Minisum Location with Closest Euclidean Distances

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Abstract

This paper considers the problem of locating a facility not among demand points, as is usually the case, but among demand regions which could be market areas. The objective is to find the location that minimizes the sum of weighted Euclidean distances to the closest points of the demand regions. It is assumed that internal distribution within the areas is “someone else’s problem”. A number of properties of the problem are derived and algorithms for solving the problem are suggested.

Résumé

Cet article traite du problème de la localisation d'une facilité non pas parmi des points de demande, comme c'est d'habitude le cas, mais parmi des régions de demande qui pourraient être des aires de marché. L'objectif est de déterminer la localisation qui minimise la somme pondérée des distances Euclidiennes aux points les plus proches des régions de demande. On suppose que la distribution interne au sein des régions est “le problème de quelqu'un d'autre”. Une série de propriétés du problème sont obtenues et des algorithmes suggérés pour le résoudre.

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1 Introduction

This paper draws on our previous work in facility location where the facilities and demands may be areas, and where the relevant distances are the closest distances between such areas (Brimberg and Wesolowsky, (2000a, 2000b)). A brief context for this problem is as follows.

The majority of traditional location problems in continuous space assume that a new facility is a point in space that is to be located with respect to some objective defined by its distances to a set of demands at specified fixed points in that space. The objective may refer to transportation costs; one example would be to minimize the sum of transportation costs, which are assumed to be proportional to distances. This objective is known as the minisum criterion. Alternatively, the objective may be to minimize the smallest time to emergency facilities, or to maximize the minimum distance to a noxious facility; these problems are known as the minimax and maximin location problems, respectively. This literature is reviewed in Francis et al. (1992), and Drezner (1995).

The idea of closest distances is well known in set theory (e.g., Hausdorff (1914)). Area demands have been used many times in location problems, but along with the assumption that travel distances within areas are relevant: e.g., Wesolowsky and Love (1971), Buchanan and Wesolowsky (1993), and Hamacher and Nickel (1992). If some form of aggregation is used, then the expected travel distance from the facility to some 'mean' point in the interior of the demand area determines the transportation cost.

We assume in this paper that the objective is to minimize the sum of weighted Euclidean closest distances between a facility and demands that may be convex polygonal areas. These polygonal areas represent separate districts responsible for their own distribution and transportation costs. In other words, the cost of travel inside any area is considered unimportant to the problem. It is assumed that the transportation system within the polygonal areas is 'dense' and may be accessed at any point in the boundary. If the facility is an area, then a similar assumption about internal transportation systems holds.

Practical examples of closest distances in location are given in our previous papers (Brimberg and Wesolowsky (2000a, 2000b)). One that was previously mentioned is a circuit board, where connections are made to the closest point of a connected component area. A limitation of these papers is that the rectangular norm is used as the distance function, while the Euclidean norm is more appropriate in many practical situations. Thus, we examine the case of closest Euclidean distance, and develop an efficient procedure to solve it.

2 The Model

We consider the problem of locating a new facility denoted by point $x \in \Re^2$ to service a set of n specified demand regions (or market areas) denoted by $A_i \subset \Re^2$, $i = 1,...,n$. The A_i are fixed, closed areas in the plane, with known demands specified by weighting constants,

$w_i > 0,\ i = 1,...,n$. The objective is to find the point x that minimizes the weighted sum of distances to the n demand regions.

What makes our problem different from the well-studied minisum problem is that the travel distance separating the facility from a demand region A_i is now defined as the Euclidean distance from x to the closest point in A_i. That is, flow from the facility will enter the given market at the closest entry point. Internal distribution costs within the market area will be 'someone else's problem'.

Denoting the closest point in A_i by $a_i(x)$, the travel distance now becomes:

$$d_i(x) = \min_{y \in A_i} \{\|x - y\|\} = \|x - a_i(x)\|, \tag{1}$$

where $\|\bullet\|$ denotes the Euclidean norm; i.e.,

$$\|x - a_i(x)\| = \left[\left(x_1 - a_{i1}(x)\right)^2 + \left(x_2 - a_{i2}(x)\right)^2 \right]^{\frac{1}{2}}, \tag{2}$$

$\forall\ x = (x_1, x_2), \quad a_i(x) = (a_{i1}(x),\ a_{i2}(x)) \in \Re^2$.

