$

$$p_{t\ell} + j q_{t\ell} = \mathbf{v}_m \left[-y_\ell \frac{\mathbf{v}_k}{t_\ell} + \left(j \frac{b'_\ell}{2} + y_\ell \right) \mathbf{v}_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (1g)$$

$$p_{f\ell} + j q_{f\ell} = v_k \left[\left(j \frac{b'_\ell}{2} + y_\ell \right) v_k - y_\ell v_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T}, \quad (1h)$$

$$p_{t\ell} + j q_{t\ell} = v_m \left[-y_\ell v_k + \left(j \frac{b'_\ell}{2} + y_\ell \right) v_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T}, \quad (1i)$$

- Generator power capacities:

$$\underline{p}_{Gg} \leq p_{Gg} \leq \bar{p}_{Gg}, \quad \underline{q}_{Gg} \leq q_{Gg} \leq \bar{q}_{Gg} \quad \forall g \in \mathcal{G}, \quad (1j)$$

- Line thermal limits:

$$|p_{f\ell} + j q_{f\ell}| \leq \bar{s}_\ell, \quad |p_{t\ell} + j q_{t\ell}| \leq \bar{s}_\ell \quad \forall \ell \in \mathcal{L}, \quad (1k)$$

- Voltage magnitude limits:

$$\underline{v}_k \leq |v_k| \leq \bar{v}_k \quad \forall k \in \mathcal{N}, \quad (1l)$$

- Reference bus constraint:

$$\angle v_1 = 0, \quad (1m)$$

- Shunt variable:

$$u_k \in \{0, 1\} \quad \forall k \in \mathcal{U}, \quad (1n)$$

- Tap ratio constraint:

$$t_\ell \in \{\underline{t}_\ell, \dots, \bar{t}_\ell\} \quad \forall \ell \in \mathcal{T}. \quad (1o)$$

The generation cost $c_{g2}p_{Gg}^2 + c_{g1}p_{Gg} + c_{g0}$ in (1a) is a convex function for all $g \in \mathcal{G}$. Constraints (1b)–(1i) derive from Kirchhoff's laws and represent power flow equations in the network. We assume $\underline{v}_k > 0$ for all $k \in \mathcal{N}$ in (1l) and $\underline{t}_\ell > 0$ for all $\ell \in \mathcal{T}$ in (1o). Constraint (1m) specifies bus $k = 1$ as the reference bus. In (1n), we assume that the shunt connected to a bus $k \in \mathcal{U}$ has an on/off switch. For all $\ell \in \mathcal{T}$, the tap ratio t_ℓ in (1o) is a discrete variable that typically takes on $2\bar{\eta} + 1$ values $\{\hat{t}_{-\bar{\eta}}, \dots, \hat{t}_0, \dots, \hat{t}_{\bar{\eta}}\}$, uniformly distributed around $\hat{t}_0$. In this paper, we assume without loss of generality that $\hat{t}_\ell$ varies between 0.9 and 1.1 pu with steps of 0.0125 pu (17 different settings) for all $\ell \in \mathcal{T}$.

3 ORPD: Convexification

3.1 Semidefinite relaxation 1 (SDR1)

Let

$$\mathbf{V} := \mathbf{v}\mathbf{v}^H, \quad (2a)$$

$$\mathbf{W}_{\{\ell\}} := \begin{bmatrix} \mathbf{V}_{kk} & \mathbf{W}_{k\ell} & \mathbf{V}_{km} \\ \mathbf{W}_{k\ell}^* & \mathbf{W}_{\ell\ell} & \mathbf{W}_{\ell m} \\ \mathbf{V}_{km}^* & \mathbf{W}_{\ell m}^* & \mathbf{V}_{mm} \end{bmatrix}$$

$$:= \begin{bmatrix} v_k \\ w_\ell \\ v_m \end{bmatrix} \begin{bmatrix} v_k \\ w_\ell \\ v_m \end{bmatrix}^H \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (2b)$$

$$w_\ell := \frac{v_k}{t_\ell} \quad \forall \ell = (k, m) \in \mathcal{T}. \quad (2c)$$

The ORPD problem (1) can be reformulated as follows

minimize (1a)

subject to (1d), (1e), (1j), (1k), (1m), (1n), (1o),

(2a), (2b), (2c),

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k u_k V_{kk} = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{U}, \quad (3a)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k u_k V_{kk} = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{U}, \quad (3b)$$

$$p_{f\ell} + j q_{f\ell} = \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) W_{\ell\ell} - y_\ell^* W_{\ell m} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (3c)$$

$$p_{t\ell} + j q_{t\ell} = -y_\ell^* W_{\ell m}^* + \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{mm} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (3d)$$

$$p_{f\ell} + j q_{f\ell} = \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{kk} - y_\ell^* V_{km} \quad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T}, \quad (3e)$$

$$p_{t\ell} + j q_{t\ell} = -y_\ell^* V_{km}^* + \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{mm} \quad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T}, \quad (3f)$$

$$\underline{v}_k^2 \leq V_{kk} \leq \bar{v}_k^2 \quad \forall k \in \mathcal{N}. \quad (3g)$$

If we define $\xi_k := u_k V_{kk} \in \{0, V_{kk}\}$ for all $k \in \mathcal{U}$, a linear formulation of (3a)–(3b) is given as

