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# Monetary Policy and Interest Rate Caps: A Regime-Shift Approach

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**Abstract**

In this paper, we derive and empirically test a regime-shifting dynamic term structure model for pricing interest rate caps. The central state variables are the target rate of the Federal Reserve and its latent regime in which it fluctuates. These state variables are driven by a discrete time inhomogeneous Markov chain that captures the timing of FOMC meetings. The Fed Funds rate and the slope of the Libor-swap term structure complete the set of state variables. Their dynamics exhibit regime-shifts based on the latent regime of the target rate in both the physical and risk-neutral measures. The modelling approach for pricing at-the-money cap prices leads to an exponential affine analytical form where the regime-shift risk is priced. Allowing for this feature significantly helps the model to empirically match cap prices. The empirical analysis also indicates that the stable regime where the target rate is frequently at inertia is a state where market operators exhibit a higher level of risk aversion. It is also in this regime that the term structure of the mean Black implied volatilities is the highest and where cap price changes are the most sensitive to jumps in the target rate.

**Key Words:** Target rate, Regime-shift, Markov chain, Risk-neutral valuation, Option caps.

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# 1 Introduction

Bonds and interest rate swaps transacted on financial markets share a common attribute: both their prices are influenced by movements in macro factors. Every day, market operators monitor their movements in an attempt to understand the mechanics at works in fixed income markets. One of the most important macro movers on these markets is unquestionably the US Federal Reserve's target rate. It is undeniable that jumps in the target rate uncovered at FOMC meetings attract substantial coverage. To shed light on the interplay between bond yield curves and the monetary policy, studies such as Chun (2011), Ang, Boivin, Dong and Loo-Kung (2010), and Piazzesi (2005) rely on no-arbitrage dynamic term structure models (DTSM) where macro factors proxy the state variables in order to capture different aspects of the monetary policy.[1,2] From the perspective of model fitting, the compromise entailed by a factor structure (partially) based on observable factors is obvious: extra economic interpretation is gained at the expense of an increasing difficulty fitting the model since latent variables are devised to help DTSM to match the yield curve. Looking at these facts and considering interest rate options, we make two observations. First, Piazzesi (2005) formally demonstrates how the release of information about target rate movements at FOMC meetings drive the yield curve. Second, interest rate caps and floors, as portfolios of European call and put options on Libor rates, are influenced by the yield curve dynamics (Trolle and Schwartz (2009), Jarrow, Li and Zhao (2007) and Han (2007) for example). From these two conjectures emerge a simple intuitive question that is the main goal of this study: How do interest rate cap prices react in a no-arbitrage framework, to discrete jumps in the target rate occurring at FOMC meetings? Our paper makes a theoretical as well as an empirical contribution by devising an option pricing model for evaluating at-the-money interest rate caps, which explicitly factors the sequential timing of FOMC meetings into the pricing.

The modelling approach derived herein is at the intersection of two different strands of research with respect to pricing and analyzing interest rate cap prices. The first strand uses the no-abritrage DTSM platform combined with latent state variables to price interest rate options. Successful pricing of at-the-money interest rate cap prices (and swaption prices) along these lines are found, for instance, in Almeida, Graveline and Joslin (2011), Joslin (2010), Almeida and Vicente (2009), and Bikbov and Chernov (2010).[3] Simultaneous pricing of interest rate cap and swaption prices is usually based on one of these three families of models: Heath, Jarrow and Morton (1992) models, Libor market models (Brace, Gatarek and Musiela (1997), amongst others) or affine DTSM (mainly based on the important contributions of Duffie and Kan (1996), Dai and Singleton (2000) and Dai and Singleton (2002)). These approaches constitute the basis over which the issue of unspanned stochastic volatility (USV) is typically debated (Collin-Dufresne and Golstein (2002)). This controversy relates to market completeness and question whether bond and swap prices depend on the same set of state variables as that of interest rate options.[4] However, a broader view of this issue suggests that the debate is provoked, at least partially, byf the fact that virtually all models invariably resort to latent factors. One can argue that the introduction of observable factors into a modelling approach would help to identify the relevant factor structure. Our study constitutes an effort in this direction although the USV issue is not the focus of this paper.

The second strand of research, as exemplified by Li and Zhao (2009) and Beber and Brandt (2006), investigates different features of interest rate options via the extraction of option-implied state price densities of the underlying interest rate instruments. Changes in these densities around key macroeconomic announcements lead to an understanding of how market operators shape their risk preferences in different economic environments. We depart from this empirically driven technology by explicitly considering a macrofinance

[1] Other studies that include observable macroeconomic fundamentals in the factor structure combined with DTSM include Bikbov and Chernov (2010), Ang, Dong and Piazzesi (2007), and Ang and Piazzesi (2004), among others.

[2] More information on the architecture of DTSM is found in Piazzesi (2010), Nawalkha, Beliaeva and Soto (2007), Singleton (2006) and Dai and Singleton (2003) which provide excellent surveys.

[3] Contrasting findings are found by Jagannathan, Kaplin and Sun (2003) that rely on a three factor CIR model. They conclude that the model is misspecified since enlarging the sample, initially composed of bond yields, with interest rate options does not materially change the magnitude of the estimated parameters while the fitting errors for short maturity interest rate caps and swaptions increase with the number of state variables considered. Li and Zhao (2006) conclude that their quadratic DTSM fails to capture some fundamental attributes of interest rate cap smiles.

[4] An incomplete list of studies that support (reject) the USV includes Anderson and Benzoni (2010), Li and Zhao (2006), and Heidari and Wu (2003) (Joslin (2010), and Fan, Gupta and Ritchken (2003)).

state variable for option pricing.[5] It is hoped that the our DTSM approach offers a different perspective of how monetary policy actions drive cap prices.

The first layer in our model is borrowed from Piazzesi (2005) where the target rate is a state variable whose jumps occur only at FOMC meetings. However, unlike Piazzesi (2005) who captures the target rate jumps via a conditional dual-jump process,[6] we resort to a conditional time-inhomogeneous Markov chain where movements are only possible on meeting dates as proposed in Ferland, Gauthier and Lalancette (2010). More precisely, it is observed that the Fed fundamentally increases (decreases) rates to slow (stimulate) the economic growth in a fashion where target rate movements clearly exhibit a trending feature as previously documented by Clarida, Gali and Gertler (2000) (Castelnuovo (2007)) for US (Euro area) data. Some researchers interpret this phenomenon as a reflection of monetary policy gradualism.[7] For the purpose of option pricing, we consider these trends as being latent regimes in which interest rate cap prices exhibit different characteristics. We apply the regime-shift approach developed in Ferland, Gauthier and Lalancette (2010) where conditioning of the Markov chain emerges from different target rate latent regimes. Given this orientation, our model falls within the category of regime-switching DTSM pioneered by Bansal and Zhou (2002) and Bansal, Tauchen and Zhou (2004) and further improved by Dai, Singleton and Yang (2007), and Ang, Bekaert, and Wei (2008).

The seminal contribution of Litterman and Scheinkman (1991) found that three factors drive virtually all movements in the variance-covariance of yield changes although the first two factors often account for more than 97% of the variability.[8] Motivated by this empirical fact and the desire to generate a parsimonious model, the first two factors driving the Libor-swap yield curve are respectively proxied by the Fed Funds rate (FFr) and the slope of the Libor-swap yield curve. The joint probability distribution of these two state variables is allowed to shift across the different latent regimes of the target rate. Based on this framework, an analytical approximation of cap prices is derived in the form of an exponential affine function of the state variables.

Our models has several advantages. First, the analytical pricing approximation allows for an estimation process that fully exploits the cross-sectional and time series properties of interest rate cap prices. Second, the regime-dependant risk-neutral pricing of cap prices makes it possible to differentiate their dynamics across different regimes of monetary activities. Furthermore, in the spirit of Dai, Singleton and Yang (2007), the option pricing model allows for the market pricing of regime-shift risk as different transition matrices are observed under the data-generating and risk-neutral measures. Third, the inclusion of observable state variables in the model opens the door to a two-step estimation strategy (as in Chun (2011) and Ang and Piazzesi (2004)) where the data-generating parameters are independently estimated from their risk-neutral counterparts. Given the model's complexity and the number of parameters involved, proceeding with the estimation in two steps should entail more benefits than costs.

The remainder of the paper is organized as follows: In the next section, the data required to conduct the empirical tests are introduced; the modelling framework is discussed in the third section while the two-step estimation approach is discussed in the fourth section; in the fifth section, the empirical findings are presented; and concluding remarks are offered in the last section.

[5]A good example of studies that perform option pricing based on macrofactors is provided by Dorion (2012) who includes that ADS business condition index in the option pricing model.

[6]See also Fontaine (2010).

[7]For instance, Mishkin (1999) motivates this practice by the needs of monetary authorities to avoid credibility problems triggered by sudden and large reverses in the target rate course. Bullard and Mitra (2002) deviced a theoretical model where monetary inertia enhances the convergence towards rational expectation equilibrium. However, Rudebush (2002) and, more recently, Consolo and Favero (2009) reject the monetary gradualism interpretation. More recently, Ang, Boivin, Dong and Loo-Kung (2010) contend that monetary authorities display policy shifts in the conduct of the monetary policy. They observe that the monetary policy loading on expected inflation changes substantially over time suggesting that the short-term rate may react quite differently to movements in that state variable. Abdymomunov and Kang (2011) further observe that the Fed's actions can be categorized between more or less active monetary policy regimes with a corresponding more or less volatile yield curve.

[8]See for instance Ang, Piazzesi and Wei (2006) and Knez, Litterman and Scheinkman (1994).

# 2 Data

Standard data sets are used in this study. FOMC meetings are mostly held on Tuesday, but sometimes on Wednesdays or Thursday. They usually last for one day and occasionally for two days though never on Thursday during the sample period chosen herein. On meeting weeks, it is assumed that monetary actions revealed at FOMC meetings are fully disseminated on markets by the end of Thursday. Thus, all data required to conduct our analysis are extracted on the Fridays of each week. The observation period begins on the first week of August 1998 and ends on the last week of December 2007 for a total of 491 data points.[9] The sample of interest rate option data is composed of US, at-the-money, Black (1976) (henceforth, Black) implied flat-volatilities of interest rate cap[10] for maturities of 2, 3, 4 and 5 years quoted on Bloomberg at the end of each Friday from August 1998 to December 2007. These implied volatilities represent averages between the best bid and ask volatility quotes of operators active in the US interest rate cap market. The time series of Black implied-flat volatilities and the corresponding interest rate cap prices are presented in Figure 1. Although in- and out-of-the-money interest cap prices are also available on Bloomberg, the more ambitious goal of a full parametrization of the option cap smile is not pursued in this study. Using a multifactor Heath, Jarrow and Morton (1992), Jarrow, Li and Zhao (2007) observe that the analysis of implied volatilities located along the smile imposes an additional level of complexity since they are driven by forces different from that of their at-the-money counterparts. While their model is more flexible than ours, as it is based on latent variables, they report that it does not fully explain the data in a satisfactory matter. It is unlikely that our DTSM based on observable state variables could match that performance. This view is further supported by the findings of Li and Zhao (2006) where multifactor quadratic affine DTSMs fail to properly hedge interest rate cap options along the smile. The challenge imposed by pricing the cap smile may result from the greater liquidity exhibited by at-the-money interest rate cap options.

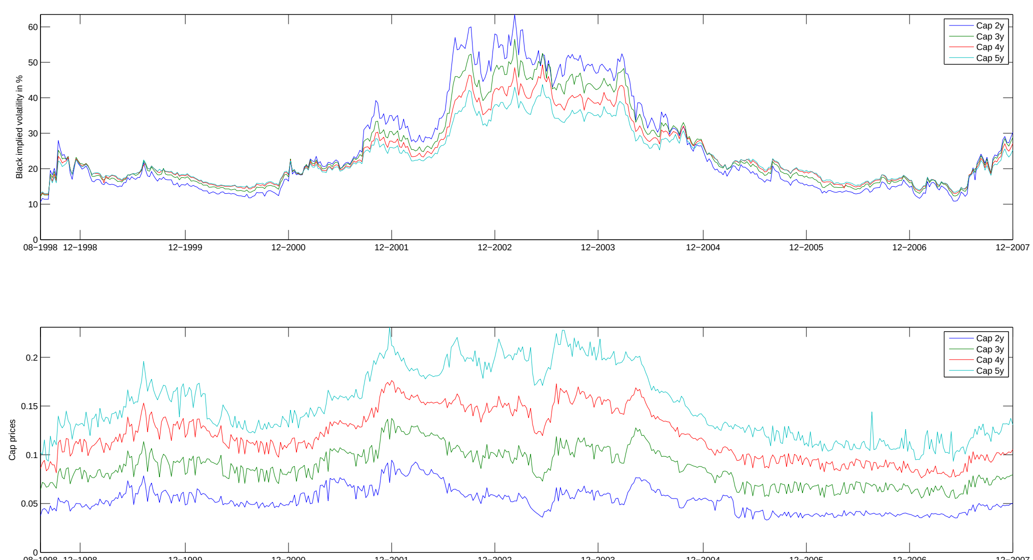


Figure 1: Interest Rate Cap Implied Volatilities and Prices

In the spirit of Joslin (2010), Han (2007) and Longstaff, Santa-Clara and Schwartz (2001), the one-year interest rate caps is also discarded from the sample. Finally, caps with maturities of 7 and 10 years are not considered for two reasons. First, given the computer resources required for the estimation of the model,

[9] Bikbov and Chernov (2010) raise a concern with respect to the number of data points available whenever interest rate options are involved. To the best of our knowledge, the length of the interest rate cap time series used herein exceeds those found in other studies using DTSM to price interest rate options.