The single facility minisum problem with closest Euclidean distances takes the following form:

$$\min_x \ W(x) = \sum_{i=1}^{n} w_i d_i(x) = \sum_{i=1}^{n} w_i \|x - a_i(x)\|. \tag{3}$$

In order to derive a solution procedure for the model in (3), it will be useful to examine the partial derivatives of $d_i(x)$. Let $\text{int}(A_i)$ and B_i denote, respectively, the interior and boundary of $A_i, i = 1,...,n$. Let $\overline{a_i(x)}$ represent the "fixed" point $a_i(x)$. It is clear that $\partial(d_i(x))/\partial x_j = 0, \forall j$, if $x \in \text{int}(A_i)$, and $\partial(d_i(x))/\partial x_j$ is undefined for at least one direction *j* if $x \in B_i$. The following result provides an interesting relation when *x* is a point outside A_i.

Property 1 *Consider any* $x \notin A_i$. *Then* $\partial(d_i(x))/\partial x_j$ *is defined* $\forall j$*; furthermore, the first-order derivative is the same when* $a_i(x)$ *is replaced by the fixed point* $\overline{a_i(x)}$.

Proof: Without loss of generality, consider the partial derivative with respect to x_1. Referring to Figure 1, two possibilities are examined:

Case 1: $a_i(x)$ is on a 'smooth' part of B_i.

$$\frac{\partial(d_i(x))}{\partial x_1} = \frac{\partial}{\partial x_1}(\|x - a_i(x)\|)$$

$$= \frac{1}{\|x - a_i(x)\|}\left[(x_1 - a_{i1}(x))\left(1 - \frac{\partial a_{i1}(x)}{\partial x_1}\right) + (x_2 - a_{i2}(x))\left(-\frac{\partial a_{i2}(x)}{\partial x_1}\right)\right]$$

and therefore:

$$\frac{\partial(d_i(x))}{\partial x_1} = \frac{x_1 - a_{i1}(x)}{\|x - a_i(x)\|} - \frac{(x - a_i(x))}{\|x - a_i(x)\|} \bullet \left(\frac{\partial a_i(x)}{\partial x_1}\right) \tag{4}$$

Since the vector $\frac{\partial a_i(x)}{\partial x_1}$ is tangent to B_i, it follows that the inner product

$$(x - a_i(x)) \bullet \frac{\partial a_i(x)}{\partial x_1} = 0, \tag{5}$$

and hence,

$$\frac{\partial(d_i(x))}{\partial x_1} = \frac{(x_1 - a_{i1}(x))}{\|x - a_i(x)\|} = \frac{\partial\left(\|x - \overline{a_i(x)}\|\right)}{\partial x_1}. \tag{6}$$

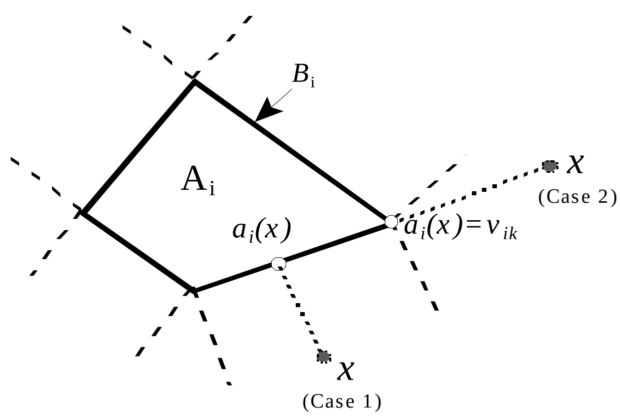

Figure 1: Calculating partial derivatives of $d_i(x)$

Case 2: $a_i(x)$ is a vertex (v_{ik}) of B_i.
If x is not on a perpendicular projection from v_{ik} (see Figure 1), then $\partial a_i(x)/\partial x_1 = 0$ (zero vector). Otherwise x is on a perpendicular projection, and either $\partial a_i(x)/\partial x_1^+$ and $\partial a_i(x)/\partial x_1^-$ are not equal, one being the zero vector and the other a tangent vector as in Case 1, or $\partial a_i(x)/\partial x_1$ is the zero vector if the perpendicular projection is parallel to the x_1 direction. Irrespectively,

$$(x - a_i(x)) \bullet \frac{\partial a_i(x)}{\partial x_1^+} = (x - a_i(x)) \bullet \frac{\partial a_i(x)}{\partial x_1^-} = 0,$$

and we conclude that equation (6) still applies.

The same argument may be made for the partial derivative with respect to x_2. (Note: the proof readily extends to N-dimensional space). Letting ∇ denote the gradient operator, we obtain

$$\nabla d_i(x) = \nabla \left\| x - \overline{a_i(x)} \right\|. \tag{7}$$

The above property is important for two reasons: first, it shows that $\partial(d_i(x))/\partial x_j$ exists, $\forall j$ and x external to A_i, even if the boundary of A_i is not smooth; second, these derivatives are calculated in the same way as if $a_i(x)$ were a fixed point. Our solution procedure will make use of this insight.