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k \xi_k = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{U}, \quad (4a)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k \xi_k = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{U}, \quad (4b)$$

$$0 \leq \xi_k \leq V_{kk} \quad \forall k \in \mathcal{U}. \quad (4c)$$

From (2b) and (2c), we have

$$W_{k\ell} = \frac{V_{kk}}{t_\ell} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (5a)$$

$$W_{\ell\ell} = \frac{V_{kk}}{t_\ell^2} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (5b)$$

which implies

$$\frac{V_{kk}}{t_\ell} \leq W_{k\ell} \leq \frac{V_{kk}}{\underline{t}_\ell} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (6a)$$

$$\frac{V_{kk}}{t_\ell^2} \leq W_{\ell\ell} \leq \frac{V_{kk}}{\underline{t}_\ell^2} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (6b)$$

since $0 < \underline{t}_\ell \leq t_\ell \leq \bar{t}_\ell$ for all $\ell \in \mathcal{T}$. Finally, we can show that V in (2a) and, for all $\ell \in \mathcal{T}$, $W_{\{\ell\}}$ in (2b) are rank-one positive semidefinite matrices. The *semidefinite relaxation 1* (SDR1) in Model 1 is obtained by dropping the rank constraints. For all $\ell = (k, m) \in \mathcal{T}$, since $W_{\{\ell\}} \succeq 0$ implies $W_{k\ell}^2 \leq V_{kk} W_{\ell\ell}$, (6) can be rewritten as

$$W_{k\ell} \geq \frac{V_{kk}}{\bar{t}_\ell} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (7a)$$

$$W_{\ell\ell} \leq \frac{V_{kk}}{\underline{t}_\ell^2} \quad \forall \ell = (k, m) \in \mathcal{T}. \quad (7b)$$

We point out that the variables u_k and t_ℓ are eliminated in Model 1 for all $k \in \mathcal{U}$ and for all $\ell \in \mathcal{T}$. This approach was already proposed in [23] and was recently used in [25]. Once the solution of Model 1 is obtained, the optimal value $\hat{u}_k$ of the shunt element connected to a bus $k \in \mathcal{U}$ and the optimal tap ratio $\hat{t}_\ell$ of a transformer $\ell = (k, m) \in \mathcal{T}$ can be determined respectively as follows:

$$\hat{u}_k = \frac{\xi_k}{V_{kk}} \in [0, 1], \quad (9a)$$

$$\hat{t}_\ell = \sqrt{\frac{V_{kk}}{W_{\ell\ell}}} \in [\underline{t}_\ell, \bar{t}_\ell]. \quad (9b)$$

Model 1 Semidefinite relaxation 1 (SDR1)

Variables:

$$\boldsymbol{\xi} \in \mathbb{R}^{|\mathcal{U}|}, \quad (8a)$$

$$\mathbf{p}_G, \mathbf{q}_G \in \mathbb{R}^{|\mathcal{G}|}, \quad (8b)$$

$$\mathbf{p}_f, \mathbf{q}_f, \mathbf{p}_t, \mathbf{q}_t \in \mathbb{R}^{|\mathcal{L}|}, \quad (8c)$$

$$\mathbf{V} \in \mathbb{H}^{|\mathcal{N}|}, \quad (8d)$$

$$(\mathbf{W}_{\ell\ell}, \mathbf{W}_{k\ell}, \mathbf{W}_{\ell m}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \quad \forall \ell = (k, m) \in \mathcal{T}. \quad (8e)$$

Minimize: (1a)

Subject to: (1d), (1e), (1j), (1k), (3c)–(3g), (4), (7), $\mathbf{V} \succeq 0$, and $\mathbf{W}_{\{\ell\}} \succeq 0 \forall \ell \in \mathcal{T}$.

If the optimal solutions $\hat{\mathbf{V}}$ and $\hat{\mathbf{W}}_{\{\ell\}}$ for all $\ell \in \mathcal{T}$ are rank-one matrices, $\hat{u}_k \in \{0, 1\}$ for all $k \in \mathcal{U}$, and $\hat{t}_\ell \in \{\underline{t}_\ell, \dots, \bar{t}_\ell\}$ for all $\ell \in \mathcal{T}$, then there exists a complex vector $\hat{\mathbf{v}}$ that is a global optimal solution of (1). We say that the relaxation SDR1 is exact.

We also consider a cheaper relaxation, called “*tight-and-cheap relaxation 1*”, given in Model 2 and obtained as follows. We replace $\mathbf{V} \succeq 0$ and $\mathbf{W}_{\{\ell\}} \succeq 0$ for all $\ell \in \mathcal{T}$ in Model 1 by

$$\begin{bmatrix} 1 & \mathbf{v}_k^* & \mathbf{v}_m^* \\ \mathbf{v}_k & \mathbf{V}_{kk} & \mathbf{V}_{km} \\ \mathbf{v}_m & \mathbf{V}_{km}^* & \mathbf{V}_{mm} \end{bmatrix} \succeq 0 \quad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T}, \quad (10a)$$

$$\begin{bmatrix} 1 & \mathbf{w}_{\{\ell\}}^H \\ \mathbf{w}_{\{\ell\}} & \mathbf{W}_{\{\ell\}} \end{bmatrix} \succeq 0 \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (10b)$$

where $\mathbf{w}_{\{\ell\}}^H = (\mathbf{v}_k^*, \mathbf{w}_\ell^*, \mathbf{v}_m^*)$ for all $\ell = (k, m) \in \mathcal{T}$, and we add the following constraints

$$\operatorname{Re}(\mathbf{v}_1) \geq \frac{\mathbf{V}_{11} + \underline{v}_1 \bar{v}_1}{\underline{v}_1 + \bar{v}_1}, \quad (10c)$$

$$\operatorname{Im}(\mathbf{v}_1) = 0, \quad (10d)$$

corresponding to the reference bus $k = 1$.