[10] A cap is priced at a *flat* implied volatility when all included caplets are priced using an identical volatility.

it becomes unmanageable using a Markov chain technology, for maturities longer than 5-years since the sequence of FOMC meetings must be preserved for the entire maturity of the interest rate caplets.[11] Second, Piazzesi (2005) reports that jumps in the target rate impact the Libor-swap yield curve for maturities up to approximately 5 years with a slowly decaying influence that eventually dies out.

We point out that although the interest rate caps with maturities of 2 to 5 years constitute only a segment of the term structure of cap implied volatilities, it nevertheless represents a cross section of 19 interest rate caplets intermingled by arbitrage relationships. From that point of view, the number of securities comprised in this cross section far exceeds that of many DTSM studies based on bond yields. Also none of these securities is segregated in the estimation process for proxying implied latent state variables. Instead, the model allows for the more plausible assumption that all caplet prices are observed with errors.

To recover the interest rate cap prices by means of the Black model from the implied volatilities, zero-coupon bonds on Libor rates with sequential quarterly maturities up to 5 years are needed. We resort to the Nelson and Seigel (2006) exponential component framework to extract them from Libor and swap rates. The Nelson and Seigel (2006) functional form is estimated for each week of the sample based on the Libor rates with maturities of 3-, 6- and 12-months and on swap rates with maturities of 2, 3, 4 and 5 years. These rates are obtained from the Bloomberg system every Friday of the samples period.

Finally, the target rate series and its companion rate, the effective FFr, are obtained from the Federal Reserve website.[12,13] Since the Federal Reserve announces the new value of the target rate at the end of each meeting and makes this information public, the target rate time series is built by accessing this published information from the Fed's website.

# 3 Model for interest rate caps under FOMC meetings

This section introduces the modelling approach. In the first section, the factor dynamics are modelled under both $\mathbb{P}$, the data generating measure, and $\mathbb{Q}$, the risk-neutral measure while the pricing of a general European contingent claim on zero coupon bond, and of interest rate caps is examined in the remaining subsections.

## 3.1 Dynamics of the state variables

As mentioned before, a level and slope factors seem are sufficient to capture most of Libor-swap curve's variability. Thus, we begin by a discussion on how these factors are proxied by observable variables. Empirical implementation of DTSM imposes working assumptions pertaining to the nature of the short-term rate. Sometimes, it consists of observable state variables such as the 1-month or 3-month T-bill rate or, alternatively, a latent variable is labelled as the short rate. For example, Piazzesi (2005) and Balduzzi, Das and Foresi (1998) associate the short rate with a latent variable whose dynamics must revert directly or indirectly to the target rate. Given our desire to attribute an unambiguous economical content to the state variables, the short rate is instead proxied by the FFr. Not only is it a rate that matches most closely the concept of instantaneous spot rate, but it also constitutes the Libor-swap curve's natural short-term extremity. Following Ang, Piazzesi and Wei (2006) and many others, the term structure's slope is proxied by the difference between the 5-year swap rate and the 3-month Libor rate.

[11] For example, the last caplet of a 5-year cap involves approximately 40 FOMC meetings, that is 8 meetings per year for 5 years.

[12] www.federalreserve.gov

[13] Concerns over the FFr time series properties are raised by Hamilton (1996) and Piazzesi (2010). They advocate the presence of seasonal measurement errors in the FFr series triggered by the so-called "settlement Wednesday" effect. Until July 1998, the reserve computation period imposed on banks overlaped with the reserve maintenance period ending on a Wednesday. Uncertainty surrounding the amount of required reserves would resolve only on the last Wednesday of the computation/maintenance period, creating seasonal spikes in the FFr. In July 1998, the Federal Reserve introduced a lagged computation period that eliminates the overlap. Kotomin and Winters (2007) mention that the FFr demonstrates a much smoother behaviour after July, 1998 and they contend that the "settlement Wednesday" effect has vanished.

Let these two state variables, the FFr and the slope of the yield curve, being respectively denoted by $r_t$ and $s_t$ with

$$Y_t = \begin{pmatrix} r_t \\ s_t \end{pmatrix}.$$

The vector of state variables $Y_t$ is confined to an autogressive Gaussian process where the coefficients change across the trends exhibited by the target rate. The mathematical handling applied on these different trends is such that the label *latent regimes* is more appropriate. The regime-shift mechanism is articulated over the state variables $(\theta_t, \rho_t)$ where $\theta_t$ is the target rate at time $t$ while $\rho_t$ is the *statistical* latent regime in which $\theta_t$ evolves. Under the $\mathbb{P}$ measure and the unobserved regime $\rho_t$, the state vector dynamics is

$$Y_{t+1} = \mu_t^{\mathbb{P}}(\theta_t, \rho_t, Y_t) + \Sigma_{\rho_t} Z_{t+1}^{\mathbb{P}} \tag{1}$$

where the conditional mean satisfies

$$\mu_t^{\mathbb{P}}(\theta_t, \rho_t, Y_t) = \begin{bmatrix} 1-\kappa_{\rho_t} & 0 \\ 0 & b_{\rho_t}^{s,\mathbb{P}} \end{bmatrix} Y_t + \begin{bmatrix} \kappa_{\rho_t} \\ c_{\rho_t}^{s,\mathbb{P}} \end{bmatrix} \theta_t, \tag{2}$$

$\Sigma_{\rho_t}$ is the Choleski decomposition of the covariance matrix $V_{\rho_t}$ with

$$V_{\rho_t} = \begin{bmatrix} \left(\sigma_{\rho_t}^r\right)^2 & \sigma_{\rho_t}^r \sigma_{\rho_t}^s \gamma_{\rho_t} \\ \sigma_{\rho_t}^r \sigma_{\rho_t}^s \gamma_{\rho_t} & \left(\sigma_{\rho_t}^s\right)^2 \end{bmatrix}, \qquad \Sigma_{\rho_t} = \begin{bmatrix} \sigma_{\rho_t}^r & 0 \\ \sigma_{\rho_t}^s \gamma_{\rho_t} & \sigma_{\rho_t}^s \sqrt{1-\gamma_{\rho_t}^2} \end{bmatrix} \tag{3}$$

and $\left\{Z_t^{\mathbb{P}}\right\}$ is a sequence of independent bivariate standard normal random variables.

Thus, under the $\mathbb{P}$ measure, $r_t$ follows a reverting process towards the target rate $\theta_t$, that fluctuates stochastically according to a discrete Markov chain while the slope factor $s_t$ is parametrized as a standard generalized autoregressive process. The reverting coefficient $\kappa_{\rho_t}$, the slope autoregressive coefficient $b_{\rho_t}^{s,\mathbb{P}}$, the target rate sensitivity coefficient in the slope equation $c_{\rho_t}^{s,\mathbb{P}}$, the variances $\sigma_{\rho_t}^r$ and $\sigma_{\rho_t}^s$, and the correlation coefficient $\gamma_{\rho_t}$, are all regime-dependent.

Under the risk-neutral measure $\mathbb{Q}$, the state vector $Y_t$ is driven by a generalized autoregressive process

$$Y_{t+1} = \mu_t^{\mathbb{Q}}(\theta_t, \rho_t, Y_t) + \Sigma_{\rho_t} Z_{t+1}^{\mathbb{Q}} \tag{4}$$

with conditional mean

$$\mu_t^{\mathbb{Q}}(\theta_t, \rho_t, Y_t) = B^{\mathbb{Q}} Y_t + c_{\rho_t}^{\mathbb{Q}} \theta_t, \quad B^{\mathbb{Q}} = \begin{bmatrix} b^{r,\mathbb{Q}} & 0 \\ 0 & b^{s,\mathbb{Q}} \end{bmatrix}, \quad c_{\rho_t}^{\mathbb{Q}} = \begin{bmatrix} c_{\rho_t}^{r,\mathbb{Q}} \\ c_{\rho_t}^{s,\mathbb{Q}} \end{bmatrix} \tag{5}$$

while $\left\{Z_t^{\mathbb{Q}}\right\}$ is a sequence of random vector which conforms to the same probability distribution as $\left\{Z_t^{\mathbb{P}}\right\}$.

As in Ang, Bekaert, and Wei (2008) and Dai, Singleton and Yang (2007), to generate a closed form solution for the caplet prices under the $\mathbb{Q}$ measure, the autoregressive coefficients, $(b^{r,\mathbb{Q}}, b^{s,\mathbb{Q}})$ are invariant across the regimes $\rho_t$. Nevertheless, the unconditional mean $\mu_t^{\mathbb{Q}}(\theta_t, \rho_t, Y_t)$ varies across regimes since the coefficients $(c_{\rho_t}^{r,\mathbb{Q}}, c_{\rho_t}^{s,\mathbb{Q}})$ exhibit a regime-shift behaviour. A comparison of equations (2) and (5) reveals that $r_t$ is not confined to identical processes under both probability measures. Under the risk-neutral measure, preferences and risk aversion of market operators may distort the dynamics of the state variables suggesting a more general process for $r_t$. On the other hand, the debate over the unspanned stochastic volatility issue has led several studies to explicitly include the short rate volatility in the factor structure in the hope of capturing the volatility nature of options. Since this approach is implemented by means of a (supplementary) latent state variable, we take a different approach to capture the stochastic volatility feature by allowing the volatilities of $r_t$ and $s_t$ and their related correlation to switch across regimes $\rho_t$.

Let us define the vector $\lambda_t$ of risk premia

$$\lambda_t = \begin{pmatrix} \lambda_t^r \\ \lambda_t^s \end{pmatrix}$$

where $\lambda_t^r$ and $\lambda_t^s$ are the risk premia associated with the state variables $r_t$ and $s_t$, respectively. Replacing $Z_{t+1}^{\mathbb{P}}$ by $Z_{t+1}^{\mathbb{Q}} - \lambda_t$ in equation (1) yields

$$Y_{t+1} = \mu_t^{\mathbb{P}}\left(\theta_t, \rho_t, Y_t\right) - \Sigma_{\rho_t}\lambda_t + \Sigma_{\rho_t} Z_{t+1}^{\mathbb{Q}}. \tag{6}$$

A comparison of equations (6) and (1) implies

$$\begin{aligned} \lambda_t &= \Sigma_{\rho_t}^{-1}\left(\mu_t^{\mathbb{P}}\left(\theta_t, \rho_t, Y_t\right) - \mu_t^{\mathbb{Q}}\left(\theta_t, \rho_t, Y_t\right)\right) \\ &= \Sigma_{\rho_t}^{-1}\begin{bmatrix} 1-\kappa_{\rho_t} - b^{r,\mathbb{Q}} & 0 \\ 0 & b_{\rho_t}^{s,\mathbb{P}} - b^{s,\mathbb{Q}} \end{bmatrix} Y_t + \Sigma_{\rho_t}^{-1}\begin{bmatrix} \kappa_{\rho_t} - c_{\rho_t}^{r,\mathbb{Q}} \\ c_{\rho_t}^{s,\mathbb{P}} - c_{\rho_t}^{s,\mathbb{Q}} \end{bmatrix}\theta_t \\ &= \Sigma_{\rho_t}^{-1}\begin{bmatrix} \lambda_{r,\rho_t}^r & 0 \\ 0 & \lambda_{s,\rho_t}^s \end{bmatrix} Y_t + \Sigma_{\rho_t}^{-1}\begin{bmatrix} \lambda_{\theta,\rho_t}^r \\ \lambda_{\theta,\rho_t}^s \end{bmatrix}\theta_t. \end{aligned} \tag{7}$$

These risk premia establish a link between investors' preferences and the state variables in different latent regimes of the target rate. In the spirit of Duffee (2002)'s essentially affine risk premia, the specification (7) demonstrates that $r_t$ and $s_t$, their related volatilities, and $\theta_t$ characterize the risk premia within each regime $\rho_t$. In other words, the risk premia are both state-dependant and regime-dependant. This is an important point as the modelling of the risk premia is a central determinant of the model ability to fit the interest rate cap data.