The basic idea behind the proposed algorithm is summarized as follows. Given an initial point x^0, we determine the closest point $a_i(x^0)$ for each demand area A_i. These $a_i(x^0)$ are then treated as fixed points $\overline{a_i(x^0)}$ replacing the respective areas A_i. In this way, the problem is converted to the standard form of the Fermat-Weber problem. This allows us to make use of the well-known Weiszfeld iterative procedure. One Weiszfeld iteration produces a new point x^1 with lower objective function value relative to the set $\{a_i(x^0);\ i = 1,...,n\}$. Using the new location x^1, we then recalculate the closest points to obtain $\{a_i(x^1);\ i = 1,...,n\}$, and a further improvement of the objective function. The whole process is repeated to provide a sequence of descent moves.

Special attention is required when an iterate lands in a demand area A_i. Since these A_i represent different customers or markets, we can reasonably assume that they do not overlap; i.e., $A_i \cap A_j = \phi, \forall i, j,\ i \neq j.$ Thus, any iterate will be located at most in one demand area. (Note, however, that the following solution procedure may be readily modified to handle overlapping demand areas.) Let the complete set $\{1,...,n\}$ be denoted by M, and let $M_i = M - \{i\},\ i = 1,...,n$.

Algorithm 1

Step 1: Select a starting point x^0 which is not on any boundary B_i. Set the iteration counter $q = 0$.

Step 2: Find $a_i(x^q)$, the closest point to x^q on A_i, $\forall\, i \in M$.

Step 3: If $x^q \in A_r$, for some $r \in M$, set $M' = M_r$; otherwise, $M' = M$. Perform a Weiszfeld iteration, treating $\{a_i(x^q), i \in M'\}$ as the given set of fixed points as follows:

$$x^{q+1} = \frac{\sum_{i \in M'} \frac{w_i a_i(x^q)}{\left\| x^q - a_i(x^q) \right\|}}{\sum_{i \in M'} \frac{w_i}{\left\| x^q - a_i(x^q) \right\|}}. \tag{8}$$

Step 4: {Ensuring the descent property} If $x^q \in A_r$ and $x^{q+1} \notin A_r$, calculate $\hat{W}(x^{q+1}) = \sum_{i \in M'} w_i \left\| x^{q+1} - a_i(x^q) \right\| + w_r \left\| x^{q+1} - b \right\|$, where b is the intersection point of the line segment (x^q, x^{q+1}) and the boundary B_r. (In case the line segment intersects B_r at more than one point, use the furthest such point from x^q as *b*). If $\hat{W}(x^{q+1}) > W(x^q)$, determine a point $\tilde{x}^{q+1}$ on the segment (b, x^{q+1}) such that $\hat{W}(\tilde{x}^{q+1}) < W(x^q)$, and set $x^{q+1} = \tilde{x}^{q+1}$.

Step 5: If $x^{q+1} \in B_t$, for some $t \in M$, relocate x^{q+1} to the mid-point of the segment (x^q, x^{q+1}) or to another convenient point on the segment, as long as it is not on any boundary B_i.

Step 6: If $\left\| x^{q+1} - x^q \right\| < \delta$ (or some other stopping criterion is satisfied), STOP; otherwise, $q \leftarrow q + 1$, and return to step 2. □

Further explanation of the algorithm is required. Suppose in the Weiszfeld iteration in step 3 that $M' = M$ (or $x^q \notin A_i, \forall i$). Using the descent property of the Weiszfeld procedure, we have for $x^{q+1} \neq x^q$:

$$W(x^q) = \sum_{i \in M} w_i \left\| x^q - a_i(x^q) \right\| > \sum_{i \in M} w_i \left\| x^{q+1} - a_i(x^q) \right\| = \hat{W}(x^{q+1}) \ . \tag{9}$$

Using the definition of closest distance, it follows that the auxiliary function,

$$\hat{W}(x) \geq \sum_{i \in M} w_i d_i(x) = W(x), \ \forall x. \tag{10}$$

Comparing (9) and (10), we conclude that

$$W(x^{q+1}) < W(x^q). \tag{11}$$

Thus we obtain a descent move if the current location x^q is external to all the A_i, and $x^{q+1} \neq x^q$.

