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A mean-field games approach**

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Dynamic marketing policies with online-review-sensitive consumers: A mean-field games approach

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Abstract: We consider a large group of consumers who can choose between two products at each purchasing occasion. Their choice is influenced by the marketing strategies of the firms, e.g., price, advertising, the reputation of the brands, and by the product reviews. The problem is modeled as a Stackelberg mean-field game, with one firm acting as leader and the consumers as followers. We determine the conditions under which an equilibrium exists and provide a numerical scheme to compute it. We give some examples to illustrate the type of insights that can be obtained with our model.

Keywords: Online reviews, advertising, pricing, goodwill, mean-field games

Résumé : On considère un grand groupe de consommateurs qui peuvent choisir entre deux produits substituables à chaque occasion d'achat. Leur choix est influencé par les stratégies marketing des deux firmes (prix, publicité et réputation de la marque) et par les avis en ligne donnés par les consommateurs. Le problème est modélisé comme un jeu à champ moyen, où une des deux firmes agit comme leader et les consommateurs comme suiveurs. On détermine les conditions d'existence de l'équilibre de Stackelberg et on fournit un algorithme pour son calcul. On illustre le modèle et l'équilibre à travers quelques exemples numériques.

Mots clés : Avis en ligne, publicité, pricing, réputation, jeux à champs moyens

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1 Introduction

We consider a large group of consumers who can choose, at each buying occasion, between two substitute goods, one focal product offered by a profit-maximizing firm and an alternative outside good supplied by a non-strategic firm, that is, an alternative product whose characteristics such as price or reputation are given. Individual discrete choice between the two products is shaped by (i) the price and other marketing variables, e.g., advertising, (ii) the product goodwill, (iii) own past consumption experience, and (iv) the aggregate product ratings by consumers. Retaining marketing instruments, brand image (reputation or goodwill) and past purchase as demand's determinants has nothing surprising. Further, there is abundant evidence that consumer reviews made available by the firms themselves, or by platforms such as TripAdvisor, Rotten Tomatoes and Yelp, have a significant impact on product sales Jiang and Guo (2015). To represent the strategic interactions between the focal firm (from now on the firm or producer) and consumers, we adopt a Stackelberg game model, where the firm acts as leader and the consumers are the followers. In this framework, the firm sets its price and makes other marketing decisions first, while consumers choose and rate the products afterwards. To account for the fact that product goodwill can only be built over time and not overnight, as well as for the impact of past purchasing on today's choices, our model is dynamic.

The vehicle of consumer interaction is the most current aggregate rating of the focal and outside-option products. At the top level, the firm optimizes its decisions based on consumers' anticipated rating, and the parameters describing the marketing decisions of the competitor. We assume that consumers care *only* about the average rating of the products. Therefore, in the limit of an infinite consumer population, an individual consumer's influence on other consumers vanishes. This situation is to be contrasted with that of the aggregate consumer rating, the impact of which persists on individual consumer decisions. It is precisely this partial asymptotic decoupling that the theory of *mean-field games* (MFG), a critical tool in our analysis, capitalizes on to achieve a simplified view of the overall game; in this case, a Stackelberg game with the firm on the top of the hierarchy, and the generic individual versus the mass of individuals represented by their average ratings in a Nash configuration, at the lower level of the hierarchy.

Our objective is to address the following research questions:

1. Under what conditions does a consumer Nash equilibrium exist?
2. Under what conditions does a producer-consumer Stackelberg equilibrium exist?
3. How can we compute the leader firm's marketing strategies?
4. How do these strategies vary with the main parameter values of the model?