The dynamics of $(\theta_t, \rho_t)$ are parameterized as in Ferland, Gauthier and Lalancette (2010) by means of a discrete inhomogeneous Markov chain. This is consistent with the target rate changing across FOMC meetings as a step function of multiples of $\pm 25$ bps, in a persistent manner. Visual examination of Figure 2 shows that for the period under study, $\theta_t$ moves within one of three possible *statistical* latent regimes: down $(d)$, stable $(s)$, and up $(u)$ implying $\rho_t = \{d, s, u\}$. In FOMC meeting weeks, under the $\mathbb{P}$ measure, transitions of $\rho_t$ are based on the transition probability matrix

$$M_t^{\mathbb{P}} = \begin{cases} M^{\mathbb{P}} & \text{if } t \text{ is a FOMC meeting week} \\ I & \text{otherwise} \end{cases}$$

where

$$M^{\mathbb{P}} = \begin{bmatrix} p_{dd} & p_{ds} & p_{du} \\ p_{sd} & p_{ss} & p_{su} \\ p_{ud} & p_{us} & p_{uu} \end{bmatrix} \tag{8}$$

and $I$ is the identity matrix. Possible movements of $\theta_t$ in FOMC meeting weeks depend on $\rho_t$. The following sequences are assumed

| regime $\rho_t$ | Possible movements of $\theta_t$ in bps |
|---|---|
| Down | $\{0, -25, -50\}$ |
| Stable | $\{-25, -50, 0, 25, 50\}$ |
| Up | $\{0, 25, 50\}$ |

for values of $\theta_t$ falling within the interval $[0.25\%, 7\%]$ given the period under study. In this framework, the target rate may remain at inertia in all three latent regimes. In the upward (downward) latent regime, $\theta_t$ cannot decrease (increase) while in the stable regime, it can move in any direction. Target rate movements are limited to 50 bps in all regimes, as a reflection of the time series fluctuations of the target rate observed in our sample. Since $\theta_t$ evolves between 0.25% and 7% leading to 27 different interest rate levels, the target rate transition matrix under the latent regime $\rho$ is

$$M_{\rho_t} = \begin{cases} M_\rho & \text{if } t \text{ is a FOMC meeting week} \\ I & \text{otherwise} \end{cases}$$

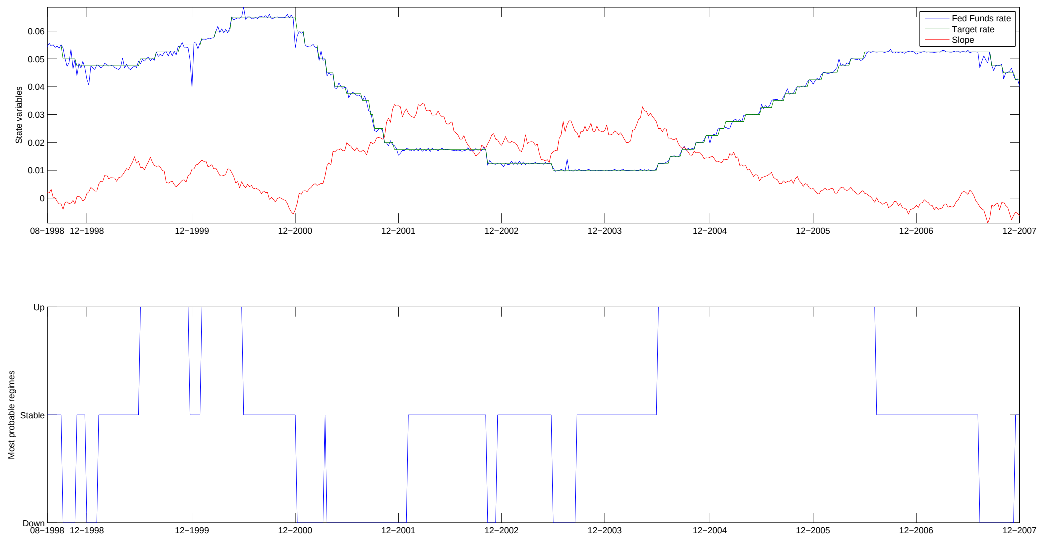


Figure 2: State Variables

where $M_\rho$ is a $27 \times 27$ transition matrix. The precise content of $M_\rho$ is best described using an example. If at week $t$, $\rho_t = \text{d}$, then the target rate evolves in the downward trending regime and $M_\rho$ becomes

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 0.25\% & 0.5\% & 0.75\% & \dots & 6.5\% & 6.75\% & 7\% \end{array} \\
M_d = & \left[\begin{array}{ccccccc}
\star & 0 & 0 & \dots & 0 & 0 & 0 \\
p_d(-0.25) & \star & 0 & \dots & 0 & 0 & 0 \\
p_d(-0.50) & p_d(-0.25) & \star & \dots & 0 & 0 & 0 \\
0 & p_d(-0.50) & p_d(-0.25) & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \star & 0 & 0 \\
0 & 0 & 0 & \dots & p_d(-0.25) & \star & 0 \\
0 & 0 & 0 & \dots & p_d(-0.50) & p_d(-0.25) & \star
\end{array}\right]
\begin{array}{c} 0.25\% \\ 0.5\% \\ 0.75\% \\ 1\% \\ \vdots \\ 6.5\% \\ 6.75\% \\ 7\% \end{array}
\end{array}
$$

and $p_d(x)$ represents the probability under the $\mathbb{P}$ measure that within the downward regime, $\theta_t$ decreases by $x$ basis points while the "$\star$" denotes the row complementary probability.[14] When $\rho_t = s$, the target rate transition matrix $M_s$, includes row-wise these non-zero elements

$$p_s(-0.50) \quad p_s(-0.25) \quad \star \quad p_s(0.25) \quad p_s(0.5).$$

If $\rho_t = u$, the following non-zero elements are included row-wise in $M_u$

$$\star \quad p_u(0.25) \quad p_u(0.5).$$

Thus, in each FOMC meeting week the pair $(\theta_t, \rho_t)$ fluctuates according to the consolidated transition matrix

$$\mathcal{M}^{\mathbb{P}} = \begin{bmatrix} p_{dd}M_d & p_{ds}M_d & p_{du}M_d \\ p_{sd}M_s & p_{ss}M_s & p_{su}M_s \\ p_{ud}M_u & p_{us}M_u & p_{uu}M_u \end{bmatrix} \tag{9}$$

with the possible sequences: $(0.25\%, d)$, $(0.5\%, d)$,..., $(7\%, d)$, $(0.25\%, s)$, ..., $(7\%, s)$, $(0.25\%, u)$, ..., $(7\%, u)$. Since there are three latent regimes and 27 possible target levels, $\mathcal{M}^{\mathbb{P}}$ is an $81 \times 81$ matrix. Empirical support

[14] Each row must sum to one.

for this approach is found in Ferland, Gauthier and Lalancette (2010). To capture the target rate dynamics at works within option cap prices, the same characterization of the pair $(\theta_t, \rho_t)$ is used under the risk-neutral measure $\mathbb{Q}$ with

$$M_t^{\mathbb{Q}} = \begin{cases} M^{\mathbb{Q}} & \text{if } t \text{ is a FOMC meeting week} \\ I & \text{otherwise} \end{cases}$$

where

$$M^{\mathbb{Q}} = \begin{bmatrix} q_{dd} & q_{ds} & q_{du} \\ q_{sd} & q_{ss} & q_{su} \\ q_{ud} & q_{us} & q_{uu} \end{bmatrix} \text{ and } \mathcal{M}^{\mathbb{Q}} = \begin{bmatrix} q_{dd}M_d & q_{ds}M_d & q_{du}M_d \\ q_{sd}M_s & q_{ss}M_s & q_{su}M_s \\ q_{ud}M_u & q_{us}M_u & q_{uu}M_u \end{bmatrix} \tag{10}$$

Comparison of $\mathcal{M}^{\mathbb{P}}$ and $\mathcal{M}^{\mathbb{Q}}$ shows that the matrices $M_d$, $M_s$ and $M_u$ are invariant across the $\mathbb{P}$ and $\mathbb{Q}$ measures indicating that the target rate risk is not priced in our model. In contrast, in the data-generating matrix of the latent regimes $\rho_t$ , $M^{\mathbb{P}}$ is allowed to differ from its risk-neutral counterpart $M^{\mathbb{Q}}$. This feature is critical as the study of Dai, Singleton and Yang (2007) insists on the importance of pricing regime-shift risk as demonstrated by their empirical findings.

## 3.2 European contingent claim

Before proceeding to the pricing of interest rate caps, it is instructive to briefly examine how the pricing of a generalized European contingent claim is performed in our framework. Given the similarity that exists between such a claim and cap pricing, the theorem presented below is used later on for cap prices. To go about the pricing of the contingent claim, it is assumed that a discrete discount factor $R_t$ is driven by an affine function of $r_t$ and $s_t$ where

$$R_t(\rho_t, \theta_t) = \Delta\alpha' Y_t \tag{11}$$

with $\alpha' = [\alpha^r \alpha^s]$ while $\Delta = \frac{1}{48}$ accommodates the weekly timeframe that emanates from the assumption that there are 48 weeks in a year with FOMC meetings held every six weeks for a total of eigth meetings a year. Although the vector $\alpha$ is state independent, the discount factor $R_t(\rho_t, \theta_t)$ is sensitive to the latent regime $\rho_t$, since the probability distribution of $Y_t$ exhibits a regime-shift feature resulting from $\mathrm{E}^{\mathbb{Q}}(Y_t) = \mu^{\mathbb{Q}}(\rho_t, \theta_t, Y_t)$ while $\Sigma_{\rho_t}$ is given by equation (3).

**Theorem 1** *Let $f(\theta_T, \rho_T)$ be the exercise value of an European contingent claim with a maturity of $T$ weeks. The time $t$ value of this contingent claim is*

$$V_t = \exp\left(B_{t,T-t}(\theta_t, \rho_t) + A'_{T-t} Y_t\right). \tag{12}$$

*where the coefficients are constructed by recursion based on the time-to-maturity,*

$$\begin{aligned} A_{T-t} &= B^{\mathbb{Q}} A_{T-t-1} - \Delta\alpha \qquad (13) \\ B_{t,T-t}(\theta, \rho) &= A'_{T-t-1} c_\rho^{\mathbb{Q}} \theta + \frac{1}{2} A'_{T-t-1} \Sigma_\rho \Sigma'_\rho A_{T-t-1} \\ &\quad + \ln \sum_{\widetilde{\rho}} \sum_{\widetilde{\theta}} \exp\left(B_{t+1,T-t-1}(\widetilde{\theta}, \widetilde{\rho})\right) Q_t\left((\theta, \rho); (\widetilde{\theta}, \widetilde{\rho})\right). \qquad (14) \end{aligned}$$

*The matrices $B^{\mathbb{Q}}$ and $c_\rho^{\mathbb{Q}}$ are defined in equation (5), and the initial conditions of the recursion are*

$$A_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textit{and} \quad B_{T,0}(\theta, \rho) = \ln f(\theta, \rho). \tag{15}$$

*$Q_t$ is an $81 \times 81$ transition matrix where*

$$Q_t = \begin{cases} \mathcal{M}^{\mathbb{Q}} & \textit{if } t \textit{ is a FOMC meeting week} \\ I & \textit{otherwise.} \end{cases}$$

Equation (14) shows that the interplay between the contingent claim price and the anticipated regimes $\widetilde{\rho}$ and target rate levels $\widetilde{\theta}$. Transitions are captured by the matrix of transition probabilities of the regime-shifts and target rate levels, $Q_t$.