Now suppose x^q is an internal point of some demand area A_r .Then the cost component $w_r d_r(x) = 0$ in a neighbourhood of x^q, and customer A_r effectively drops out of the picture. (Also note that including r in the index set leads to division by zero in (8), since $\left\| x^q - a_r(x^q) \right\| = \left\| x^q - x^q \right\| = 0$. This duplicates the problem which occurs in the standard Weiszfeld procedure when an iterate coincides with one of the fixed points.) We accommodate the case where an x^q lands in A_r by removing r from the index set (M') in (8). The associated Weiszfeld iteration results in a descent move relative to the subset of fixed points $\{a_i(x^q), i \in M_r\}$. This is fine as long as x^{q+1} remains in A_r, but if x^{q+1} lands outside of A_r, the descent property of the algorithm may be violated since the cost component for A_r is no longer equal to zero.

The algorithm resolves this difficulty in step 4. Consider the line segment (x^q, x^{q+1}), and again let b denote the last intersection with the boundary B_r. For the points $x \in (x^q, b]$ that are internal to A_r, it is clear that the descent property holds. Thus, a point $\tilde{x}^{q+1}$ external to A_r on the segment (b, x^{q+1}) can always be found which also allows a descent move. A suitable $\tilde{x}^{q+1}$ may be obtained by conducting a simple one-dimensional search as required in step 4 as follows:

Choose the mid-point y_M of the segment (b, x^{q+1}).

If $\sum_{i \in M_r} w_i \left\| y_M - a_i(x^q) \right\| + w_r \left\| y_M - b \right\| < W(x^q)$, set $\tilde{x}^{q+1} = y_M$; else continue to divide (b, y_M) to obtain a new mid-point y_M, until the preceding condition is satisfied.

Step 5 handles the unlikely event of an iterate landing on a boundary of one of the demand areas. This is analogous to the case where an iterate coincides with a fixed point in the standard Weiszfeld procedure; except now the iteration function in (8) is discontinuous at the points along any boundary B_t. To see this, we note that $M' = M_t$ on one side of B_t (the interior of A_t) while $M' = M$ on the other side (the exterior of A_t). Thus, the iteration function has two limiting values at a boundary point depending on the direction of approach. To circumvent this problem, the iterate x^{q+1} in step 5 is simply relocated to a more 'convenient' point on the line segment (x^q, x^{q+1}). Assuming that at most one intersection occurs with B_r whenever step 4 is invoked, any such point provides a descent move, as shown in the next result.

Property 2 $W(x) < W(x^q), \ \forall x \in \left(x^q, x^{q+1}\right]$.

Proof: First suppose that x^q is an external point ($x^q \notin A_i, \forall i$). It is well known (e.g., see Love et al. (1988)) that when the customers are fixed demand points, the weighted-sum objective function is convex in x. Thus we have that $\hat{W}(x)$ is a convex function of x.

It now follows from (9) that $\hat{W}(x) < W(x^q), \forall x \in \left(x^q, x^{q+1}\right]$. Furthermore, from (10), $\hat{W}(x)$ is an upper bound on $W(x)$.

If $x^q \in A_r$ for some *r*, define $\hat{W}_r(x) = \sum_{i \in M_r} w_i \left\| x - a_i(x^q) \right\|$, and let

$$\hat{W}(x) = \begin{cases} \hat{W}_r(x), \ \forall x \in A_r \cap \left[x^q, x^{q+1}\right], \\ \hat{W}_r(x) + w_r \left\| x - b \right\|, \ \forall x \in \bar{A}_r \cap \left[x^q, x^{q+1}\right], \end{cases}$$

where $\bar{A}_r$ is the complement of A_r. It follows that $\hat{W}(x)$ is convex on $\left[x^q, x^{q+1}\right]$, and similarly as above, $W(x) \le \hat{W}(x) < W(x^q)$.

The stopping criterion is given in step 6 of the algorithm. If two successive iterates are virtually identical, or some other condition such as an upper limit on execution time is satisfied, the algorithm is terminated.

3 Convergence Properties

Consider the iteration function given in (8),

$$T(x) = \frac{\sum_{i \in M'} \frac{w_i a_i(x)}{\left\| x - a_i(x) \right\|}}{\sum_{i \in M'} \frac{w_i}{\left\| x - a_i(x) \right\|}}. \tag{12}$$

Letting $B = \bigcup_{i=1}^{n} B_i$, it readily follows that $T(x)$ is continuous and infinitely differentiable, $\forall x \notin B$, while T is discontinuous, $\forall x \in B$. Hence, B comprises the set of singular points of the mapping T.

Let $Q(x^0) = \left\{x^0, x^1, x^2, \ldots\right\}$ denote the sequence generated by Algorithm 1 with initial point x^0. Since any iterate that lands on a boundary is conveniently relocated in step 5 of the algorithm, it follows that all points in $Q(x^0)$ are non-singular. Using the terminology in Brimberg and Love

(1993), we may thus say that $Q(x^0)$ is a regular sequence, $\forall x^0 \notin B$. Additional properties of the sequence are given below.