Our work draws from and contributes to different research areas, namely: (i) dynamic advertising and goodwill models; (ii) consumer ratings; and (iii) applied mean-field games. Following the seminal contributions by Lanchester (1916) (Kimball (1957) was probably the first to recognize the applicability of Lanchester's models of military combat to problems of dynamic advertising competition. He wrote that the model "has been used with great success to describe the effect of advertising"), Nerlove and Arrow (1962), Vidale and Wolfe (1957) and Ozga (1960), a large literature developed with the aim of characterizing optimal dynamic advertising policies in the monopoly case and competitive equilibrium advertising strategies in oligopoly. We shall refrain from reviewing this literature and refer the reader to the surveys in Feichtinger et al. (1994); Huang et al. (2012); Jørgensen and Zaccour (2014), and to the books by Erickson (2012) and Jørgensen and Zaccour (2004). In this literature, the state variable is either sales volume, market share or the brand goodwill. In the specific stream with goodwill dynamics, to which this paper belongs, the evolution of the state variable depends on advertising expenditures, and sometimes on another control variable, e.g., the price(s) or quality of the product(s). With very few exceptions, the dynamics are linear in both the state and control variables. In our model, there is no restriction on the number of variables that can enter the goodwill dynamics, nor on its functional form. Indeed, it can be nonlinear. We only require continuity of the dynamics with respect to its

parameters. However, this flexible and richer model of the goodwill dynamics comes at a cost, that is, the game does not admit an analytical solution and the equilibrium has to be determined numerically.

Online consumer review websites have proliferated during the last two decades or so. Nowadays, with a couple of clicks, a consumer can quickly obtain from others' experience with particular goods or services an overall evaluation of the quality of food in an all-inclusive resort, the atmosphere in a restaurant or the performance of a sport equipment. In any event, whether totally reliable or not, these reviews have had a tremendous impact on business. According to TripAdvisor, 90% of hotel managers think that review websites are very important to their business and 81% monitor their reviews at least weekly Crapis et al. (2017). Empirical research has found that positive reviews increase sales. To illustrate, on a five-star scale, Luca (2016) determines that a one-star improvement translates to a 5%–9% increase in sales for restaurants in Seattle, Washington. Looking at the impact of consumer reviews on sales of books at Amazon.com and Barnesandnoble.com, Chevalier and Mayzlin (2006) find that an improvement in book reviews leads to an increase in sales, and, interestingly, that the impact of one-star reviews is greater than the impact of five-star reviews.

Recent literature looks at online reviews from different perspectives. For instance, Liu et al. (2017) analyze how online reviews and past sales volume information jointly affect consumer purchase decisions as well as firms' pricing strategies. Jiang and Guo (2015) look at how the design itself of consumer review systems impacts review outcomes and product sales. Crapis et al. (2017) investigate monopoly pricing in the presence of social interactions among heterogeneous consumers and obtain, numerically, that accounting for these interactions improves profits. Pekgün et al. (2018) assess the impact of unequal weighting of positive and negative reviews on price competition. Chen and Xie (2008) argue that consumer reviews are a form of word-of-mouth that works as free sales assistants to help consumers match their needs. Our contribution to the literature is in the modeling process of the interactions between consumers. Indeed, we consider a large number of rational consumers that both rely on the average product rating when choosing a product, and rate the chosen product. As a result, this average is generated endogenously by the model as an "equilibrium outcome". More precisely, it will be equal to the average product rating when the consumers' choices constitute a Nash equilibrium. We use the MFG methodology to compute approximate Nash equilibria that converge to exact Nash equilibria as the number of consumers increases to infinity. The approximate equilibria are simpler to compute. More importantly, the MFG methodology allows us to anticipate the dynamics of the average product rating and the corresponding market share. The latter is then used by the leader firm to design optimal marketing and pricing strategies.

The MFG theory was originally developed in a series of papers by Huang et al. (2003, 2007, 2006), and independently by Lasry and Lions (2006a,b, 2007). Since then, it has found applications in many different areas, e.g., transportation, energy systems (smart grids) and finance. To the best of our knowledge, the only published application in marketing is Salhab et al. (2018), where a mean-field model is introduced, whereby a producer makes advertising investments to sway consumer choices in favor of its product. The choices are made at the end of the game, and are influenced by both the level of advertising investments and the aggregate consumers' opinions. Pricing and other marketing instruments, repeat purchase behavior and goodwill dynamics were not accounted for in Salhab et al. (2018), whereas our model includes these relevant features. We believe that such a model can help understand markets where social interactions, in the form of reviews, play a crucial role. It gives firms a tool to make optimal decisions and to assess the impact of parameter values on their policies and outcomes, such as market share and profit.