**Proof:** The result is established using an induction argument based on the time-to-maturity $T-t$. Indeed, if $T-t=0$, then

$$V_T = f(\theta_T, \rho_T) = \exp(\ln f(\theta_T, \rho_T)).$$

A straightforward comparison with equation (12) determines the initial coefficients (15). The induction hypothesis presumes that equation (12) holds true at time $t+1$, that is

$$V_{t+1} = \exp\left(B_{t+1,T-t-1}(\theta_{t+1}, \rho_{t+1}) + A'_{T-t-1} Y_{t+1}\right).$$

Therefore,[15]

$$\begin{aligned}
V_t &= \mathrm{E}_t^{\mathbb{Q}}\left[\exp(-\Delta\alpha' Y_t) V_{t+1}\right] \\
&= \exp(-\Delta\alpha' Y_t)\, \mathrm{E}_t^{\mathbb{Q}}\left[\exp\left(B_{t+1,T-t-1}(\theta_{t+1}, \rho_{t+1}) + A'_{T-t-1} Y_{t+1}\right)\right] \\
&= \exp(-\Delta\alpha' Y_t)\, \mathrm{E}_t^{\mathbb{Q}}\left[\exp\left(A'_{T-t-1} Y_{t+1}\right)\right] \sum_{\widetilde{\theta},\widetilde{\rho}} \exp\left(B_{t+1,T-t-1}\left(\widetilde{\theta}, \widetilde{\rho}\right)\right) Q_t\left((\theta_t, \rho_t); \left(\widetilde{\theta}, \widetilde{\rho}\right)\right) \\
&= \exp\left(\left(A'_{T-t-1} B^{\mathbb{Q}} - \Delta\alpha'\right) Y_t + A'_{T-t-1} c^{\mathbb{Q}}_{\rho_t} \theta_t + \frac{1}{2} A'_{T-t-1} \Sigma_{\rho_t} \Sigma'_{\rho_t} A_{T-t-1}\right) \\
&\quad \times \sum_{\widetilde{\theta},\widetilde{\rho}} \exp\left(B_{t+1,T-t-1}\left(\widetilde{\theta}, \widetilde{\rho}\right)\right) Q_t\left((\theta_t, \rho_t); \left(\widetilde{\theta}, \widetilde{\rho}\right)\right)
\end{aligned}$$

since $\mathrm{E}_t^{\mathbb{Q}}\left[\exp\left(A'_{T-t-1} Y_{t+1}\right)\right] = \exp\left(A'_{T-t-1}\left(B^{\mathbb{Q}} Y_t + c^{\mathbb{Q}}_{\rho_t} \theta_t\right) + \frac{1}{2} A'_{T-t-1} \Sigma_{\rho_t} \Sigma'_{\rho_t} A_{T-t-1}\right)$. A term-by-term matching with equation (12) leads to equations (13) and (14). $\square$

**Corollary 1** *The value at week $t$, of a zero-coupon bond expiring $n$ weeks later is*

$$P(t, n; \theta_t, \rho_t, Y_t) = \mathrm{E}_t^{\mathbb{Q}}\left[\exp\left\{-\sum_{i=0}^{n} \left(\alpha^r r_{t+i} + \alpha^s s_{t+i}\right)\right\}\right] = \exp\left(B_{t,n}(\theta_t, \rho_t) + A'_n Y_t\right) \tag{16}$$

*where $B_{n,0}(\theta, \rho) = \ln 1 = 0$.*

## 3.3 Cap pricing methodology

In this section, an analytical expression is presented to (approximately) compute the no-arbitrage price of an option caplet in a regime-shift environment. Recall that a caplet is a European option whose payoff at maturity $T_{k+1}$, expressed in number of weeks, depends on the 3-month Libor $L(T_k, T_{k+1})$. Formally, the caplet payoff is

$$0.25 \max\left(L(T_k, T_{k+1}) - K, 0\right) = \frac{\max\left(1 - (1 + 0.25K)\, P(T_k, 12), 0\right)}{P(T_k, 12)} \tag{17}$$

when the option's notional amount is set to unity. In equation (17), $P(T_k, 12)$ corresponds to the shorthand notation of the zero-coupon price $P(T_k, 12; \theta_{T_k}, \rho_{T_k}, Y_{T_k})$. In conformity with the convention observed on the forward rate agreement and swap markets, equation (17) shows that the value of the 3-month Libor is determined at time $T_k$ but the cash flows are exchanged at $T_{k+1}$. Although the option caplet is the object of the pricing technology, only cap prices are available on markets. Each cap is a portfolio of caplets with sequential maturities of $T_0 = 12$ weeks $< T_1 < ... < T_m < T_{m+1}$ where $T_i - T_{i-1} = 0.25$, that is 12 weeks in this framework. All caplets embodied in an at-the-money cap share the same strike $K$ associated with the swap rate whose maturity matches that of the cap.

[15] $\mathrm{E}_t^{\mathbb{Q}}$ denotes the conditional expectation with respect to the information available at time $t$.

As mentioned in the previous section, to ease the management of the time frame, it is assumed under the $\mathbb{Q}$ measure that FOMC meetings take place according to a standardized schedule of 8 meetings a year, every 6 weeks. This assumption is required to obtain a tractable pricing set up.[16] The discount factor for the time period $]T_i, T_j]$ is

$$D(T_i, T_j) = \exp\left(-\Delta \sum_{t=T_i}^{T_j-1} \alpha' Y_t\right) = \exp\left(-\Delta\alpha^r \sum_{t=T_i}^{T_j-1} r_t - \Delta\alpha^s \sum_{t=T_i}^{T_j-1} s_t\right).$$

To ease the notation, we drop the subscript that refers to a particular quarter whereby $T_k$ becomes simply $T$. Hence, the week $t$ value of a caplet with a strike rate $K$ and a maturity (in number of weeks) $T > t$, is given by

$$\begin{aligned} c(t, T+12) &= 0.25\, \mathrm{E}_t^{\mathbb{Q}}\left[D(t, T+12)\max\left(L(T, T+12) - K, 0\right)\right] \\ &= \mathrm{E}_t^{\mathbb{Q}}\left[D(t, T)\max\left(1 - (1+0.25K)\, P(T, 12), 0\right)\right]. \end{aligned} \tag{18}$$

There is no tractable closed-form solution to the expectation (18) in our regime-shift environment, but it is possible to derive an analytical approximation for the price. By using the law of iterated expectations, equation (18) becomes

$$c(t, T+12) = \mathrm{E}_t^{\mathbb{Q}}\left[D(t, T-6)\, c(T-6, T+12)\right], \tag{19}$$

$$c(T-6, T+12) = \mathrm{E}_{T-6}^{\mathbb{Q}}\left[D(T-6, T)\max\left(1 - (1+0.25K)\, P(T, 12), 0\right)\right]. \tag{20}$$

Equations (19) and (20) give rise to a pricing strategy articulated over two observations. First, if there is no regime shift nor any target rate movement in the interval $]T-6, T]$, it is possible to generate an exact analytical expression for equation (20) based on the assumption that $D(T-6, T)$ and $P(T, 12) = \exp\left(B_{T,12}(\theta_T, \rho_T) + A'_{12} Y_T\right)$ are jointly lognormally distributed. This is the aim of Proposition 1. Second, if $c(T-6, T+12)$ is approximated by a function that depends only on $\rho_{T-6}$ and $\theta_{T-6}$ and not on $Y_{T-6}$, then Theorem 1 implies that the caplet price $c(T-6, T+12)$ is approximated by an exponential affine equation with coefficients recursively computed. This idea is formalized in Proposition 2. The analytical expression of $c(T-6, T+12)$ requires the computation of a conditional expectation as seen in equation (18). This is the goal of Lemma 1.

**Lemma 1** *Let $(X_1, X_2)$ be a Gaussian random vector with a mean vector $\mu$ and variance-covariance matrix $\Sigma$ given by*

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \eta\sigma_1\sigma_2 \\ \eta\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

*Then for any real number $a > 0$, one has*

$$\mathrm{E}\left[e^{X_1}\max\left(1 - a e^{X_2}, 0\right)\right] = F_1 N(d_1) - a F_2 N(d_2). \tag{21}$$

*with*

$$\begin{aligned} d_1 &= -\frac{\mu_2 + \log a}{\sigma_2} - \sigma_1\eta, \qquad d_2 = d_1 - \sigma_2, \\ F_1 &= \exp\left(\mu_1 + \frac{\sigma_1^2}{2}\right), \qquad F_2 = F_1 \exp\left(\mu_2 + \eta\sigma_1\sigma_2 + \frac{\sigma_2^2}{2}\right). \end{aligned}$$

*where $N(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.*

The proof of Lemma 1 is in the Appendix. Combining equation (20) and lemma 1 lead to the following proposition to value a caplet under the assumption that the regime $\rho_t$, and the target $\theta_t$, skip one possible movement over $]T-6, T]$.

[16] In computing zero-coupon bond prices based on term structure model replicating the monetary policy, Piazzesi (2005) only matches the number of days until the next FOMC meeting while assuming that subsequent meetings are equally spaced over a year. She points out that the pricing errors emanating from this approximation are negligible.

**Proposition 1** *If there is no regime shift nor any target rate movement in the interval* $]U,T]$,[17] *the week* $U$ *value of caplet of maturity* $T+12$ *(in number of weeks) is directly provided by equation (21):*

$$\begin{aligned}\mathcal{C}\left(\rho_U,\theta_U,Y_U;T+12\right) &= \mathrm{E}_U^{\mathbb{Q}}\left[D\left(U,T\right)\max\left(1-\left(1+0.25K\right)P\left(T,12\right),0\right)\right] \\ &= F_1N(d_1)-aF_2N(d_2)\end{aligned} \tag{22}$$

*with parameters* $a$, $F_1$ *and* $F_2$ *defined within the proof presented in the Appendix. These parameters depend on* $\rho_U$, $\theta_U$ *and* $Y_U=\left[r_U,s_U\right]'$.

**Proposition 2** *The caplet price of maturity* $T_{k+1}$ *is approximated by*

$$\widetilde{c}(0,T_{k+1})=\exp\left(\frac{-T_{k+1}}{48}\right)\widetilde{c}_1(0,T_{k+1})+\left(1-\exp\left(\frac{-T_{k+1}}{48}\right)\right)\widetilde{c}_2(0,T_{k+1}) \tag{23}$$

*with*

$$\begin{aligned}\widetilde{c}_1(0,T_{k+1}) &= \exp\left(B^{(1)}_{0,T_k-6}(\theta_0,\rho_0)+A'_{T_k-6}Y_0\right) \\ \widetilde{c}_2(0,T_{k+1}) &= \exp\left(B^{(2)}_{0,T_k-6}(\theta_0,\rho_0)+A'_{T_k-6}Y_0\right)\end{aligned}$$

*where the coefficients* $A$, $B^{(1)}$ *and* $B^{(2)}$ *are defined as in Theorem 1, although* $B^{(1)}$ *and* $B^{(2)}$ *have different initial values:*

$$\begin{aligned}B^{(1)}_{T_k-6,0} &= \ln\mathcal{C}\left(\rho_{T_k-6},\theta_{T_k-6},Y_0;T_{k+1}\right) \\ \text{and}\ \ B^{(2)}_{T_k-6,0} &= \ln\mathcal{C}\left(\rho_{T_k-6},\theta_{T_k-6},m\left(\rho_{T_k-6},\theta_{T_k-6}\right);T_{k+1}\right).\end{aligned}$$

$$m'\left(\rho_{T-6},\theta_{T-6}\right)=\left[\frac{1}{1-b^{r,\mathbb{Q}}}\ c^{r,\mathbb{Q}}_{\rho_{T-6}}\ \theta_{T-6},\frac{1}{1-b^{s,\mathbb{Q}}}\ c^{s,\mathbb{Q}}_{\rho_{T-6}}\ \theta_{T-6}\right]'.$$

*where* $\mathcal{C}\left(\rho_{T_k-6},\theta_{T_k-6},\ \cdot\ ;T_{k+1}\right)$ *is given by equation (22) in Proposition 1.*

**Proof:** The pricing formula builds upon two approximations. Starting from equation (19) and assuming that there is no regime shift nor any target variation during $]T-6,T]$, then Proposition 1 applies and leads to the first approximation:

$$c(t,T+12)\cong\mathrm{E}_t^{\mathbb{Q}}\left[D\left(t,T-6\right)\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_{T-6},T+12\right)\right] \tag{24}$$

where $\mathcal{C}$ is defined at equation (22). At this stage, since $\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_{T-6},T+12\right)$ depends on the state variable $Y_{T-6}$, Theorem 1 cannot be applied. The obvious way to circumvent the problem is to build up an approximation of $\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_{T-6},T+12\right)$ that does not involved $Y_{T-6}$. If $T-6$ is close to $t$, then the observed value $Y_t$ may be used as a first approximation, that is

$$\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_{T-6},T+12\right)\cong\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_t,T+12\right).$$

Thus, a first approximation of the caplet price is

$$\widetilde{c}_1(t,T+12)=\exp\left(B^{(1)}_{t,T-6-t}(\theta_t,\rho_t)+A'_{T-6-t}Y_t\right)\ \text{with}\ B^{(1)}_{T-6,t}=\ln\mathcal{C}\left(\rho_{T-6},\theta_{T-6},Y_t,T+12\right). \tag{25}$$

However, if $T$ is much larger than $t$, equation (25) is likely to produce a poor approximation. In this case, we suggest replacing $Y_{T_k-6}$ with its *stationary* mean $m'\left(\rho_{T-6},\theta_{T-6}\right)$. Providing a second approximation

$$\begin{aligned}\widetilde{c}_2(t,T+12) &= \exp\left(B^{(2)}_{t,T-6-t}(\theta_t,\rho_t)+A'_{T-6-t}Y_t\right) \\ \text{with}\ B^{(2)}_{T-6,t} &= \ln\mathcal{C}\left(\rho_{T-6},\theta_{T-6},m\left(\rho_{T-6},\theta_{T-6}\right),T+12\right).\end{aligned} \tag{26}$$

[17] It should be pointed out that since regime and target rate changes are reflected in the bond pricing formula, the caplet value accounts for regime shift during $]T,T+12]$.