Property 3 *Let* $ch\{A_1,...,A_n\}$ *denote the smallest convex hull containing the customers* $A_1,...,A_n$. *Let* $x^0 \in \mathrm{ch}\{A_1,...,A_n\}$*; then* $Q(x^0) \subset \mathrm{ch}\{A_1,...,A_n\}$.

Proof: Referring to (8), it is well known that x^{q+1} lies within the convex hull of $\{a_i(x^q), i \in M'\}$ (e.g. see the seminal paper by Kuhn (1973)). Thus, x^q, x^{q+1}, and all points along the line segment (x^q, x^{q+1}) lie within $ch\{A_1,\ldots,A_n\}$, $q = 0,1,2,\ldots$.

We conclude from the above result that, in general, except for possibly the first few iterates, all points in $Q(x^0)$ belong to $ch\{A_1,...,A_n\}$.

Property 4 *Let* x^* *denote a stationary point of the objective function* $(\partial W(x^*)/\partial x_j = 0, \forall j)$. *If* $x^q = x^*$, *then* $x^{q+1} = x^*$, *as well as all subsequent iterations. If* $x^{q+1} = x^q$, *then* $x^q = x^*$.

Proof: Since x^* is a stationary (differentiable) point, this implies $x^* \notin B$. We have that

$$\frac{\partial W(x^*)}{\partial x_j} = \frac{\partial W(x^q)}{\partial x_j} = 0, \quad \forall j.$$

Using the fundamental insight in Property 1, that the first-order derivatives are the same when the $a_i(x^q)$ are treated as fixed points, it follows that $x^{q+1} = T(x^q) = x^q$, as well as all subsequent iterations. Similarly, if $x^{q+1} = x^q$ in the regular sequence $Q(x^0)$, it must be that $T(x^q) = x^q$, and thus $x^q = x^*$.

Property 5 *(Descent property) If* $x^{q+1} \neq x^q$, *then* $W(x^{q+1}) < W(x^q)$.

Proof: Suppose x^q is an external point and $x^{q+1} = T(x^q)$ without relocation in step 5; we see from (9), (10), and (11) that $W(x^{q+1}) < W(x^q)$. If $x^q \in A_r$ for some r and $x^{q+1} = T(x^q)$, then by step 4, a descent move has occurred; otherwise x^{q+1} is relocated on the segment (b, x^{q+1}) to guarantee the descent move using the upper bound $\hat{W}(x)$. If relocation occurs in step 5, any point on the segment (x^q, x^{q+1}) will provide a descent move by Property 2.

Property 6 $Q(x^0)$ *converges to a single attraction point.*

Proof: First, consider the case where $x^{q+1} = x^q$ for some q. Since $x^q \notin B$, we must have $x^{q+1} = T(x^q) = x^q$. Hence, x^q is a stationary point, and all subsequent iterates remain at this point.

Otherwise, $x^{q+1} \neq x^q, \forall q$. Since $Q(x^0)$ belongs in a compact set (Property3), it follows by the Bolzano-Weierstrasz theorem that there exists at least one point P with subsequence x^{r_l} such that $\lim_{l\to\infty} x^{r_l} = P$.

Consider two subsequences x^{r_l} and x^{t_l} with respective attraction points P_1 and P_2 ($P_1 \neq P_2$).We have $lim_{l\to\infty} W(x^{r_l}) = W(P_1)$ and $lim_{l\to\infty} W(x^{t_l}) = W(P_2)$. It must be concluded that $W(P_1) = W(P_2)$; otherwise, the descent property over the entire sequence $Q(x^0)$ would be violated (see Property 5). If $P_1 \notin B$, $\lim T(x^{r_l})$ exists and is given by some point $P_1' \neq P_1$. If $P_1 \in B$, the subsequence x^{r_l} may be restricted to either only external or internal type points, and it may be shown (see for example Brimberg and Love, 1993, for a fixed-point analogy) once again that $lim T(x^{r_l}) = P_1'$. We conclude that $W(P_1') < W(P_1)$ (where P_1' may have been relocated in step 4), the vicinity of P_1 will not be revisited, and the existence of subsequence x^{r_l} is contradicted.

When the customer areas are allowed to assume general shapes, the objective function is generally nonconvex, and as a result, may contain several local minima. Convergence of the algorithm to the global solution cannot be guaranteed. On the other hand, if each A_i is given by a convex region, the convexity of $W(x)$ readily follows (e.g., see Brimberg and Wesolowsky 2000a). Under this assumption, we obtain the following main result.