Model 2 Tight-and-cheap relaxation 1 (TCR1)Variables: (8), $\mathbf{v} \in \mathbb{C}^{|\mathcal{N}|}$, $\mathbf{w} \in \mathbb{C}^{|\mathcal{T}|}$.

Minimize: (1a)

Subject to: (1d), (1e), (1j), (1k), (3c)–(3g), (4), (6), (10).

The tight-and-cheap relaxation (TCR) was first proposed in [26] for the ACOPF problem. It was shown in [26] that TCR is stronger than the standard SOCP relaxation and nearly as tight as the standard SDP relaxation. Moreover, computation experiments on standard test cases with up to 6515 buses showed that solving TCR for large-scale instances is much less expensive than solving the chordal relaxation, a SDP relaxation technique that exploits the sparsity of power networks.

3.2 Semidefinite relaxation 2 (SDR2)

For all $\ell = (k, m) \in \mathcal{T}$, variables $\mathbf{W}_{k\ell}$ and $\mathbf{W}_{\ell\ell}$ in (5) are respectively described by constraints of the form $z_n = x/y^n$, $n = 1, 2$. Consider the set

$$\mathcal{S}_1 = \{(x, y, z_1, z_2) \in \mathbb{R}^4 : \underline{x} \leq x \leq \bar{x}, \underline{y} \leq y \leq \bar{y}, z_1 = x/y, z_2 = x/y^2\},$$

where $0 < \underline{x} < \bar{x}$ and $0 < \underline{y} < \bar{y}$. We can show that $\mathcal{S}_1$ is equivalent to

$$\mathcal{S}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 : z_2 = z_1^2/x, (x, z_1) \in \Omega\},$$

where $\Omega = \{(x, z_1) \in \mathbb{R}^2 : \underline{x} \leq x \leq \bar{x}, x/\bar{y} \leq z_1 \leq x/\underline{y}\}$ is the convex quadrilateral with vertices $(\underline{x}, \underline{x}/\bar{y})$, $(\bar{x}, \bar{x}/\bar{y})$, $(\bar{x}, \bar{x}/\underline{y})$, and $(\underline{x}, \underline{x}/\underline{y})$.

In Section 3.1, the proposed convex set that contains $\mathcal{S}_2$ is

$$\mathcal{S}_3 = \{(x, z_1, z_2) \in \mathbb{R}^3 : \underline{x} \leq x \leq \bar{x}, z_2 \geq z_1^2/x, z_1 \geq x/\bar{y}, z_2 \leq x/\underline{y}\}.$$

In this section, we propose a tighter convex set containing $\mathcal{S}_2$.

Lemma 1 *Let $f(x, z_1) = z_1^2/x$ defined over $\mathbb{R}_+ \times \mathbb{R}$. Then f is convex.*

Proof. For all $(x, z_1) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\nabla^2 f(x, z_1) = \frac{2}{x^3} \begin{bmatrix} z_1^2 & -xz_1 \\ -xz_1 & x^2 \end{bmatrix} = \frac{2}{x^3} \begin{bmatrix} z_1 \\ -x \end{bmatrix} \begin{bmatrix} z_1 \\ -x \end{bmatrix}^T \succeq 0.$$

Then f is convex. $\square$

Lemma 2 *Let $(x, z_1) \in \Omega$. If $z_2 = z_1^2/x$, then $x + \underline{y}\bar{y}z_2 \leq (\underline{y} + \bar{y})z_1$.*

Proof. Let $(x, z_1) \in \Omega$ and $z_2 = z_1^2/x$. We have $x/\bar{y} \leq z_1 \leq x/\underline{y}$ with $0 < \underline{y} < \bar{y}$. Therefore, $(x - \underline{y}z_1)(x - \bar{y}z_1) \leq 0$. On the other hand,

$$(x - \underline{y}z_1)(x - \bar{y}z_1) = x^2 - (\underline{y} + \bar{y})xz_1 + \underline{y}\bar{y}z_1^2 = x[x - (\underline{y} + \bar{y})z_1 + \underline{y}\bar{y}z_2] \leq 0.$$

Since $\underline{x} > 0$, it follows that $x - (\underline{y} + \bar{y})z_1 + \underline{y}\bar{y}z_2 \leq 0$. $\square$

Proposition 1 *Let $f(x, z_1) = z_1^2/x$ defined over Ω . The concave envelope or the tightest concave overestimator of f on Ω is $h(x, z_1) = \frac{1}{\underline{y}\bar{y}}[(\underline{y} + \bar{y})z_1 - x]$.*

Proof. The function h is affine and, from Lemma 2, overestimates f on Ω . Then h is a concave overestimator of f on Ω . We should note that $h(x, z_1) = f(x, z_1)$ for all $(x, z_1) \in \Omega$ such that $z_1 = x/\bar{y}$ or $z_1 = x/\underline{y}$.

Suppose that there exists a concave overestimator $\tilde{h}$ of f on Ω such that $f(x, z_1) \leq \tilde{h}(x, z_1) \leq h(x, z_1)$ for all $(x, z_1) \in \Omega$ and $\tilde{h}(\tilde{x}, \tilde{z}_1) < h(\tilde{x}, \tilde{z}_1)$ for some $(\tilde{x}, \tilde{z}_1) \in \Omega$. Since Ω is the convex quadrilateral with vertices $(\underline{x}, \underline{x}/\bar{y})$, $(\bar{x}, \bar{x}/\bar{y})$, $(\bar{x}, \bar{x}/\underline{y})$, and $(\underline{x}, \underline{x}/\underline{y})$, there exist nonnegative scalars $\alpha_1, \alpha_2, \alpha_3$, and α_4 such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ and $(\tilde{x}, \tilde{z}_1) = \alpha_1(\underline{x}, \underline{x}/\bar{y}) + \alpha_2(\bar{x}, \bar{x}/\bar{y}) + \alpha_3(\bar{x}, \bar{x}/\underline{y}) + \alpha_4(\underline{x}, \underline{x}/\underline{y})$. Therefore,