The final caplet price is a convex combination of the approximations (25) and (26) where the weights are determined by the maturity. □

The centre piece of this study is the analytical approximation (23) presented in Proposition 2. The pricing precision of this approximation is challenged by an exact Monte Carlo approach based on 1000 different sets of parameters randomly drawn from uniform distributions (as in Broadie and Detemple (1996)). The findings of this experiment are reported in Table 1. The upper part shows that the median pricing errors are very small for all maturities even though the mean absolute errors stand at higher level. The error distribution quantiles reveal that in most cases, the analytical approximation performs very well while in about 1% of the scenarios considered, some combinations of parameters leads to anomalous pricing. This point is illustrated by the $Q(99\%)$ quantile estimates, which shows huge pricing errors occurring in the tail of the distribution, particularly for the 5-year cap. An examination of the relative pricing errors presented in the lower part of Table 1 offers a similar perspective and reveals that significant relative pricing errors are observed at the 95% and 99% quantiles.[18] Interestingly, the relative pricing errors decrease as the maturity of the interest rate cap increases. This may reflect the fact that the analytical approximation omits one regime-shift just prior the maturity of the underlying caplets in contrast to the Monte Carlo approach which is not subject to that feature. This attribute impact more short-term caplets that embody only a few regime-shift possibilities.

# 4 Estimation

The observability of the state variables $\theta_t$, $r_t$ and $s_t$ allows us to split the parameter space between the $\mathbb{P}$- and $\mathbb{Q}$- measure parameters. Every week of our sample, the state variables $r_t$ and $s_t$ fluctuate according to the dynamics captured by equations (1) and (2) while at each FOMC meeting, the pair $(\theta_t, \rho_t)$ discretely transit from one state to the other based on the probability matrix (9). The joint estimation of the parameters driving these processes and the Markov chain probabilities, can proceed via maximum likelihood. As previously mentioned, similar estimations are performed in Ferland, Gauthier and Lalancette (2010) based only $\theta_t$ and $r_t$ when the latter is confined to a discrete mean-reverting process. Extending their methodology to our context, the joint maximum likelihood estimation of the Markov chain driving the pair $(\theta_t, \rho_t)$ and the bivariate system of equations for $r_t$ and $s_t$, under regime-shifts follows directly from the methodology presented in Hamilton (1994). However, the full estimation approach is a daunting task as 29 parameters are estimated simultaneously, all cast into a highly nonlinear objective function. To circumvent this obstacle, we first estimate the semi-hidden Markov chain $(\theta_t, \rho_t)$ and, in turn, use the resulting estimates to optimize the global likelihood function.

The estimation results are reported in Table 2 while the most probable regime $\rho_t$ observed at each week $t$ computed from the filtered probabilities $p_t(\rho_t)$, is reported in the lower plot of Figure 2. Parameters in bold are significantly different from 0 at the 5% level. The majority of the parameter estimates are significant. Note however that the conditional correlations are very close to 0 irrespective of the latent regimes. The volatility estimates of $r_t$ and $s_t$ are significant and display a regime-shift behaviour. Globally, the findings reported in Table 2 support the regime-shift framework. As expected, variations in $s_t$ are inversely related to the target's rate level although the relation seems imprecisely estimated as it is significant only in the stable regime.

The estimation of the parameters belonging to the $\mathbb{Q}$ measure raises two issues. First, under the $\mathbb{P}$ measure, the FOMC meeting dates are observable and constitute a fundamental determinant of the estimations reported in Table 2. In contrast, the sequence of FOMC meetings over the risk-neutral $\mathbb{Q}$ measure is inherently unobservable but is presumably discounted by market operators when setting and quoting cap prices. To compute the caplet prices via equations (23), it is assumed that a year comprises 48 weeks with 8 FOMC meeting per year, every 6 weeks. Second, since cap prices are generally quoted in terms of Black's implied volatilities, minimization of the Sum-Squares Errors (SSE) is achieved by minimizing the Euclidean distance between the market- and model-implied volatilities or, alternatively, between the market- and model-cap

[18] It is worth emphasizing that some of the important relative pricing errors, computed as the price differences divided by benchmark cap prices, occur since the latter are very small. Thus, no benchmark price filters were imposed in this experiment.

Table 1: Analytical Approximation for Cap Prices

| Cap Maturities (years) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Pricing Errors of Cap Options | | | | |
| Median | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Mae | 0.0029 | 0.0082 | 0.0491 | 0.7390 |
| Q(1%) | −0002 | −0.0003 | −0.0004 | −0.0005 |
| Q(5%) | −0.0001 | −0.0001 | −0.0001 | −0.0002 |
| Q(10%) | −0.0001 | −0.0001 | −0.0001 | −0.0001 |
| Q(90%) | 0.0018 | 0.0017 | 0.0019 | 0.0020 |
| Q(95%) | 0.0075 | 0.0094 | 0.0108 | 0.0121 |
| Q(99%) | 0.0557 | 0.1042 | 0.1971 | 0.3904 |
| Relative Absolute Pricing Errors of Cap Options | | | | |
| Median | 0.0086 | 0.0074 | 0.0069 | 0.0067 |
| Mae | 0.0479 | 0.0381 | 0.0333 | 0.0304 |
| Q(1%) | 0.0002 | 0.0001 | 0.0001 | 0.0001 |
| Q(5%) | 0.0007 | 0.0007 | 0.0004 | 0.0004 |
| Q(10%) | 0.0013 | 0.0012 | 0.0010 | 0.0009 |
| Q(90%) | 0.1217 | 0.0898 | 0.0783 | 0.0668 |
| Q(95%) | 0.2959 | 0.2172 | 0.1905 | 0.1678 |
| Q(99%) | 0.6211 | 0.5116 | 0.4911 | 0.4306 |

The median, mean absolute pricing errors (MAE) and the empirical quantiles of order $\alpha$ ($Q(\alpha\%)$) for cap maturities of 2 to 5 years, are reported in the upper part of the Table. Cap pricing errors are based on option caplets evaluated using a Monte Carlo approach (10 000 trajectories) and using the analytical approximation (23). The second part reports similar statistics for relative pricing errors. The pricing errors are based on 1000 sets of simulated parameters drawn from uniform distributions:

$$
\begin{array}{rcl}
q_{uu},\ q_{ss},\ q_{dd} & \sim & U(0,1)\,, q_{su} = (1-q_{ss}) * \text{rand} \\
b^r & \sim & U(0,1)\,,\ c^r_\bullet \sim U(-1.5, 1.5)\,,\ \sigma^r_\bullet \sim U(0.0001, 0.0051) \\
b^s & \sim & U(0,1)\,,\ c^s_\bullet \sim U(0,3)\,,\ \sigma^s_\bullet \sim U(0.0001, 0.0051) \\
\gamma_\bullet & \sim & U(-1,1) \\
\alpha^r & \sim & U(0,1)\,,\ \alpha^s \sim U(0,1) \\
\theta_0 & = & U(4,28)\cdot 0.0025, \rho_0 = U(d,s,u) \\
r_0 & = & \theta_0, s_0 = 0 \\
\mid b^r & + & c^r_\bullet \mid\ <\ 1,\ \text{(This constraint is imposed to avoid non-stationary dynamics of } r_t\text{).}
\end{array}
$$

prices. Our choice is dictated by Christofferson and Jacobs (2004) who advocate that implied volatility-based estimates are significantly noisier than option priced-based estimates.

To formally define the least square errors under regime-shifts, let $\widetilde{\text{cap}}(t, T_k; \theta_t, \rho_t, Y_t)$ be the price at week $t$ of an interest rate cap expressed as a portfolio of caplets whose prices are given by the analytical approximation (23) presented in Proposition 2. Since the computation of cap prices requires the identification of the latent regime $\{\rho_t = d, s, u\}$, let $\text{cap}^{\text{model}}(t, T_k)$, the cap price at week $t$ given by the model, be defined as

$$
\begin{aligned}
\text{cap}^{\text{model}}(t, T_k) &= \mathrm{E}_t^{\mathbb{Q}}\left[\widetilde{\text{cap}}(t, T_k; \theta_t, \rho_t, Y_t)\right] \\
&= \sum_\rho \widetilde{\text{cap}}(t, T_k; \theta_t, \rho, Y_t)\ \Pi_t(\rho_t = \rho)
\end{aligned} \tag{27}
$$

where $\Pi_t(\rho_t = \rho)$ is the risk-neutral probability that the latent regime $\rho_t = \rho$ at week $t$. Thus, to compute equation (27), the time-varying probabilities $\Pi_t(\rho_t = \rho)$ must be estimated. A first look at equation (27) suggests that these probabilities resemble filtered probabilities as defined in standard regime-shift applications. This view raises some concern. The vast majority of regime-shift estimations that we are aware of,

Table 2: Estimated Parameters Under the $\mathbb{P}$ measure

| Target Rate Regime-switch Probabilities | | | |
|---|---|---|---|
| $p_{dd}$ | $p_{su}$ | $p_{ss}$ | $p_{uu}$ |
| **0.8395 (0.1495)** | 0.0524 (0.0396) | **0.8994 (0.0757)** | **0.8969 (0.0645)** |

| Parameters \ Regime($\rho$): | Downward | Stable | Upward |
|---|---|---|---|
| Target Rate Transition Probabilities | | | |
| $p_\rho\,(-50)$ | **0.5745 (0.1432)** | 0.0215 (0.0397) | - |
| $p_\rho\,(-25)$ | **0.2378 (0.1164)** | 0.0271 (0.0349) | - |
| $p_\rho\,(0)$ | 0.1877 | 0.9403 | 0.1038 |
| $p_\rho\,(25)$ | - | 0.0111 (0.0262) | **0.8580 (0.0803)** |
| $p_\rho\,(50)$ | - | 0.0000 | 0.0382 (0.0377) |
| Fed Funds rate: $r_{t+1} = r_t + \kappa_{\rho_t}\,(\theta_t - r_t) + \sigma^r_{\rho_t} Z^{r,\mathbb{P}}_{t+1}$ | | | |
| $\kappa_\rho$ | **0.5828**(0.1152) | **1.1207** (0.0493) | **0.6340** (0.0868) |
| $\sigma^r_\rho$ | **0.0023** (0.0002) | **0.0004** (0.0000) | **0.0017** (0.0001) |
| Slope: $s_{t+1} = b^{s,\mathbb{P}}_{\rho_t} s_t + c^{s,\mathbb{P}}_{\rho_t}\theta_t + \sigma^s_{\rho_t} Z^{s,\mathbb{P}}_{t+1}$ | | | |
| $b^{s,\mathbb{P}}_\rho$ | **0.9844**(0.0244) | **0.9632** (0.0178) | **0.9552** (0.0167) |
| $c^{s,\mathbb{P}}_\rho$ | -0.0238 (0.0190) | **-0.0212** (0.0104) | -0.0039 (0.0072) |
| $\sigma^s_\rho$ | **0.0018** (0.0001) | **0.0013** (0.0001) | **0.0011**(0.0001) |
| $\gamma^{Z^r,Z^s}_\rho$ | -0.1531 (0.09638) | -0.0273 (0.0744) | -0.0968(0.0785) |

The regime-switch transition probabilities based on the matrix (8), the target rate transition probabilities based on the matrix $M_\rho$, the parameter estimates for state variables dynamics based on (1) and (2), and their related standard deviations (in parentheses), are reported in this table under the three target rate's latent regime-shifts (down, stable and up). Estimates in bold are significant at the 5% level.

are performed under the $\mathbb{P}$ measure. Confining the set of probabilities $\Pi_t(\rho_t = \rho)$ to $\mathbb{P}$-measure filtered probabilities, is equivalent to assuming that regime-shift classifications at each week $t$, are identical under the $\mathbb{P}$ and $\mathbb{Q}$ measures. This is clearly inconsistent with our pricing strategy. Alternatively, the estimation of $\mathbb{Q}$-measure filtered probabilities at each week $t$ requires a nonlinear recursive procedure that depends on the estimated parameters. This is unmanageable in the context of our pricing model. Furthermore, the concept of filtered probabilities in a risk-neutral setting considerably challenges our comprehension of the nature of these quantities. These limitations lead us to believe that the structural strategy proposed by Bansal, Tauchen and Zhou (2004) and Bansal and Zhou (2002) to recover the weekly latent regime seems well suited. Their approach implies that under the $\mathbb{Q}$ measure, the observed regime $\rho_t$ at week $t$ (with max $\Pi_t(\rho_t = \rho)$), is the latent regime whose corresponding cap price $\widetilde{\text{cap}}(t, T_k; \theta_t, \rho, Y_t)$ entails the smallest pricing error.