The rest of the paper is organized as follows. We introduce our mathematical model in Section 2. In Section 3, we discuss the MFG theory, which constitutes the methodology of this work. We analyze the problem in Section 4, and develop a Stackelberg solution. In Section 5, we propose a numerical scheme to compute the producer's optimal marketing activities and pricing policies. Section 6 provides a numerical example and sensitivity analysis, and Section 7 concludes the paper.

2 Model

Denote by t the time index and let T be the planning horizon. At each time $t \in \{1, \dots, T\}$, consumers can choose between two substitutable products, offered by the firm under consideration (product 0) and a non-strategic seller (product 1), respectively. By non-strategic seller, we mean a producer whose decisions, e.g., price and product quality, are fixed throughout the planning horizon. In our framework, product 1 is an outside option available to consumers, which allows us to capture in the most parsimonious way a competitive environment. In fact, product 1 does not need to be the same at each time period.

We formulate our producer-consumer problem as a Stackelberg dynamic game, with the producer acting as leader and a large number N of consumers behaving as non-cooperative followers. At each time $1 \leq t \leq T-1$, the leader chooses a price $p_t^0 \in [0, 1]$ for its product and the expenditures in its different marketing activities, e.g., advertising, promotion, quality. Denote by $m_t^0 = (m_{1t}^0, \dots, m_{dt}^0)$ the vector of per-consumer marketing activities at time $1 \leq t \leq T-1$, where $m_t^0 \in [0, 1]^d$, and d is a positive integer. Next, each consumer decides to buy product 0 or product 1. Let $c_t^i \in \{0, 1\}$ be consumer's i decision variable at time $1 \leq t \leq T-1$, where $c_t^i = j$ means that consumer i buys product j at time t . When making her decision, consumer i will be influenced by the other consumers' choices. This social interaction will be modeled using a mean-field game approach.

Denote by $G_t^0 \in [0, 1]$ the goodwill (reputation or attractiveness) of product 0 at time $1 \leq t \leq T$. The product goodwill summarizes the (perceptual) information that consumers derive from catalogs, word-of-mouth, websites and publicity. The evolution over time of the goodwill is described by the following difference equation:

$$G_{t+1}^0 = h_t(G_t^0, m_t^0, p_t^0), \quad G_1^0 \text{ given}, \quad (1)$$

where $h_t : [0, 1] \times [0, 1]^d \times [0, 1] \mapsto [0, 1]$ is a continuous function and G_1^0 is the initial goodwill value. The price and the goodwill of product 1 at time t are denoted $p_t^1 \in [0, 1]$ and $G_t^1 \in [0, 1]$, respectively. Following our assumption that the seller of product 1 is non-strategic (or passive), $p_{1:T-1}^1$ and $G_{1:T}^1$ are given parameters known to all agents. Here and in the remainder of the paper, $y_{s:t}$, $1 \leq s < t \leq T$, denotes the vector $(y_s, \dots, y_t)$.

Denote by $r_t^i \in [0, 1]$ consumer i 's preference or *rating* of the two products at time $1 \leq t \leq T$, where $r_t^i = 0$, $r_t^i = 1$, and $r_t^i = 0.5$ refer respectively to "consumer i totally prefers product 0", "consumer i totally prefers product 1" and "consumer i is indifferent between the two products". The rating r_t^i is a state variable whose value is the result of the consumer's personal experience with the products and the most recently observed goodwill of the products. It evolves according to the following rule. At time t , consumer i buys one of the two products, that is, product $j \in \{0, 1\}$. After consuming this product, she changes her rating to r_{t+1}^i . The new rating depends on the current goodwill value G_t^j of product $j \in \{0, 1\}$, and also on some personal attributes, unobservable by the producers and other consumers. As a result, r_{t+1}^i will be a noisy version of the goodwill for the product chosen at time t . To describe this update process, we introduce a family of probability density functions $\{f_G\}_G$ on $[0, 1]$ indexed by $G \in [0, 1]$. In general, f_G are such that their supports are concentrated around G . If consumer i chooses product 0 at time t , then her next rating r_{t+1}^i has a density function $f_{1-G_t^0}$, i.e., tends to be in the vicinity of $1 - G_t^0$. This means that if G_t^0 is high and individual i consumes product 0 at time t , then she will tend to maintain a preference for product 0 at time $t+1$. If consumer i chooses at time t product 1, then her next rating r_{t+1}^i has a density function $f_{G_t^1}$. Thus, the variable r_t^i evolves according to the probability density kernel