$$\begin{aligned} \tilde{h}(\tilde{x}, \tilde{z}_1) &= \tilde{h}(\alpha_1(\underline{x}, \underline{x}/\bar{y}) + \alpha_2(\bar{x}, \bar{x}/\bar{y}) + \alpha_3(\bar{x}, \bar{x}/\underline{y}) + \alpha_4(\underline{x}, \underline{x}/\underline{y})) \\ &\geq \alpha_1\tilde{h}(\underline{x}, \underline{x}/\bar{y}) + \alpha_2\tilde{h}(\bar{x}, \bar{x}/\bar{y}) + \alpha_3\tilde{h}(\bar{x}, \bar{x}/\underline{y}) + \alpha_4\tilde{h}(\underline{x}, \underline{x}/\underline{y}) \\ &= \alpha_1f(\underline{x}, \underline{x}/\bar{y}) + \alpha_2f(\bar{x}, \bar{x}/\bar{y}) + \alpha_3f(\bar{x}, \bar{x}/\underline{y}) + \alpha_4f(\underline{x}, \underline{x}/\underline{y}) \\ &= \alpha_1h(\underline{x}, \underline{x}/\bar{y}) + \alpha_2h(\bar{x}, \bar{x}/\bar{y}) + \alpha_3h(\bar{x}, \bar{x}/\underline{y}) + \alpha_4h(\underline{x}, \underline{x}/\underline{y}) \\ &= h(\alpha_1(\underline{x}, \underline{x}/\bar{y}) + \alpha_2(\bar{x}, \bar{x}/\bar{y}) + \alpha_3(\bar{x}, \bar{x}/\underline{y}) + \alpha_4(\underline{x}, \underline{x}/\underline{y})) \\ &= h(\tilde{x}, \tilde{z}_1). \end{aligned}$$

The inequality follows from the definition of a concave function; the two subsequent equalities from the fact that h agrees with f at the vertices of Ω and $f(x, z_1) \leq \tilde{h}(x, z_1) \leq h(x, z_1)$ for all $(x, z_1) \in \Omega$; and the last equality from the fact that h is affine. The relation obtained $\tilde{h}(\tilde{x}, \tilde{z}_1) \geq h(\tilde{x}, \tilde{z}_1)$ contradicts $\tilde{h}(\tilde{x}, \tilde{z}_1) < h(\tilde{x}, \tilde{z}_1)$. So h is the concave envelope of f on Ω . $\square$

A similar proof can be found in [27] on convex and concave envelopes of functions defined on convex quadrilaterals.

Proposition 2 *The convex hull of $\mathcal{S}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 : \underline{x} \leq x \leq \bar{x}, x/\bar{y} \leq z_1 \leq x/\underline{y}, z_2 = z_1^2/x\}$, where $0 < \underline{x} < \bar{x}$, $0 < \underline{y} < \bar{y}$, is $\bar{\mathcal{S}}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 : \underline{x} \leq x \leq \bar{x}, z_1^2 \leq xz_2, x + \underline{y}\bar{y}z_2 \leq (\underline{y} + \bar{y})z_1\}$.*

Proof. The result follows from Lemma 1 and Proposition 1. $\square$

Applying Proposition 2 to (5), we replace (7) in Model 1 by

$$V_{kk} + \underline{t}_\ell \bar{t}_\ell W_{\ell\ell} \leq (\underline{t}_\ell + \bar{t}_\ell) W_{k\ell} \quad \forall \ell = (k, m) \in \mathcal{T}, \quad (12)$$

and define a new formulation of SDP relaxation in Model 3. We will refer to this relaxation as “*semidefinite relaxation 2*” (SDR2). The “*tight-and-cheap relaxation 2*” is given in Model 4. From Proposition 2, $\bar{\mathcal{S}}_2 \subset \mathcal{S}_3$ and then SDR2 (respectively TCR2) is stronger than SDR1 (TCR1).

Model 3 Semidefinite relaxation 2 (SDR2)

Variables: (8).

Minimize: (1a)

Subject to: (1d), (1e), (1j), (1k), (3c)–(3g), (4), (12), $V \succeq 0$, and $W_{\{\ell\}} \succeq 0 \quad \forall \ell \in \mathcal{T}$.

Model 4 Tight-and-cheap relaxation 2 (TCR2)

Variables: (8), $\mathbf{v} \in \mathbb{C}^{|\mathcal{N}|}$, $\mathbf{w} \in \mathbb{C}^{|\mathcal{T}|}$.

Minimize: (1a)

Subject to: (1d), (1e), (1j), (1k), (3c)–(3g), (4), (12), (10).