Theorem 1 *Sufficient conditions for the convergence of* $Q(x^0)$ *to an optimal solution are*

(i) $W(x)$ *is a convex function,*

(ii) $\lim_{q\to\infty} x^q \notin B$.

Proof: We know from Property 6 that $Q(x^0)$ is convergent; thus, $\lim_{q\to\infty} x^q$ exists and is given by some unique attraction point P. Since $P \notin B$, it follows from Property 4 that if $x^q = x^{q+1}$ (and all subsequent iterations), then P is a stationary point. Otherwise, $x^q \neq x^{q+1}, \forall q$, and by continuity of the iteration function,

$$P = \lim_{q\to\infty} x^q = \lim_{q\to\infty} x^{q+1} = \lim_{q\to\infty} T(x^q) = T(P). \tag{13}$$

Again, P must be a stationary point. The convexity of W implies that $P = x^*$ is an optimal solution.

The main advantage of Algorithm 1 lies in its simplicity. By adapting the well-known Weiszfeld procedure we are able to avoid the messy calculation of descent directions and step sizes. However, there is an interesting difference to be noted. In the standard (fixed point) problem, the sequence generated by the Wieszfeld procedure cannot converge to a non-optimal fixed point. (See Kuhn (1973) and Brimberg and Love (1993) for two different proofs.) In our extended model with demand areas and closest distances, the sequence generated by Algorithm 1 may converge to a non-optimal point on some boundary B_i. This is because the replacement of demand areas by fixed points $\left(\overline{a_i(x^q)}\right)$ in step 3 significantly distorts the objective function external to and in the close vicinity of any A_i. Thus a fixed point $\left(\overline{a_i(x^q)}\right)$ may be optimal (or near optimal) in the standard problem but not so in the extended model.

When convergence to a boundary point is detected, additional steps must be taken as outlined below:

(i) determine if the sequence is converging to an optimal solution by examining directional derivatives;

(ii) if not, move along the steepest descent direction to a better solution; and

(iii) resume Algorithm 1 using this solution as the new starting point.

As we can see, the calculation of descent directions and step sizes may not be eliminated altogether.

The steps of Algorithm 1 are illustrated in the following numerical example. The five unit squares in Figure 2 represent customers with equal demands $\left(w_i = 1, i = 1, \ldots, 5\right)$.

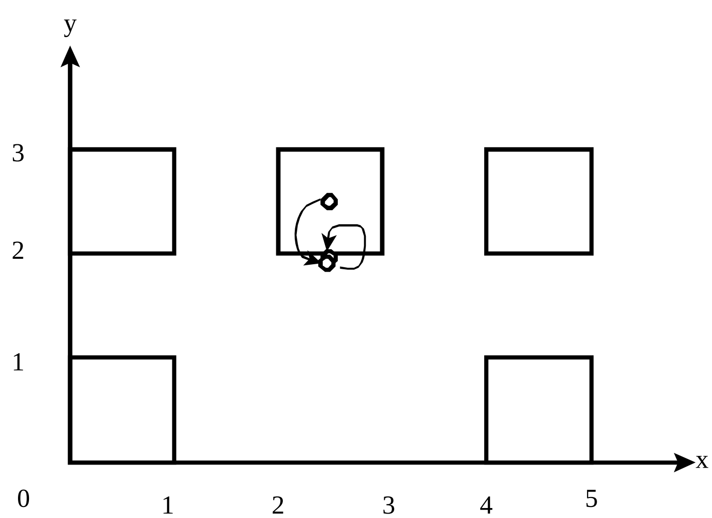

Figure 2: Steps in algorithm 1

The initial position of the facility (x^0) is taken as (2.5, 2.5), which is in the interior of an area. The closest points on the other areas are (1, 1), (1, 2.5), (4, 2.5), and (4,1). A Weiszfeld iteration then gives the point (2.5,1.893), which is exterior to all areas. Since $\hat{W}(x^1) = 6.8347 < W(x^0) = 7.2426$, a relocation of x^1 in step 4 is not required. The closest points are now (1, 1), (1, 2), (2.5, 2), (4, 2), and (4,1). Weiszfeld iterations subsequently converge to (2.5, 1.948).

4 Area Facility

Suppose that the new facility is given by an area of fixed dimensions and orientation. Let $S(x)$ denote the new facility, where "x" is the position of a specified center point of the associated area. We now define the closest distance between the facility and customer *i* as:

$$d_i(x) = \min_{z \in S(x), y \in A_i} \{\|z - y\|\} = \|c_i(x) - a_i(x)\|, \qquad (14)$$

where $c_i(x)$ and $a_i(x)$ are the closest points in $S(x)$ and A_i, respectively (see Figure 3).