$$\mathbb{P}(r_{t+1}^i \in [r, r + dr] | c_t^i = j, G_t^0 = G^0, G_t^1 = G^1) = p(r, j, G^0, G^1) dr, \quad (2)$$

where $p(r, j, G^0, G^1) = f_{1-G^0}(r) 1_{\{0\}}(j) + f_{G^1}(r) 1_{\{1\}}(j)$, and 1_A denotes the indicator function of set A . It is possible to add a dependency on prices in these transition kernels, with no change in the analysis.

Example 1 Assume that $f_G(r) = U_\epsilon(r - G(1 - \epsilon))$, where U_ϵ is the uniform probability distribution with support $[0, \epsilon]$, with $\epsilon < 1$. Suppose that consumer i buys product 0 at time t . If the goodwill value was maximum at t , i.e., $G_t^0 = 1$, then r_{t+1}^i belongs to $[0, \epsilon]$ with probability 1. In other words, consumer i prefers product 0 at time $t + 1$. On the other hand, if $G_t^0 = 0$, then r_{t+1}^i belongs to $[1 - \epsilon, 1]$ with probability 1. In case G_t^0 is between 0 and 1, then $r_{t+1}^i \in [G_t^0(1 - \epsilon), G_t^0(1 - \epsilon) + \epsilon]$ with probability 1.

We summarize the notation introduced above in Table 1.

Table 1: Notation

Producer 0		
$m_t^0 \in [0, 1]^d$		Marketing activities (decision variables)
$p_t^0 \in [0, 1]$		Price (decision variable)
$G_t^0 \in [0, 1]$		Goodwill (state variable)
Producer 1		
$p_t^1 \in [0, 1]$		Price (parameter)
$G_t^1 \in [0, 1]$		Goodwill (parameter)
Consumer i		
$c_t^i \in \{0, 1\}$		Decision variable / Choose product 0 or 1
$r_t^i \in [0, 1]$		Rating (preference)
Population		
$\bar{r}_t = \frac{1}{N} \sum_{i=1}^N r_t^i$		Average rating
$1 - \bar{c}_t = 1 - \frac{1}{N} \sum_{i=1}^N c_t^i$		Market share of product 0

We suppose that a consumer's binary choice at each purchasing occasion depends on the following factors: (i) the prices p_t^0 and p_t^1 ; (ii) her current rating r_t^i (it should be noted that the rating r_t^i at time t is a result of the goodwill of the product purchased at time $t - 1$); (iii) the average rating $\bar{r}_t := \frac{1}{N} \sum_{i=1}^N r_t^i$, which can be seen as an aggregate measure of the preferences of consumers belonging to the same market segment; (iv) the goodwill of the products (G_t^0 and G_t^1); and finally (v) her past choice c_{t-1}^i , which is one way of accounting for brand loyalty. We adopt the following per-step utility function for consumer i at time $1 \leq t \leq T - 1$:

$$l_t(c_{t-1:t}^i, r_t^i, \bar{r}_t, p_t^0, p_t^1, G_t^0, G_t^1) = L_0(1 - c_t^i) + L_1 c_t^i = \begin{cases} L_0, & \text{if product 0 is chosen,} \\ L_1, & \text{if product 1 is chosen,} \end{cases} \quad (3)$$

where $L_0 = \alpha_1(1 - r_t^i) + \alpha_2(1 - \bar{r}_t) + \alpha_3 G_t^0 + \alpha_4(1 - p_t^0) + \alpha_5(1 - c_{t-1}^i)$ and $L_1 = \alpha_1 r_t^i + \alpha_2 \bar{r}_t + \alpha_3 G_t^1 + \alpha_4(1 - p_t^1) + \alpha_5 c_{t-1}^i$, $l_T = 0$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are some nonnegative parameters such that $\sum_{k=1}^5 \alpha_k = 1$.