4 ORPD: Solution approach

Following [12], a mixed-integer nonlinear program (MINLP) is an optimization problem of the form

$$\hat{v} = \text{minimize } f_0(\mathbf{x}, \mathbf{y}) \quad (15a)$$

$$\text{subject to } f_i(\mathbf{x}, \mathbf{y}) \leq 0, \quad i = 1, 2, \dots, m, \quad (15b)$$

$$\mathbf{x} \in \mathbb{R}^{n_1}, \mathbf{y} \in \mathbb{Z}^{n_2}. \quad (15c)$$

By fixing all integer variables, i.e. $\mathbf{y} = \tilde{\mathbf{y}} \in \mathbb{Z}^{n_2}$, we obtain the following *subproblem* (SP) of (15)

$$\hat{v}_{SP} = \text{minimize } f_0(\mathbf{x}, \tilde{\mathbf{y}}) \quad (16a)$$

$$\text{subject to } f_i(\mathbf{x}, \tilde{\mathbf{y}}) \leq 0, \quad i = 1, 2, \dots, m, \quad (16b)$$

$$\mathbf{x} \in \mathbb{R}^{n_1}. \quad (16c)$$

Then we can rewrite the MINLP (15) as

$$\hat{v} = \min_{\tilde{\mathbf{y}} \in \mathbb{Z}^{n_2}} \left\{ \min_{\mathbf{x} \in \mathbb{R}^{n_1}} \{f_0(\mathbf{x}, \tilde{\mathbf{y}}) : f_i(\mathbf{x}, \tilde{\mathbf{y}}) \leq 0, i = 1, 2, \dots, m\} \right\}$$

If (16) is feasible, its global optimal value $\hat{v}_{SP}$ provides an upper bound for $\hat{v}$ and the optimal solution $(\hat{\mathbf{x}}_{SP}, \tilde{\mathbf{y}})$ of (16) is called a *suboptimal* solution of MINLP (15). The subproblem (16) may be nonconvex and solving it to global optimality may be hard. In this case, any local optimal value $\bar{v}_{SP}$ of (16) can be considered as an upper bound of $\hat{v}$.

Consider a relaxation (R) of (15) of the form

$$\hat{v}_R = \text{minimize } \underline{f}_0(\mathbf{x}, \mathbf{y}) \quad (17a)$$

$$\text{subject to } (\mathbf{x}, \mathbf{y}) \in \bar{\Omega} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (17b)$$

where $\bar{\Omega}$ contains the feasible set of (15) and $\underline{f}_0$ underestimates f_0 for all feasible solutions of (15). The global optimal value $\hat{v}_R$ is a lower bound of $\hat{v}$. Since it is necessary to solve (17) to global optimality to obtain a lower bound on $\hat{v}$, it is more advantageous for the relaxation (17) to be convex.

Let $(\hat{\mathbf{x}}_R, \hat{\mathbf{y}}_R)$ and $\hat{v}_R$ be respectively the global optimal solution and the global optimal value of a convex relaxation (R) of a MINLP. Now, consider the subproblem (SP) associated to $\hat{\mathbf{y}}_R$, the closest integer solution to $\hat{\mathbf{y}}_R$. We define the optimality gap of the relaxation (R) by $100(1 - \hat{v}_R/\bar{v}_{SP})$, where $\bar{v}_{SP}$ is a local optimal value of the subproblem (SP). If the optimality gap is close to zero, we say that the relaxation is *tight* and the suboptimal solution $(\hat{\mathbf{x}}_{SP}, \hat{\mathbf{y}}_R)$ is *near-global optimal* for the MINLP.

This leads to the following approach to solve the ORPD problem (1):

1. Solve SDR1, TCR1, SDR2 or TCR2 and find corresponding shunt solutions $\hat{\mathbf{u}} \in [0, 1]^{|\mathcal{U}|}$ and tap ratios $\hat{\mathbf{t}} \in \prod_{\ell \in \mathcal{T}} [\underline{t}_\ell, \bar{t}_\ell]$ with formulas (9),
2. Round-off $\mathbf{u}$ and $\mathbf{t}$ to their respective nearest discrete values $\tilde{\mathbf{u}} \in \{0, 1\}^{|\mathcal{U}|}$ and $\tilde{\mathbf{t}} \in \prod_{\ell \in \mathcal{T}} \{\underline{t}_\ell, \dots, \bar{t}_\ell\}$,
3. Fix $\mathbf{u} = \tilde{\mathbf{u}}$ and $\mathbf{t} = \tilde{\mathbf{t}}$ and solve the ACOPF subproblem with a nonlinear (local) solver.

5 Computational results

In this section, we evaluate the accuracy and the computational efficiency of SDR1 and TCR1 as compared to SDR2 and TCR2.

In order to increase the computational speed of solving SDR1 and SDR2, we exploited the sparsity of a power network by replacing the SDP constraint $\mathbf{V} \succeq 0$ by small SDP constraints defined on a chordal extension of the power network. More details about exploiting the sparsity of power networks in SDP relaxations of the OPF problem can be found in [26, 28, 29, 30, 31].

Let us interpret the network $\mathcal{P} = (\mathcal{N}, \mathcal{L})$ as a connected, simple and undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \dots, n\}$ represents the set of vertices and $\mathcal{E} = \{\{k, m\} : (k, m) \text{ or } (m, k) \in \mathcal{L}\}$, the set of edges. It was proved in [32] that the SDP constraint $\mathbf{V} \succeq 0$ in SDR1 or SDR2 is equivalent to $\mathbf{V}_\mathcal{K} \succeq 0$ for every maximal clique $\mathcal{K}$ of a chordal extension $\mathcal{G}'$ of $\mathcal{G}$. $\mathbf{V}_\mathcal{K}$ is the submatrix of $\mathbf{V}$ in which the set of row indices that remain and the set of column indices that remain are both $\mathcal{K}$. To compute the maximal cliques of a chordal extension of $\mathcal{G}$, we used the same algorithm as in [26].

We tested the models 1, 2, 3, 4, on standard test cases available from MATPOWER [24, 33, 34]. We assigned a shunt variable u_k to each shunt element connected to a bus $k \in \mathcal{U}$ and we assumed that the tap ratio t_ℓ of each transformer $\ell \in \mathcal{T}$ varies from 0.9 to 1.1 pu by steps of 0.0125 pu.