Introducing

$$a_i'(x) = a_i(x) + (x - c_i(x)), \forall i, \qquad (15)$$

allows us to rewrite the closest distance in terms of center point *x* and a translation of $a_i(x)$ to $a_i'(x)$ (see Figure 3):

$$d_i(x) = \left\| x - a_i'(x) \right\|. \qquad (16)$$

The objective function becomes

$$W(x) = \sum_{i=1}^{n} w_i \|c_i(x) - a_i(x)\| = \sum_{i=1}^{n} w_i \left\| x - a_i'(x) \right\|, \qquad (17)$$

the last expression being identical in form as for a point facility "x".

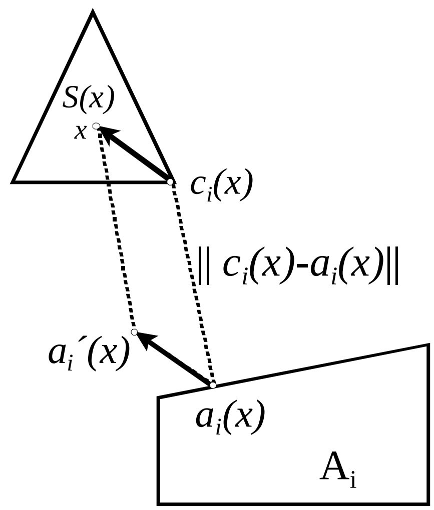

Figure 3: Closest distance with an area facility

The fundamental result given by Property 1 may be replicated for an area facility. Details of the proof are omitted, since a similar reasoning applies as for a point facility after replacing $a_i(x)$ by $a_i'(x)$.

Property 7 *Consider any* $x \notin A_i' = \{y \mid d_i(y) = 0\}$. *Then* $\partial(d_i(x))/\partial x_j$ *is defined* $\forall j$ *; furthermore, the first-order derivative is the same when* $a_i'(x)$ *is replaced by the fixed point* $\overline{a_i'(x)}$.

Let us examine the set A_i' in more detail. We may define an associated closest distance function,

$$d_i'(x) = \min_{y \in A_i'} \|x - y\| = \left\|x - a_i''(x)\right\|, \tag{18}$$

where $a_i''(x)$ is the closest point to *x* in A_i'.

Property 8 *The two distance functions,* $d_i(x)$ *and* $d_i'(x)$, *are one and the same.*

Proof: Clearly, $a_i'(x)$ and $a_i''(x)$ both belong to A_i'. Thus, $d_i'(x) < d_i(x)$ implies that $a_i'(x)$ is not a closest point on A_i' to x. In turn, this implies that $\|c_i(x) - a_i(x)\|$ can not be the closest distance between $S(x)$ and A_i. We conclude that $d_i'(x) = d_i(x), \forall x$.

The set A_i' is obtained by rotating $S(x)$ all around A_i such that $S(x)$ is always outside and just touching A_i. (See the illustration in Figure 4). The trajectory of the center *(x)* traces the boundary of A_i', which also corresponds to $\left\{a_i'(x) \mid x \notin A_i'\right\}$. Clearly, $A_i' \supset A_i$; also, if $S(x)$ and A_i are both convex areas (polygons), A_i' will itself be a convex area (polygon).

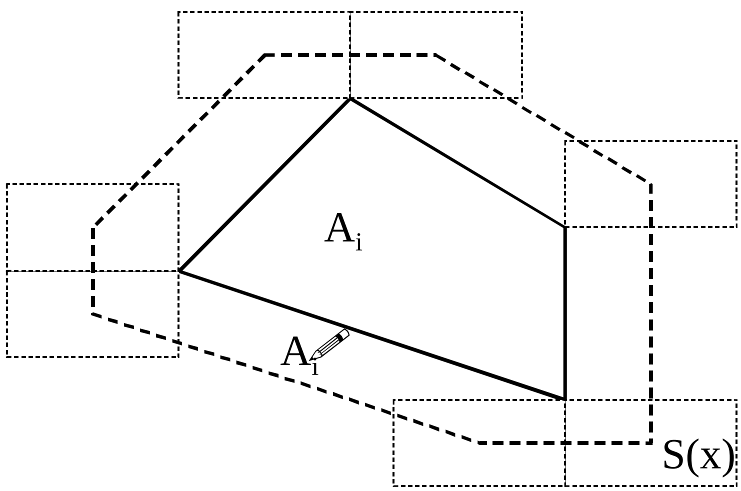

Figure 4: Enlarged area A_i'

From Property 8, it follows that by replacing each A_i with A_i', the original problem is converted into an equivalent point facility problem. The solution approach would then be to use Algorithm 1 with $\left\{a_i'(x)\right\}$ taking the place of $\{a_i(x)\}$ in the Weiszfeld iteration in step 3. We would also have to address the possibility of overlapping areas among the A_i', and how to handle iterates that land in and depart from such areas (steps 3 and 4). Coincidence of an iterate

x^q with the boundary of an A_i' (step 5) implies in the original problem that $S(x^q)$ is externally just touching the A_i.