Denote the vector of risk-neutral parameters by $\Theta$ and the week $t$ percentage pricing error of a cap with maturity $T_i$ as

$$e_{it}(\Theta) = \left[\frac{\text{cap}^{\text{market}}(t, T_i) - \text{cap}^{\text{model}}(t, T_i)}{\text{cap}^{\text{market}}(t, T_i)}\right] \tag{28}$$

where $\text{cap}^{\text{market}}(t, T_i)$ is the corresponding option price observed on the cap market. Estimation is performed by minimizing the SSE over the entire time frame and set of cap prices such that

$$\hat{\Theta} = \arg\ \min_{\Theta} \sum_{t=1}^{T} e_t(\Theta)' e_t(\Theta) \tag{29}$$

where $e_t(\Theta)$ is a $4 \times 1$ vector of pricing errors, $e_{it}(\Theta)$. To estimate (29), we follow the traces of Jarrow, Li and Zhao (2007) by resorting to nonlinear least squares. Properties of these estimators such as asymptotic consistency are discussed in Davidson and MacKinnon (1993). A robust variance-covariance matrix of the estimated parameters must account for the strong autocorrelation and cross-autocorrelation that characterize the time series of $\hat{e}_{it}(\Theta)$, as it is usually observed with DTSM estimations. This is achieved by computing the Newey and West (1987) variance-covariance matrix with 25 lags, weighted by the factor score matrices obtained at the minimum SSE.

The estimation results are reported in Table 3. The autoregressive parameters $b^{r,\mathbb{Q}}$ and $b^{s,\mathbb{Q}}$ where fixed in order to ease the estimation process as they were rapidly converging towards 0.99 suggesting that $r_t$ and $s_t$ may not be stationary stochastic processes under $\mathbb{Q}$. In short, Dai, Singleton and Yang (2007) point out that this behaviour is frequently observed under the risk-neutral measure as a reflection of the tension that may exists between the model's flexibility and some features of the data. In other words, given the discount factor $D\left(0, T_{k+1}; \theta_0, \rho_0, r_0, s_0\right) = \exp\left[-\Delta \sum_{t=0}^{T_{k+1}} (\alpha^r r_t + \alpha^s s_t)\right]$, $r_t$ or $s_t$ may be required to exhibit a near or slightly explosive behaviour to adequately price the caplets. In turn, this may reflect the fact that the parameter $b^{r,\mathbb{Q}}$ is invariant across regimes while its $\mathbb{P}$ measure counterpart, $\kappa_\rho$ clearly displays a regime-dependant dynamics as shown in Table 2. Collecting the estimates $\lambda^{\bullet}_{\bullet,\rho_t}$ from the difference between the state variable processes under equations (1) and (4), shows that these quantities regime-shift. This provides an indication that the market operators trading interest rate caps change their tolerance for risk according to the latent regime in which the target rate evolves.

The probability estimates of the matrix $M^{\mathbb{Q}}$ are also reported in Table 3. They are highly significant and differ substantially from their $\mathbb{P}$ counterparts. In short, the relative magnitude of the diagonal probabilities, $q_{dd}$, $q_{ss}$, and $q_{uu}$, suggests that there is considerably less regime persistence under the $\mathbb{Q}$ measure. Furthermore, a likelihood ratio test (LRT) of the null hypothesis that regime-shift risk is not priced, that is $M^{\mathbb{Q}} = M^{\mathbb{P}}$, generates a likelihood statistics of 1.6395e+03 leading to a strong rejection at the 1% level. Dai, Singleton and Yang (2007) emphasize the importance of pricing regime-shift risk. These findings are consistent with that view and extend their conjecture to the case of interest rate options.

# 5 Empirical findings

The pricing of regime-shift risk is a mean by which interest cap traders express their risk attitude with respect to the Fed's monetary actions. To acquire deeper insights into the fact that $M^{\mathbb{Q}} \neq M^{\mathbb{P}}$, we compute the stationary distribution's regime probabilities under both measures. Table 4 reports the estimated matrices $M^{\mathbb{P}}$ and $M^{\mathbb{Q}}$ along with their related vectors of stationary probabilities. A comparison between $\pi^{\mathbb{P}}$ and $\pi^{\mathbb{Q}}$ shows that the stable regime is more (less) probable under the risk-neutral (physical) measure with a probability of 68.78% (55.29%). Li and Zhao (2009) point out that periods of very high or very low interest rates are associated with bad economic states (such as recessions and high inflation periods). In those states, they observe high state prices. This finding has clear connections with our model as a quick glance at the upper plot of Figure 2 demonstrates that in periods where $\theta_t$ stands at very high or low levels, the target rate is very often at inertia. In turn, the lower plot in Figure 2 reveals that estimated filtered probabilities of regime-shifts show quite distinctively that these long inertial periods typically occur within the stable regime. For example, the second half of year 2000 is a period where the most probable state is the stable regime, and where the target rate reached the highest levels, above 6%. This period was immediately followed by an economic decline eventually leading to a target rate level standing at less than 2%.[19] Thus, if the stable regime is perceived as being less desirable and embodying more uncertainty, a higher probability of occurrence under the $\mathbb{Q}$ measure may indicate that risk-averse market operators attribute a higher market price of risk in that state. This is in line with Dai, Singleton and Yang (2007) who observe that bad economic states are more risk-neutral probable.

[19] Fontaine (2010) mentions that the December 2000 FOMC statement emphasizes higher energy costs, a declining consumer confidence, and lowering business' earnings and sales. Then, on January 2001, the FOMC unexpectedly lowered the target rate by 50 bps.

Table 3: Estimated Parameters Under the $\mathbb{Q}$ measure

| Parameters | Regime | Coefficients | Std |
|---|---|---|---|
| $\alpha_r$ | | 0.2715 | 0.5682 |
| $\alpha_s$ | | **0.1865** | 0.0637 |
| $b^{r,\mathbb{Q}}$ | | 0.9959 | — |
| $b^{s,\mathbb{Q}}$ | | 0.9999 | — |
| $c^{r,\mathbb{Q}}_{\rho_t}$ | Downward | -1.4438 | 3.3018 |
| | Stable | -1.5459 | 3.0284 |
| | Upward | -0.9712 | 2.0284 |
| $c^{s,\mathbb{Q}}_{\rho_t}$ | Downward | **0.0361** | 0.0142 |
| | Stable | **2.7795** | 0.9376 |
| | Upward | **0.0824** | 0.0284 |
| $q_{uu}$ | | **0.4396** | 0.0116 |
| $q_{su}$ | | **0.0171** | 0.0011 |
| $q_{ss}$ | | **0.6703** | 0.0994 |
| $q_{ds}$ | | **0.7382** | 0.2038 |
| | | $\lambda^r_{r,\rho_t}$ | $\lambda^r_{\theta,\rho_t}$ |
| | Downward | -0.5787 | 2.0266 |
| | Stable | -1.1166 | 2.6663 |
| | Upward | -0.6299 | 1.6052 |
| | | $\lambda^s_{s,\rho_t}$ | $\lambda^s_{\theta,\rho_t}$ |
| | Downward | -0.0155 | 0.0123 |
| | Stable | -0.0367 | 2.7583 |
| | Upward | -0.0447 | 0.0785 |
| SSE | 15.5781 | LRT | 1.6395e+03 |

The Nonlinear least square estimates of the $\mathbb{Q}$ parameters based on the analytical approximation (23) based on the risk-neutral processes (4) and (5) along with their standard deviations are reported in this Table. Estimates in bold are significant at the 5% level. The table also reports the resulting estimates $\lambda^{\bullet}_{\bullet,\rho_t}$ based on equation (7). SSE indicates the sum square errors based on the objective function (29). LRT indicates the value of the log ratio statistic for null hypothesis that regime-shift risk is not priced.

Table 4: Transition and Stationary Regime-Shift Probabilities

$$M^{\mathbb{P}} = \begin{pmatrix} 0.8395 & 0.1605 & 0 \\ 0.0483 & 0.8994 & 0.0524 \\ 0 & 0.1031 & 0.8969 \end{pmatrix} \qquad \pi^{\mathbb{P}} = \begin{pmatrix} 0.1661 \\ 0.5529 \\ 0.2810 \end{pmatrix}$$

$$M^{\mathbb{Q}} = \begin{pmatrix} 0.2168 & 0.7382 & 0 \\ 0.3126 & 0.6703 & 0.0171 \\ 0 & 0.5604 & 0.4396 \end{pmatrix} \qquad \pi^{\mathbb{Q}} = \begin{pmatrix} 0.2912 \\ 0.6878 \\ 0.0210 \end{pmatrix}$$

The estimated matrices of transition probabilities, $M^{\mathbb{P}}$ (reported in Table 2) and $M^{\mathbb{Q}}$ (reported in Table 3), and their corresponding vectors of probabilities drawn from the stationary distribution, $\pi^{\mathbb{P}}$ and $\pi^{\mathbb{Q}}$, are reported in this Table. Matrix entries are located following the convention down-stable-up latent-regimes.

Table 5 reports pricing error statistics based on the difference between the fitted model price and the market price, divided by the market price for interest rate caps for maturities of 2 to 5 years. The analysis is also presented in terms of Black (1976) flat-implied volatilities. The mean pricing errors appear relatively small with a maximum value of 1.93%, for the 2-year cap. This cap exhibits the largest mean absolute

Table 5: Pricing Performance of Interest Rate Caps and Related Black Volatilities

| | Interest rate Caps | | | |
|---|---|---|---|---|
| | 2 y | 3 y | 4y | 5y |
| Mean pricing error | 0.0193 | -0.0055 | -0.0047 | 0.0191 |
| Mean absolute pricing error | 0.0955 | 0.0628 | 0.0510 | 0.0671 |
| RMSE | 0.1201 | 0.0803 | 0.0646 | 0.0818 |
| Mean market Black vol (1) | 0.2584 | 0.2513 | 0.2411 | 0.2312 |
| Mean model Black vol (2) | 0.2265 | 0.2511 | 0.2407 | 0.2206 |
| Mean abs((1) - (2)) | 0.0848 | 0.0328 | 0.0200 | 0.0249 |

This pricing performance of the interest cap pricing model is reported in this Table. The pricing error corresponds to the difference between the market value of the cap and its model value divided by the market value. The set of statistics is based on pricing error while the second group is based on Black implied volatilities computed from the Black (1976) model.

error and RMSE statistics. The estimated empirical distribution of the nonlinear least squares residuals for each cap are illustrated in Figure 3. It is apparent that for the 2-year cap, there are several occurrences where the model generates cap prices substantially inferior to their market counterpart but this tendency progressively vanishes with the 3-year and 4-year caps. This indicates that the model exhibits a superior pricing performance for longer interest rate caplets. Overall, the model's pricing performance appears to be in line with previous studies even though the model relies on a single latent discrete variable with tree possible states.[20] An examination of the performance based on the corresponding implied volatilities leads to consistent findings. The largest difference in mean Black volatilities is observed for the 2-year interest rate cap where the mean market (model) implied Black volatility is 25.84% (22.65%). For the remaining caps, the differences amount to approximately 100 bps or less.

So far our findings indicate that monetary policy actions resulting into regime-shifts of the target rate, are associated with different levels of risk tolerance from cap market operators. In Figure 4, we push the investigation one step further by computing the mean implied Black volatilities under the target rate's regimes. To construct the three plots of Figure 4, we first compute the set of filtered probabilities $\{p_t(\rho_t)\}$ where $\rho_t = \{d, s, u\}$ from the transition probability matrix $M^{\mathbb{P}}$ for each week of the sample period. The most probable regime at week $t$, $\rho_t$ is sequentially identified by finding $\max\{p_t(\rho_t)\}$. All weeks are then sorted according to the most probable regime $\rho_t$, from which the Black implied volatilities computed each week are obtained for all cap maturities. This is a unique feature of our pricing approach as it allows the examination of the cap attributes under different latent regimes.