We solved SDR1, TCR1, SDR2 and TCR2 in MATLAB using CVX 2.1 [35] with the solver MOSEK 8.0.0.60 and `default precision` (tolerance $\epsilon = 1.49 \times 10^{-8}$). MOSEK numerically failed to solve SDR1 for `case3120sp`.

All ACOPF subproblems were solved with the MATPOWER-solver MIPS. When MIPS numerically failed to solve subproblems for some test cases, marked with “*” in Table 1 and Table 2, we then used the solver FMINCON. Among these subproblems, FMINCON did not converge to a feasible solution after 1000 iterations for `case3012wp` (SDR1), `case3120sp` (TCR1) in cost minimization, and `case3120sp` (SDR1, TCR1), `case3375wp` (SDR1, TCR1) in loss minimization.

All the computations were carried out on an Intel Core i7-6700 CPU @ 3.40 GHz computing platform.

We considered two objective functions: the generation cost [\$/h] (1a) and the active loss [MW] where $c_{g2} = 0$, $c_{g1} = 1$ and $c_{g0} = 0$ for all $g \in \mathcal{G}$ in (1a). Both objective functions of test cases from [34] are the same. We denote $\underline{v}$ the best lower bound which is the maximum value among $\hat{v}_{SDR1}$, $\hat{v}_{SDR2}$, $\hat{v}_{TCR1}$, $\hat{v}_{TCR2}$, respective optimal values of SDR1, TCR1, SDR2, and TCR2. The normalized $\hat{v}_R$ of a relaxation is measured as $\hat{v}_R/\underline{v} \leq 1$, where $\hat{v}_R$ is the relaxation’s optimal value. Table 1 and Table 2 summarize the normalized optimal values of relaxations. The results support the following key points:

1. Among all relaxations, SDR2 is the strongest.
2. SDR1 (respectively SDR2) is stronger than TCR1 (respectively TCR2).
3. SDR1 and TCR2 are comparable.

For a relaxation, the local optimal value of the ACOPF subproblem obtained from the relaxation's optimal solution after rounding-off the discrete variables $\mathbf{u}$ and $\mathbf{t}$ is denoted $\bar{v}_{SP}$. The normalized $\bar{v}_{SP}$ in Table 1 and Table 2 is the value $\bar{v}_{SP}/\bar{v} \geq 1$, where $\bar{v}$ the best upper bound, i.e. the minimum value among all $\bar{v}_{SP}$. In Table 1 and Table 2, we observe that:

1. In general, applying the round-off technique with SDR2 (respectively TCR2) provide tighter upper bounds than with SDR1 (respectively TCR1) for large-scale instances.
2. Applying the round-off technique with TCR2 is comparable to using SDR1.

Optimality gaps of the four relaxations are given in Table 3 and Table 4. The optimality gap of a relaxation is measured as $100(1 - \hat{v}_R/\bar{v}_{SP})$, where $\hat{v}_R$ is the relaxation's optimal value and $\bar{v}_{SP}$ is a local optimal value of the corresponding subproblem. With the optimality gap, we can guarantee the near-global optimality of the suboptimal solution obtained with MIPS. Results in Table 3 and Table 4 show that:

1. Suboptimal solutions obtained from SDR2's optimal solutions have the lowest guaranteed optimality gaps. We can say that they are near-global optimal for all but one test case: **case-*ACTIV_SG-500*** (cost minimization).
2. In general, suboptimal solutions obtained from SDR1's and TCR2's optimal solutions have almost the same guaranteed optimality gaps.
3. Suboptimal solutions obtained from TCR1's optimal solutions have slightly larger optimality gaps.

The computation times required to solve SDR1, TCR1, SDR2, and TCR2 as reported by MOSEK are also shown in Table 3 and Table 4. We see that

1. Solving SDR1 (respectively TCR1) is as expensive as solving SDR2 (respectively TCR2) in general.
2. Solving SDR1 (respectively SDR2) is much more expensive than solving TCR1 (respectively TCR2) for large-scale instances. The TCRs are on average 30 times faster than the SDRs.

Overall, we can see that TCR2 offers an interesting trade-off between the optimality gap and the computation time.

6 Conclusion

We proposed a tight SDP relaxation (SDR2) for the ORPD problem. This formulation is based on a standard SDP relaxation (SDR1) combined with a tight convex tap changer model. Experiments on selected MATPOWER instances with up to 3375 buses show that SDR2 is stronger than SDR1, and computationally comparable. From both SDR1 and SDR2, we derived tight-and-cheap relaxations TCR1 and TCR2, respectively.

A round-off technique based on a SDP relaxation of the ORPD problem instead of the continuous relaxation was also proposed. Computational results show that applying the round-off technique with SDR2 and TCR2 provides suboptimal solutions which are near-global optimal. Both provide almost the same guaranteed optimality gaps, but TCR2 is computationally much less expensive for large-scale instances. In summary, the proposed TCR-based approach provides the best trade-off between optimality gap and computation time.