Assumption 1 We assume that $c_0^i = 0$ if $r_1^i \leq 1/2$ and $c_0^i = 1$ if $r_1^i > 1/2$, i.e., the consumers who initially prefer product 0 (resp. product 1) chose product 0 (resp. product 1) before the start of the game.

Remark 1 We normalize the utility function parameter values to sum up to one to simplify the interpretation. The model could have been calibrated differently without any conceptual change.

We make the following two remarks:

1. Since $c_t^i \in \{0, 1\}$ at any time t , it is apparent that the utility is equal to L_j when product $j \in \{0, 1\}$ is chosen. The utility-maximizing consumer makes her choice by comparing L_0 and L_1 .
2. As expected, each product's utility is decreasing in its price and increasing in its goodwill. Recalling that the rating scale is from 0 to 1, with $r_t^i = 0$ meaning that consumer totally prefers product 0, the lower (larger) the value of r_t^i , the higher (lower) preference for product 0 (product 1). The same reasoning applies to the average rating. Therefore, our formulation takes into account a crowd effect, that is, the consumer does not seek exclusivity. Finally, brand loyalty matters, that is, the consumer prefers buying the product she chose at the previous buying occasion.

The per-step profit of producer 0 at time $1 \leq t \leq T-1$ is

$$l_t^0(m_t^0, p_t^0, \bar{c}_t) = \alpha_0(1 - \bar{c}_t)p_t^0 - \beta_0(m_t^0), \quad (4)$$

and $l_T^0 = 0$, where $\alpha_0 \in \mathbb{R}$ is a positive scaling parameter, β_0 is a continuous function increasing in each component of vector m_t^0 , and $1 - \bar{c}_t = 1 - \frac{1}{N} \sum_{i=1}^N c_t^i$ is the fraction of consumers choosing product 0 at time t , i.e., the market share of this product. This profit is equal to the difference between the average sales per consumer of product 0 and the cost per consumer in marketing activities. We assume that the consumers' initial ratings r_1^i , $1 \leq i \leq N$, are independent and identically distributed according to a known initial probability density function $\pi(r)$. We also assume that the decision variables of producer 0, $(m_{1:T-1}^0, p_{1:T-1}^0)$, belong to following set,

$$\mathcal{U}_0 = \left\{ (m_{1:T-1}^0, p_{1:T-1}^0) \in [0, 1]^{(T-1)(d+1)} \mid B_1(m_{1:T-1}^0, p_{1:T-1}^0) = 0, B_2(m_{1:T-1}^0, p_{1:T-1}^0) \leq 0 \right\}. \quad (5)$$

Here, $B_j : [0, 1]^{(T-1)(d+1)} \mapsto \mathbb{R}^{d_j}$, with $d_j \in \mathbb{N}$, $j = 1, 2$, are continuous functions that model the constraints on the prices and on marketing expenditures. Note that $\mathcal{U}_0$ is a compact set. For example, suppose that the price must be higher than a lower bound p_l (for brand image or cost reasons) and that the total available budget for marketing activities is b . In this case, $B_1(m_{1:T-1}^0, p_{1:T-1}^0) = b - \sum_{t=1}^{T-1} \sum_{l=1}^d \gamma_l m_{lt}^0$ and $B_2(m_{1:T-1}^0, p_{1:T-1}^0) = (p_l - p_1^0, \dots, p_l - p_{T-1}^0)$, where γ_l is the unit cost of marketing activity l .