Although the three plots share a similar shape, it shows that the implied volatilities stand at much higher level when the target rate evolves within a stable or downward regimes. There is a clear cut difference between the mean Black implied volatilities computed under the stable and downward regimes than computed under the upward regime. In the former case, the mean volatilities stand at approximately 25% or more while in the latter case, they do not exceed 20%. This conjecture is a reflection that in some monetary policy states, risk-averse operators holding short positions on cap options require a higher premium while hedgers seeking

[20] For instance, Almeida, Graveline and Joslin (2011) estimate $A_1(3)$ and $A_2(3)$ models based on latent factors, a joint a set of swap rates and interest rate cap prices. Their RMSE statistics range from 9% to 14.6% for caps with maturities of 2 to 5 years. Jarrow, Li and Zhao (2007) fit a multifactor market model based on latent factors, with stochastic volatility and jump, to the entire cap volatility surface. For at-the-money differential caps up to 5 years, the mean pricing errors range from -0.04% to 8.39% for different model specifications. Li and Zhao (2006) examine the performance of quadratic-affine DTSM to price and hedge cap prices. Given the complexity of their model which includes three latent factors, estimations rely on swap rates and, in turn, are fitted to cap prices with various strikes. For at-the-money differential caps with maturities of 2 to 5 years, the observed RMSE range between 9.1% to 17.3% for the best specification. Based on a DTSM belonging to the Heath-Jarrow-Morton family with a principal component approach, Driessen, Klassen and Melenber (2003) point out that their model yields an average absolute mean pricing error of around 7%.

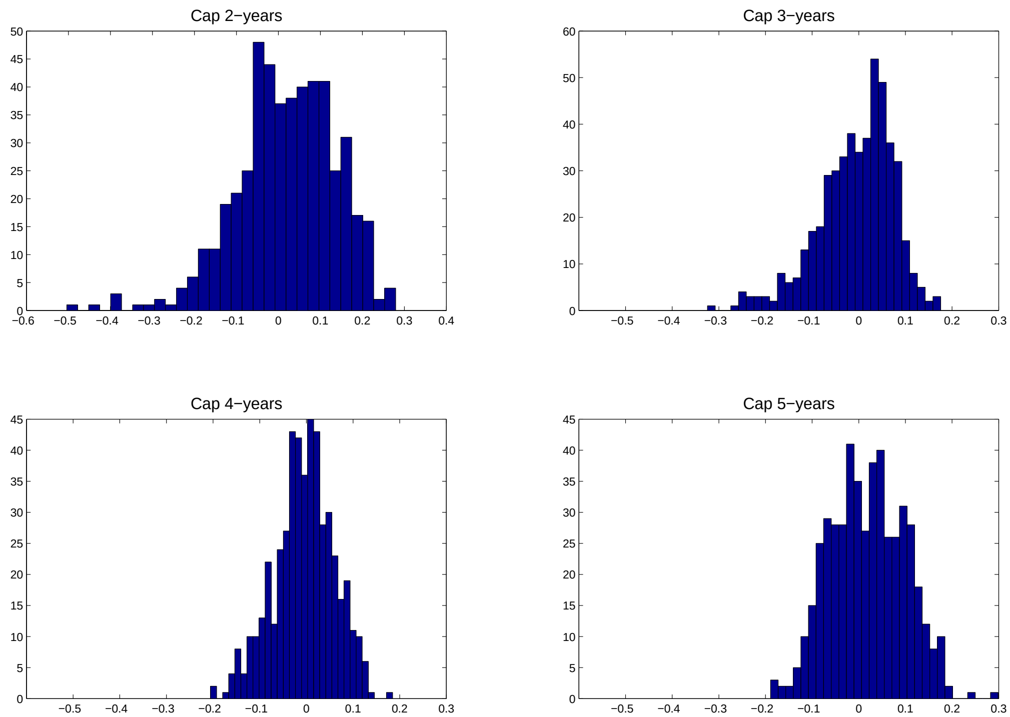


Figure 3: Pricing Errors

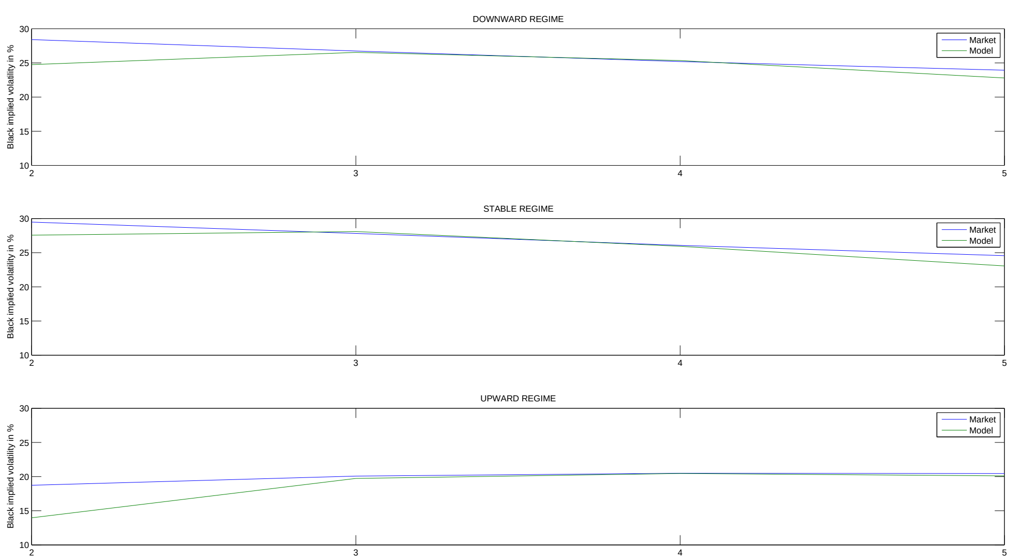


Figure 4: Term Structures of Black Implied Volatilities

to reduce their interest rate risk exposures accept to pay more. These plots are consistent with the findings in Table 4 suggesting that the upward latent regime is a state of lower risk aversion.

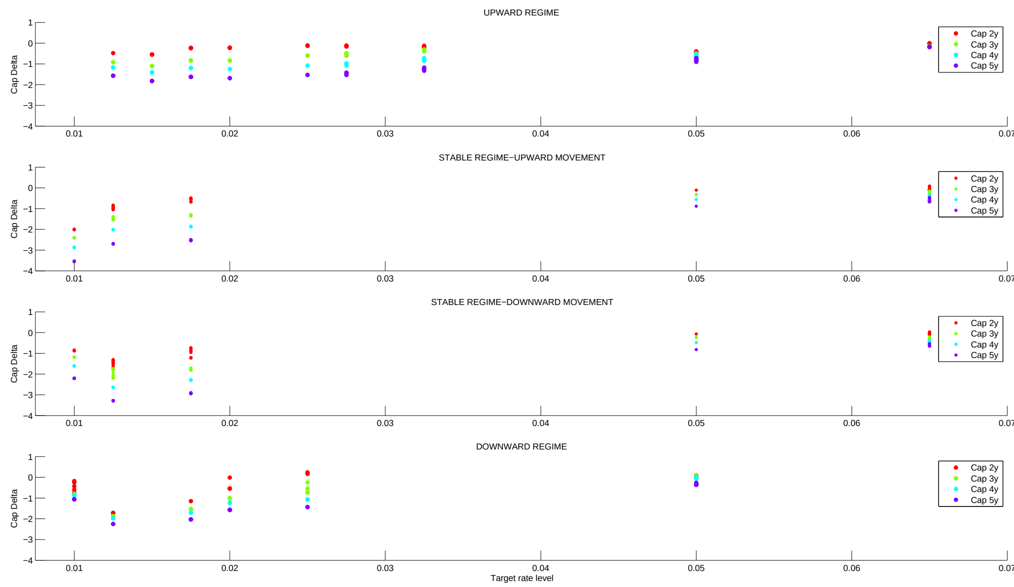


Figure 5: Option Cap Delta with Respect to the Target Rate

Figure 5 illustrates the response of option caps prices to a jump of 25 bps by the target rate under each regime. To compute the sensitivity measure, we first identify the most probable regime at each time interval following the procedure described in Figure 4, based on the filtered probabilities, $\{p_t(\rho_t)\}$. Then, at each time interval the sensitivity measure

$$\frac{d\ \widetilde{\text{cap}}(t, T_k; \theta, \rho, Y_t)}{d\theta} \tag{30}$$

is computed. It is not possible to derive a complete closed-form solution for equation (30) since the term $B^{(\bullet)}_{t,T_k-6}(\theta, \rho)$ found in equation (23) is built on a recursive process as implied by equation (14). To bypass this obstacle, the finite difference $B^{(\bullet)}_{t,T_k-6}(\theta + 25\text{ bps}, \rho) - B^{(\bullet)}_{t,T_k-6}(\theta, \rho)$ is computed. The sensitivity coefficient (30), is computed for each of the 491 weekly sets of state variables $\{\theta_t, \rho_t, r_t, s_t\}$ when the target rate is shocked by 25 bps.

The first (last) plot displayed in Figure 5 shows the response of cap prices to an upward (downward) jump of 25 bps by $\theta_t$ within the upward (downward) latent regime.[21] The two centre plots at Figure 5 respectively show the response of cap prices to a jump of 25 bps by $\theta_t$ within the stable regime. In all plots, cap price changes exhibit the counterintuitive response of varying inversely with changes in $\theta_t$. All things being equal, an increase in $\theta_t$ should induce an upward (potentially asymmetrical, see Piazzesi (2005)) movement in the Libor-swap curve leading, in turn, to a positive cap price change resulting in an increase in the moneyness of the underlying caplets. However, it appears that an opposite effect is at work. More interesting however, is the influence of regime-shifts in this context. Figure 5 shows that in absolute value, the magnitude of the target-rate delta is higher within the stable regime such that a jump of 25 bps of $\theta_t$, irrespective of its directionality, induces larger cap price changes in that state. This finding leads to a coherent story with the observations made in Table 4 and Figure 4 where the stable regime constitutes a less desirable state for market operators active the option cap market. It shows that monetary actions undertaken by the FED when $\theta_t$ is located in that state, are surrounded by potentially higher risk aversion magnifying the cap price responses.

[21] As mentioned before, the Markov dynamics of $\theta_t$ are devised such that a reduction (increase) of $\theta_t$ is not possible within the upward (downward) regime.

# 6 Concluding remarks

Previous studies based on dynamic term structure modelling underline the importance of accounting for monetary policy actions. Inspired by this trend, this study examines the role played by the Fed's target rate dynamics on option cap pricing in a regime-shift environment. The proposed model is based on a joint modelling of the target rate and the latent regime in which it evolves, as a discrete time-inhomogeneous Markov chain where movements occur only on at FOMC meetings. The set of state variables is complemented by the Fed Funds rate and the slope of the Libor-swap curve's slope. Since the state variables are observable, their dynamics are estimated under the physical measures. Under the risk-neutral measure, an analytical expression for pricing caplets is derived from the estimated risk-neutral parameters. The model allows for the pricing of regime-shift risk.

The model's empirical performance significantly improves by allowing the regime transition probability matrix to differ under the data-generating and risk-neutral measures. The risk-neutral transition probabilities indicate that regime-shift are substantially more probable under this measure. The empirical analysis also shows that the target rate's stable (upward) regime is perceived by market operators as being the least (most) desirable state. In addition, regrouping the term structures of mean Black implied volatilities according to the most probable regime reveals that these volatilities stand at a higher level under the stable and downward regimes. Finally, cap deltas with respect to the target rate show that jumps in that state variable entail the largest (smallest) price changes under the stable (upward) latent regime. Overall, we conclude that interest rate cap prices exhibit different properties across the target rate's various latent regimes.

Our approach was driven by important modelling choices. For example, our model does not allow for the pricing of target rate movements; the same set of transition probabilities are applied under both probability measures. Preliminary trials to relax this constraint quickly led to a proliferation of parameters and overfitting. Nevertheless, in light of our finding, this appears to be an interesting avenue to explore. However, it is likely that the model's ability to match the term structure of implied volatilities could have been improved by integrating latent state variables into the factor structure. However, we have purposely focused on observable state variables in order to gain a better comprehension of the forces at work in the pricing model. It also seems that this approach lends itself more easily into practical applications for market operators. Finally, the model imposes constant volatilities on the processes of the Fed Funds rate and the yield curve slope within each regime. Relaxing this constraint may provide the model with additional flexibility but constitutes a serious impediment to its implementation.

# Appendix

Lemma 1 stated in Section 3 is a key step in the caplet pricing formula in the absence of regime transition.