Table 1: Cost minimization: Normalized optimal values

Test case	$\underline{v}$ [\$/h]	$\bar{v}$ [\$/h]	Normalized $\hat{v}_R$				Normalized $\bar{v}_{SP}$			
			SDR1	SDR2	TCR1	TCR2	SDR1	SDR2	TCR1	TCR2
Small-scale instances										
case14	8 078.62	8 078.75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case24_ieee_rts	63 333.39	63 335.67	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
case30	576.89	576.89	1.0000	1.0000	0.9993	0.9993	1.0000	1.0000	1.0000	1.0000
case_ieee30	8 902.67	8 902.75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case39	41 850.24	41 852.45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case57	41 682.14	41 688.61	0.9997	1.0000	0.9996	0.9999	1.0000	1.0000	1.0000	1.0000
case89pegase	5 803.58	5 804.23	0.9996	1.0000	0.9992	0.9999	1.0001	1.0000	1.0001	1.0001
Average			0.9998	1.0000	0.9996	0.9999	1.0000	1.0000	1.0000	1.0000
Medium-scale instances										
case118	129 526.00	129 662.15	0.9987	1.0000	0.9987	0.9997	1.0003	1.0000	1.0003	1.0000
case_ACTIV_SG_200	27 552.84	27 553.52	1.0000	1.0000	1.0000	1.0000	*1.0000	1.0000	*1.0000	1.0000
case_illinois200	36 738.07	36 743.37	1.0000	1.0000	0.9998	0.9998	*1.0000	1.0000	1.0000	1.0000
case300	718 938.90	719 154.24	0.9994	1.0000	0.9989	0.9992	1.0002	1.0000	1.0002	1.0002
case_ACTIV_SG_500	70 316.47	72 454.07	0.9875	1.0000	0.9751	0.9792	1.0016	1.0033	1.0000	1.0018
Average			0.9950	1.0000	0.9902	0.9919	1.0007	1.0013	1.0001	1.0007
Large-scale instances										
case1354pegase	73 999.72	74 005.77	0.9998	1.0000	0.9995	0.9998	1.0000	1.0000	1.0000	1.0000
case2383wp	1 856 849.35	1 862 626.70	0.9996	1.0000	0.9981	0.9992	1.0039	1.0000	1.0068	1.0033
case2736sp	1 306 771.84	1 307 134.47	0.9997	1.0000	0.9988	0.9998	1.0001	1.0000	1.0002	1.0001
case2737sop	777 020.01	777 337.82	0.9997	1.0000	0.9990	0.9997	1.0001	1.0000	1.0001	1.0001
case2746wop	1 206 874.60	1 207 872.71	0.9994	1.0000	0.9983	0.9994	1.0001	1.0000	1.0002	1.0001
case2746wp	1 630 617.28	1 631 018.94	0.9997	1.0000	0.9988	0.9996	1.0001	1.0000	1.0002	1.0000
case2869pegase	133 867.21	133 877.91	0.9998	1.0000	0.9994	0.9997	1.0000	1.0000	1.0000	1.0000
case3012wp	2 578 533.79	2 590 254.80	0.9987	1.0000	0.9968	0.9985	–	1.0000	1.0009	1.0013
case3120sp	2 135 933.93	2 143 654.67	–	1.0000	0.9975	0.9988	–	1.0002	–	1.0000
case3375wp	7 397 491.46	*7 409 223.47	0.9998	1.0000	0.9988	0.9994	*1.0002	*1.0000	*1.0002	*1.0004
Average			–	1.0000	0.9985	0.9994	–	1.0000	–	1.0005

Table 2: Loss minimization: Normalized optimal values

Test case	$\underline{v}$ [MW]	$\overline{v}$ [MW]	Normalized $\hat{v}_R$				Normalized $\overline{v}_{SP}$			
			SDR1	SDR2	TCR1	TCR2	SDR1	SDR2	TCR1	TCR2
Small-scale instances										
case14	259.49	259.49	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case24_ieee_rts	2 875.33	2 875.37	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
case30	191.09	191.09	1.0000	1.0000	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000
case_ieee30	284.68	284.71	0.9997	1.0000	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000
case39	6 283.20	6 283.44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case57	1 260.83	1 260.98	0.9998	1.0000	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
case89pegase	5 803.58	5 804.23	0.9996	1.0000	0.9992	0.9999	1.0001	1.0000	1.0001	1.0001
Average			0.9998	1.0000	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000
Medium-scale instances										
case118	4 250.75	4 251.17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
case_ACTIV_SG_200	1 483.23	1 483.32	1.0000	1.0000	0.9999	0.9999	1.0000	1.0000	*1.0000	1.0000
case_illinois200	2 245.96	2 246.22	1.0000	1.0000	0.9999	0.9999	*1.0000	1.0000	1.0000	1.0000
case300	23 723.13	23 725.42	0.9999	1.0000	0.9996	0.9997	1.0000	1.0001	1.0001	1.0001
case_ACTIV_SG_500	7 815.64	7 817.25	1.0000	1.0000	0.9998	0.9998	1.0000	1.0000	1.0000	1.0000
Average			1.0000	1.0000	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000
Large-scale instances										
case1354pegase	73 999.72	74 005.77	0.9998	1.0000	0.9995	0.9998	1.0000	1.0000	1.0000	1.0000
case2383wp	24 967.98	24 983.29	1.0000	1.0000	0.9992	0.9997	1.0000	1.0000	1.0001	1.0001
case2736sp	18 324.27	18 329.28	0.9999	1.0000	0.9993	0.9998	1.0001	1.0000	1.0001	1.0000
case2737sop	11 391.09	11 394.38	0.9997	1.0000	0.9992	0.9997	1.0001	1.0000	1.0001	1.0001
case2746wop	19 197.71	19 209.58	0.9996	1.0000	0.9989	0.9996	1.0001	1.0000	1.0001	1.0001
case2746wp	25 255.49	25 260.72	0.9999	1.0000	0.9992	0.9997	1.0001	1.0000	1.0001	1.0000
case2869pegase	133 867.21	133 877.91	0.9998	1.0000	0.9994	0.9997	1.0000	1.0000	1.0000	1.0000
case3012wp	27 611.41	27 648.22	0.9995	1.0000	0.9986	0.9994	1.0001	1.0000	1.0001	1.0003
case3120sp	21 463.22	21 524.52	0.9998	1.0000	0.9991	0.9997	–	1.0000	–	1.0000
case3375wp	48 950.55	49 002.59	0.9997	1.0000	0.9993	0.9995	–	1.0027	–	1.0000
Average			0.9998	1.0000	0.9992	0.9996	–	1.0003	–	1.0001