The solution procedure is outlined below. We let B_i' denote the boundary of $A_i', \forall i \in M$; $J := \left\{ j \mid x^q \in A_j' \right\}$ for any iteration q, and $J' := \left\{ j \in J \mid x^{q+1} \notin A_j' \right\}$. Let $\{b_1, ..., b_\tau\}$ represent the set of intersection points of (x^q, x^{q+1}) with boundaries $B_j', j \in J'$, where the b_k are sequenced in order from x^q to x^{q+1}, b_1 being the closest intersection point from x^q. For simplicity, we assume that multiple intersection points with the same boundary cannot occur, as is the case when the A_j' are convex areas. Otherwise, the procedure is readily modified by incorporating a more detailed bookkeeping of exit and re-entry points of the $A_j', \forall j \in J$. Finally, $[k] :=$ the index of the boundary containing b_k (i.e., $b_k \in B_{[k]}'$, $k = 1, ..., \tau$).

Algorithm 2 {Area Facility}

Step 1: Select a starting point x^0 which is not on any B_i'. Set $q = 0$.

Step 2: Find $a_i(x^q)$, $c_i(x^q)$ and $a_i'(x^q) = a_i(x^q) + x^q - c_i(x^q), \forall i \in M$.

Step 3: Determine $M' = M - J$. Perform a Weiszfeld iteration, treating $\left\{ a_i'(x^q), i \in M' \right\}$ as the given set of fixed points as follows:

$$x^{q+1} = \frac{\sum_{i \in M'} \frac{w_i a_i'(x^q)}{\left\| x^q - a_i'(x^q) \right\|}}{\sum_{i \in M'} \frac{w_i}{\left\| x^q - a_i'(x^q) \right\|}}. \tag{19}$$

Step 4: {Ensuring the descent property} If $J = \phi$, or $J' = \phi$, go to step 5. Otherwise, calculate $\hat{W}(x^{q+1}) = \sum_{i \in M'} w_i \left\| x^{q+1} - a_i(x^q) \right\| + \sum_{k=1}^{\tau} w_{[k]} \left\| x^{q+1} - b_k \right\|$.

If $\hat{W}(x^{q+1}) > W(x^q)$, find $\hat{W}_{t^*}(b_{t^*}) = \min_{1 \le t \le \tau} \{\hat{W}_t(b_t)\}$, where $\hat{W}_t(x) = \sum_{i \in M'} w_i \|x - a_i(x^q)\| + \sum_{k=1}^{t} w_{[k]} \|x - b_k\|, t = 1, \ldots, \tau$. Determine a point $\tilde{x}^{q+1}$ on the segment (b_{t^*}, b_{t^*+1}), where $b_{t^*+1} = x^{q+1}$ if $t^* = \tau$, such that $\hat{W}_{t^*}(\tilde{x}^{q+1}) < W(x^q)$, and set $x^{q+1} = \tilde{x}^{q+1}$.

Step 5: If $x^{q+1} \in B_t'$, for some $t \in M$, relocate x^{q+1} to the mid-point of the segment (x^q, x^{q+1}) or another convenient point on the segment, as long as it is not on any boundary B_i'.

Step 6: If $\|x^{q+1} - x^q\| < \delta$ (or some other stopping criterion is satisfied), STOP; otherwise, $q \leftarrow q+1$, and return to step 2. □

It is easy to see that if the facility in Figure 2 were a square with side .5, the algorithm would give a solution of (2.5,1.75)

The convergence properties are readily extended from the point facility to the area facility. Thus, we obtain an analogous result as follows.

Theorem 2 *Given that* $A_i, i = 1, \ldots, n,$ *and* $S(x)$ *are convex areas, and that* $\lim_{q \to \infty} x^q \notin B' = \bigcup_{i=1}^{n} B_i'$, *the sequence generated by Algorithm 2 converges to an optimal solution.*

5 Conclusions

A single facility minisum location problem is formulated, where the set of customers and even the new facility may be represented as areas on the plane $\Re^2$. The novelty is that distance is measured as the minimum Euclidean distance between any point in the customer region and any point in the facility. Properties of the model are investigated which provide an interesting link with the standard Weber problem. An efficient solution procedure is proposed that exploits this insight. Future research may extend the model and algorithms to other distance metrics such as the l_p norm and multi-facilities.

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