**Proof of Lemma 1**. The proof extensively uses a property of Gaussian variables whereby $\mathrm{E}[\exp(X)] = \exp\left(\mathrm{E}[X] + \frac{1}{2}\mathrm{Var}[X]\right)$. We have

$$\mathrm{E}\left[e^{X_1}\max\left(1 - ae^{X_2}, 0\right)\right] = \int_{-\infty}^{-\log a}\left(\int_{-\infty}^{\infty} e^{x_1}(1 - ae^{x_2}) f(x_1, x_2)\,\mathrm{d}x_2\right)\mathrm{d}x_1$$

with $f$, the joint probability density function of $(X_1, X_2)$. The integral breaks down in two. Because the conditional distribution of $X_1$ given $X_2$ is Gaussian with expectation $\mu_1 + \eta\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$ and variance $\sigma_1^2\left(1 - \eta^2\right)$, the first integral becomes

$$\int_{-\infty}^{-\log a}\left(\int_{-\infty}^{-\infty} e^{x_1} f(x_1, x_2)\,\mathrm{d}x_2\right)\mathrm{d}x_1 = \int_{-\infty}^{-\log a} f_2(x_2)\,\mathrm{E}\left[e^{X_1}\,\middle|\, X_2 = x_2\right]\mathrm{d}x_2$$

$$= \int_{-\infty}^{-\log a} f_2(x_2)\exp\left\{\mu_1 + \eta\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2) + \frac{\sigma_1^2}{2}\left(1 - \eta^2\right)\right\}\mathrm{d}x_2$$

$$= \exp\left\{\mu_1 + \frac{\sigma_1^2}{2}\right\}\int_{-\infty}^{-\log a}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 + \eta\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \frac{(\eta\sigma_1)^2}{2}\right\}\mathrm{d}x_2$$

where $f_2$ denotes the marginal density function of $X_2$. Applying the change of variable $z = (x_2-\mu_2)/\sigma_2$, the integral becomes

$$\exp\left\{\mu_1 + \frac{\sigma_1^2}{2}\right\}\int_{-\infty}^{-(\mu_2+\log a)/\sigma_2}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(z-\eta\sigma_1)^2\right\}\mathrm{d}z.$$

The latter integral is just equal to $N\left(\frac{\mu_2+\log a}{\sigma_2} - \sigma_1\eta\right)$. That gives the first term in (21). Similarly, the second integral is

$$a\int_{-\infty}^{-\log a}\left(\int_{-\infty}^{\infty} e^{x_1+x_2}f(x_1,x_2)\,\mathrm{d}x_2\right)\mathrm{d}x_1 = a\int_{-\infty}^{-\log a} e^{x_2}f_2(x_2)\,\mathrm{E}\left[e^{X_1}\middle| X_2 = x_2\right]\mathrm{d}x_2$$

$$= \exp\left\{\mu_1 + \frac{\sigma_1^2}{2}\right\}\int_{-\infty}^{-\log a}\frac{1}{\sqrt{2\pi}}\exp\left\{x_2 - \frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 + \eta\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \frac{(\eta\sigma_1)^2}{2}\right\}\mathrm{d}x_2\,.$$

The same change of variable gives the second term in equation (21), and is equal to

$$\begin{aligned}
&\exp\left\{\mu_1+\mu_2 + \frac{\sigma_1^2}{2}\right\}\int_{-\infty}^{-(\mu_2+\log a)/\sigma_2}\frac{1}{\sqrt{2\pi}}\exp\left\{\sigma_2 z - \frac{1}{2}(z-\eta\sigma_1)^2\right\}\mathrm{d}z\\
=\;& \exp\left\{\mu_1+\mu_2+\frac{\sigma_1^2(1-\eta^2)}{2} + \frac{(\eta\sigma_1+\sigma_2)^2}{2}\right\}\int_{-\infty}^{-\frac{\mu_2+\log a}{\sigma_2}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(z-\eta\sigma_1-\sigma_2)^2}{2}\right\}\mathrm{d}z\\
=\;& F_2 N(d_1-\sigma_2)\,. \qquad \square
\end{aligned}$$

**Proof of Proposition 1**. For any integers $U < T$, the week $U$ caplet value satisfies

$$c(U,T+12) = \mathrm{E}_U^{\mathbb{Q}}\left[D\left(U,T\right)\max\left(1-(1+0.25K)\exp\left(B_{T,12}(\theta_T,\rho_T) + A_{12}'Y_T\right),0\right)\right]$$

where $B_{T,12}(\theta_T,\rho_T)$ and $A_{12}'$ are defined in Theorem 1. If there is no regime or target rate movement in the interval $]U,T]$, then $\theta_T = \theta_U$, $\rho_T = \rho_U$ and the caplet value becomes

$$c(U,T) = \mathrm{E}_U^{\mathbb{Q}}\left[D\left(U,T\right)\max\left(1-a\exp\left(A_{12}'Y_T\right),0\right)\right]$$

where

$$a = (1+0.25K)\exp\left(B_{T,12}(\theta_U,\rho_U)\right). \tag{31}$$

Because $D\left(U,T\right) = \exp\left(-\Delta\sum_{t=U}^{T-1}\alpha' Y_t\right)$, a comparison with equation (21) suggests that $X_1 = -\Delta\sum_{t=U}^{T-1}\alpha' Y_t$ while $X_2 = A_{12}'Y_T$. The remainder of the proof consists in finding the two first moments of these Gaussian random variables. Let $y_t$ denotes any of the two state variables $r_t$ or $s_t$, and $\tau = T-U$. A recursion argument based on equation (4) and (5) and the absence of target rate and regime movements yields

$$\begin{aligned}
y_{U+\tau} &= \left(b^{y,\mathbb{Q}}\right)^{\tau} y_U + c_{\rho_U}^{y,\mathbb{Q}}\,\beta_\tau^{y,\mathbb{Q}}\,\theta_U + \sigma_{\rho_U}^{y}\sum_{t=0}^{\tau-1}\left(b^{y,\mathbb{Q}}\right)^{t} Z_{U+\tau-t}^{y,\mathbb{Q}}\\
&= \left(b^{y,\mathbb{Q}}\right)^{\tau} y_U + c_{\rho_U}^{y,\mathbb{Q}}\,\beta_\tau^{y,\mathbb{Q}}\,\theta_U + \sigma_{\rho_U}^{y}\sum_{t=1}^{\tau}\left(b^{y,\mathbb{Q}}\right)^{\tau-t} Z_{U+t}^{y,\mathbb{Q}}
\end{aligned}$$

and

$$\sum_{u=U}^{U+\tau-1} y_u = \sum_{u=0}^{\tau-1}\left(b^{y,\mathbb{Q}}\right)^{u} y_U + c_{\rho_U}^{y,\mathbb{Q}}\left(\sum_{u=1}^{\tau-1}\beta_u^{y,\mathbb{Q}}\right)\theta_U + \sigma_{\rho_U}^{y}\sum_{u=1}^{\tau-1}\sum_{t=1}^{u}\left(b^{y,\mathbb{Q}}\right)^{u-t} Z_{U+t}^{y,\mathbb{Q}}$$

$$= \beta_\tau^{y,\mathbb{Q}} y_U + \frac{c_\rho^{y,\mathbb{Q}}}{1-b^{y,\mathbb{Q}}}\left(\tau - \beta_\tau^{y,\mathbb{Q}}\right)\theta_U + \sigma_{\rho_U}^y \sum_{t=1}^{\tau-1} \beta_{\tau-t}^{y,\mathbb{Q}} Z_{U+t}^{y,\mathbb{Q}}.$$

where $\beta_\tau^{y,\mathbb{Q}} = \frac{1-\left(b^{y,\mathbb{Q}}\right)^\tau}{1-b^{y,\mathbb{Q}}}$.

The laborious calculation of first moments of $X_1$ and $X_2$ is a direct consequence of these two expressions. More precisely, the conditional first moments are

$$\mu_1 = -\Delta\alpha' \begin{pmatrix} \beta_\tau^{r,\mathbb{Q}} r_U + \frac{c_{\rho_U}^{r,\mathbb{Q}}}{1-b^{r,\mathbb{Q}}}\left(\tau - \beta_\tau^{r,\mathbb{Q}}\right)\theta_U \\ \beta_\tau^{s,\mathbb{Q}} s_U + \frac{c_{\rho_U}^{s,\mathbb{Q}}}{1-b^{s,\mathbb{Q}}}\left(\tau - \beta_\tau^{s,\mathbb{Q}}\right)\theta_U \end{pmatrix} \text{ and } \mu_2 = A_{12}' \begin{pmatrix} \left(b^{r,\mathbb{Q}}\right)^\tau r_U + c_{\rho_U}^{r,\mathbb{Q}}\ \beta_\tau^{r,\mathbb{Q}}\theta_U \\ \left(b^{s,\mathbb{Q}}\right)^\tau s_U + c_{\rho_U}^{s,\mathbb{Q}}\ \beta_\tau^{s,\mathbb{Q}}\theta_U \end{pmatrix} \tag{32}$$

The variances satisfy

$$\sigma_1^2 = \Delta \begin{pmatrix} \alpha^r \frac{\sigma_{\rho_U}^r}{1-b^{r,\mathbb{Q}}} & \alpha^s \frac{\sigma_{\rho_U}^s}{1-b^{s,\mathbb{Q}}} \end{pmatrix} \Sigma_1 \begin{pmatrix} \alpha^r \frac{\sigma_\rho^r}{1-b^{r,\mathbb{Q}}} \\ \alpha^s \frac{\sigma_\rho^s}{1-b^{s,\mathbb{Q}}} \end{pmatrix} \text{ and } \sigma_2^2 = A_{12}'\Sigma_2 A_{12} \tag{33}$$

with

$$\Sigma_1 = \begin{pmatrix} \tau - 2\beta_\tau^{r,\mathbb{Q}} + \widetilde{\beta}_\tau^{r,\mathbb{Q}} & \gamma_{\rho_U}\left(\tau - \widetilde{\beta}_\tau^{r,\mathbb{Q}} - \widetilde{\beta}_\tau^{s,\mathbb{Q}} + \frac{1-\left(b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}}\right) \\ \gamma_{\rho_U}\left(\tau - \widetilde{\beta}_\tau^{r,\mathbb{Q}} - \widetilde{\beta}_\tau^{s,\mathbb{Q}} + \frac{1-\left(b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}}\right) & \tau - 2\beta_\tau^{s,\mathbb{Q}} + \widetilde{\beta}_\tau^{s,\mathbb{Q}} \end{pmatrix},$$

$$\Sigma_2 = \begin{pmatrix} \left(\sigma_{\rho_U}^r\right)^2 \widetilde{\beta}_\tau^{r,\mathbb{Q}} & \sigma_{\rho_U}^r \sigma_{\rho_U}^s \gamma_{\rho_U} \frac{1-\left(b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}} \\ \sigma_{\rho_U}^r \sigma_{\rho_U}^s \gamma_{\rho_U} \frac{1-\left(b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{y,\mathbb{Q}}b^{s,\mathbb{Q}}} & \left(\sigma_{\rho_U}^s\right)^2 \widetilde{\beta}_\tau^{s,\mathbb{Q}} \end{pmatrix}$$

and

$$\widetilde{\beta}_\tau^{y,\mathbb{Q}} = \frac{1-\left(b^{y,\mathbb{Q}}\right)^{2\tau}}{1-\left(b^{y,\mathbb{Q}}\right)^2}.$$

Finally, the correlation parameter is

$$\begin{aligned} \eta \quad = \quad & -\frac{\left(\sigma_{\rho_U}^r\right)^2}{\sigma_1\sigma_2}\frac{\alpha^r A_{12}^r}{1-b^{r,\mathbb{Q}}}\left(\beta_\tau^{r,\mathbb{Q}} - \widetilde{\beta}_\tau^{r,\mathbb{Q}}\right) - \frac{\left(\sigma_{\rho_U}^s\right)^2}{\sigma_1\sigma_2}\frac{\alpha^s A_{12}^s}{1-b^{s,\mathbb{Q}}}\left(\beta_\tau^{s,\mathbb{Q}} - \widetilde{\beta}_\tau^{s,\mathbb{Q}}\right) \\ & -\frac{\sigma_{\rho_U}^r\sigma_{\rho_U}^s\gamma_{\rho_U}}{\sigma_1\sigma_2}\left(\frac{\alpha^r A_{12}^s}{1-b^{r,\mathbb{Q}}}\left(\beta_\tau^{s,\mathbb{Q}} - \frac{1-\left(b^{r,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{r,\mathbb{Q}}b^{s,\mathbb{Q}}}\right) - \frac{\alpha^s A_{12}^r}{1-b^{s,\mathbb{Q}}}\left(\beta_\tau^{r,\mathbb{Q}} - \frac{1-\left(b^{r,\mathbb{Q}}b^{s,\mathbb{Q}}\right)^\tau}{1-b^{r,\mathbb{Q}}b^{s,\mathbb{Q}}}\right)\right). \end{aligned} \tag{34}$$

□

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