Table 3: Cost minimization: Optimality gaps and computation times

Test case	Optimality gap [%]				Computation time [s]			
	SDR1	SDR2	TCR1	TCR2	SDR1	SDR2	TCR1	TCR2
<i>Small-scale instances</i>								
case14	0.00	0.00	0.00	0.00	0.15	0.13	0.12	0.36
case24_ieee_rts	0.01	0.00	0.01	0.00	0.19	0.55	0.54	0.28
case30	0.00	0.00	0.07	0.07	0.12	0.12	0.20	0.20
case_ieee30	0.00	0.00	0.00	0.00	0.12	0.11	0.13	0.14
case39	0.01	0.01	0.01	0.01	0.43	0.41	0.61	0.52
case57	0.05	0.02	0.05	0.03	0.47	0.21	0.20	0.24
case89pegase	0.06	0.01	0.10	0.03	2.26	1.64	0.76	0.72
Average	0.03	0.01	0.05	0.02	0.91	0.69	0.45	0.42
<i>Medium-scale instances</i>								
case118	0.27	0.11	0.27	0.13	0.30	0.36	0.38	0.40
case_ACTIV_SG_200	*0.00	0.00	*0.01	0.01	0.82	0.79	0.57	0.55
case_illinois200	*0.02	0.01	0.03	0.03	0.94	1.41	0.73	0.75
case300	0.11	0.03	0.16	0.13	1.23	2.25	0.99	0.89
case_ACTIV_SG_500	4.31	3.27	5.36	5.14	4.37	4.53	3.85	3.14
Average	1.69	1.26	2.10	2.00	2.23	2.60	1.92	1.63
<i>Large-scale instances</i>								
case1354pegase	0.03	0.01	0.06	0.03	13.35	11.95	7.53	6.66
case2383wp	0.74	0.31	1.17	0.71	307.06	325.67	14.19	12.97
case2736sp	0.07	0.03	0.16	0.07	426.85	398.85	13.84	12.05
case2737sop	0.08	0.04	0.16	0.08	381.34	372.61	11.23	12.75
case2746wop	0.15	0.08	0.27	0.15	477.80	425.23	10.96	11.68
case2746wp	0.06	0.02	0.16	0.07	418.71	414.08	11.27	11.19
case2869pegase	0.03	0.01	0.07	0.04	55.71	80.07	21.27	19.78
case3012wp	—	0.45	0.87	0.73	1 670.60	1 618.99	12.27	12.17
case3120sp	—	0.38	—	0.48	763.19	672.87	13.64	13.22
case3375wp	*0.20	*0.16	*0.29	*0.24	1 638.91	1 763.90	20.47	18.89
Average	—	0.16	—	0.27	684.22	678.29	14.15	13.63

Table 4: Loss minimization: Optimality gaps and computation times

Test case	Optimality gap [%]				Computation time [s]			
	SDR1	SDR2	TCR1	TCR2	SDR1	SDR2	TCR1	TCR2
<i>Small-scale instances</i>								
case14	0.00	0.00	0.00	0.00	0.12	0.12	0.12	0.12
case24_ieee_rts	0.00	0.00	0.00	0.00	0.16	0.18	0.39	0.20
case30	0.00	0.00	0.01	0.01	0.13	0.13	0.17	0.17
case_ieee30	0.04	0.01	0.05	0.02	0.16	0.34	0.49	0.28
case39	0.00	0.00	0.00	0.00	0.17	0.19	0.22	0.22
case57	0.03	0.01	0.03	0.01	0.19	0.23	0.18	0.49
case89pegase	0.06	0.01	0.10	0.03	2.26	1.64	0.76	0.72
Average	0.03	0.01	0.04	0.02	0.82	0.66	0.41	0.43
<i>Medium-scale instances</i>								
case118	0.01	0.01	0.01	0.01	0.29	0.27	0.30	0.30
case_ACTIV_SG_200	0.01	0.01	*0.01	0.01	0.73	0.74	0.48	1.06
case_illinois200	*0.01	0.01	0.02	0.02	1.83	0.92	1.70	0.74
case300	0.02	0.02	0.06	0.05	1.02	0.91	0.98	0.88
case_ACTIV_SG_500	0.03	0.02	0.04	0.04	5.82	4.56	4.59	3.95
Average	0.02	0.02	0.04	0.03	2.85	2.21	2.32	2.00
<i>Large-scale instances</i>								
case1354pegase	0.03	0.01	0.06	0.03	13.35	11.95	7.53	6.66
case2383wp	0.07	0.06	0.15	0.10	252.55	271.06	13.94	12.56
case2736sp	0.05	0.03	0.11	0.05	434.61	280.45	10.52	9.58
case2737sop	0.06	0.03	0.12	0.06	210.99	215.05	8.67	8.55
case2746wop	0.10	0.06	0.18	0.12	394.81	444.52	9.88	11.20
case2746wp	0.04	0.02	0.11	0.05	431.06	403.43	9.70	9.72
case2869pegase	0.03	0.01	0.07	0.04	55.71	80.07	21.27	19.78
case3012wp	0.19	0.13	0.28	0.22	2 354.15	2 426.65	16.82	17.45
case3120sp	—	0.28	—	0.32	1 193.18	973.71	16.66	19.16
case3375wp	—	0.38	—	0.15	1 415.74	1 588.19	28.25	39.33
Average	—	0.12	—	0.12	753.59	749.07	15.09	16.54

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