We assume throughout the paper that producer 0 acts as a leader and the consumers as followers. We look for open-loop (global) Stackelberg solutions Basar and Olsder (1999) (See also Definition 2 below) where producer 0 announces his strategies for all periods $(m_{1:T-1}^0, p_{1:T-1}^0)$ before the start of the game, and the consumers react to these strategies while playing Nash between each others. Thus, the information states for the different players are as follows. Consumer i observes at time t the goodwill of the products $G_{1:T}^0$ and $G_{1:T}^1$ (here $G_{1:T}^0$ is given by (1)), the prices $p_{1:T-1}^0$ and $p_{1:T-1}^1$, his choice's past history $c_{0:t-1}^i$, his rating's history $r_{1:t}^i$, the average rating's history $\bar{r}_{1:t}$, and the initial ratings' probability distribution $\pi(r)$. Hence, the information state of consumer i at time t is $\mathcal{I}_t^i = \{c_{0:t-1}^i, r_{1:t}^i, \bar{r}_{1:t}, \pi, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$. Since producer 0 announces his strategies before the start of the game, then his information state at time t is $\mathcal{I}_t^0 = \{\pi, p_1^0, p_{1:T-1}^1, G_1^0, G_{1:T}^1\}$. As we shall see later, the MFG methodology allows the consumers to reduce their information state at time t to $\{c_{t-1}^i, r_t^i, \pi, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$.

3 A mean-field games approach

The dynamic game defined above involves a leader (the firm or manufacturer of product 0) facing a passive competitor whose decisions are taken as known parameters, and N non-cooperative followers (consumers). Three approaches to the lower-level equilibrium can be considered in such a hierarchical structure. First, if each follower's decision is independent of all other followers, then the problem is very much similar to the one-follower case, the only difference being that we would have N independent reaction functions. Second, if the followers interact strategically, then one would solve for a Nash equilibrium of the N -player non-cooperative game to obtain the followers' reaction functions. Here, the assumption would be that *each* follower's decision depends on *every single one* of the other followers' decisions. To give an analogy in our context, it is as if every consumer accounts for each of the other ratings when making a purchasing decision, and in turn, has an influence on each other consumer's rating. A third possibility is that the followers' decisions are interdependent, but what really matters to each of them is only the mass behavior, and not any particular individual behavior. To illustrate a situation where this decision model would be relevant, suppose one is to choose the time to start driving from home to work. In that case, the piece of information that really matters is the estimated traffic density on the road at each admissible starting time and not the identity and the driving behavior of any particular driver. It is this last mode of thinking that is adopted here in modeling the influence consumers exert on others and in determining the resulting equilibrium.

In our model, the vehicle of consumer interaction is the aggregate rating. At the top level, the active producer optimizes its decisions based on the anticipation of the goodwill of the products and the market share consequent to such decisions. The consumers' dynamic coupling remains weak, carried as it is only through the average rating. Thus, in the limit of an infinite consumer population, an individual consumer's influence on other consumers vanishes. This situation is to be contrasted with that of the aggregate consumer rating, which continues to impact individual consumer decisions. Thus, given the large size of customer populations, we shall resort to the simplifications made possible by the *mean-field games* (MFG) methodology to solve the game.

The MFG methodology is concerned with a class of dynamic games involving a large number of agents interacting through their empirical distribution. In the limit of an infinite number of agents, the vanishing influence of isolated individual decisions, ensuing agent independence, and the Law of Large Numbers Durrett (2010) induce a deterministic probability measure, which must for consistency be the probability distribution of a generic agent's state under its best response to the probability measure in question. As a result, a Nash equilibrium (if it exists) is characterized by a backward propagating Hamilton-Jacobi-Bellman (HJB) equation, or a dynamic programming equation in the discrete-time setting, coupled with a forward propagating Fokker-Planck equation for agents evolving as diffusion processes, or a Chapman-Kolmogorov (CK) equation in the discrete-time Markov case. The backward equation describes a generic agent's best response to its state's probability distribution, while the forward equation propagates the probability distribution under the generic agent's best response. The corresponding best responses, when applied to the more realistic finite population situation, constitute under appropriate conditions an ϵ -Nash equilibrium Huang et al. (2006, 2007), which is defined as follows:

Definition 1 *Let S be a set. For all $N \in \mathbb{N}$, define on the set S^N the utility function of agent k , $1 \leq k \leq N$, $J_k^{(N)}(s_1, \dots, s_N)$. Let ϵ_N , $N \geq 0$, be a sequence of real numbers converging to 0 as $N \rightarrow \infty$. A strategy profile $\{s_i^*, i \in \mathbb{N}\}$ is called an ϵ_N -Nash equilibrium with respect to the utilities $J_k^{(N)}$, if for all N , for any $1 \leq i \leq N$ and any $s_i \in S$, we have $J_i^{(N)}(s_i, s_{-i}^*) \leq J_i^{(N)}(s_i^*, s_{-i}^*) + \epsilon_N$, where $s_{-i}^* = (s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_N^*)$.*

The advantages of the MFG methodology are twofold. First, it leverages the game symmetries to reduce the large number of equations governing the Nash equilibrium to two (at least in the homogeneous agent case). Second, tractable solutions are often possible. In particular, the MFG methodology provides a machinery to compute the evolution of the agents' distribution, which, as we shall see in the paper, allows us to describe the dynamics of the average rating and demand. Further extensions of the aforementioned MFG models were developed later. In particular, Huang (2010) introduced a class of so-called major-minor agents MFGs, where all agents except a single major one have vanishing individual influence of order $1/N$, while the major agent, although dominant in terms of influence, has no particular priority in its decision making. The appropriate equilibrium concept in this case is Nash. By contrast, Bensoussan et al. (2014, 2015) recently developed in a major-minor MFG model where the major agent plays *first* and only then are the minor agents allowed to make their decisions. In this setup, on which we build here, the appropriate solution concept is the Stackelberg solution Von Stackelberg (2010); Basar and Olsder (1999).

Definition 2 *Consider $N+1$ agents, a set of strategy profiles $S = S_0 \times \dots \times S_N$, and for each agent k , a utility function $J_k(s_0, \dots, s_N)$ defined for all $(s_0, \dots, s_N)$ in S . Suppose that agent 0 is the major agent. A strategy profile $(s_0^*, \dots, s_N^*) \in S$ is called a Stackelberg solution with respect to the utilities J_k , if there exists a map T from S_0 to $S_1 \times \dots \times S_N$, such that for all $s_0 \in S_0$, $T(s_0)$ is a Nash Equilibrium with respect to J_k , $k = 1, \dots, N$, and $s_0^* = \max_{s_0 \in S_0} J_0(s_0, T(s_0))$, with $T(s_0^*) = (s_1^*, \dots, s_N^*)$.*

4 Results

In this section, we provide a solution to the consumer-producer game using the MFG methodology. The key feature of the problem on which we rely is the fact that the consumers interact with each other

and producer 0 only through the mean field term $\bar{r}_{1:T}$, i.e., the average rating trajectory. Following the discussions on the MFG methodology in Section 3, we assume throughout this section an infinite number of consumers and a *deterministic* trajectory $\bar{r}_{1:T}$, which corresponds to the mean of the deterministic probability measure in the MFG discussion. It is legitimate to search for a deterministic rating under the consumers' Nash strategies (if they exist) for the following reason. If all the consumers optimally respond to a deterministic $\bar{r}_{1:T}$, then their coupling through $\bar{r}_{1:T}$, the independence of their initial states r_1^i and the Law of Large Numbers ensure that the average rating converges to a deterministic quantity as the number of consumers goes to infinity.

The players solve the game as follows. Given the marketing activities $m_{1:T-1}^0$ and prices $p_{1:T-1}^0$ the consumers play a non-cooperative game parametrized by $(m_{1:T-1}^0, p_{1:T-1}^0)$. As we show below, the consumers' Nash equilibria are determined by their mean rating trajectory $\bar{r}_{1:T}$. If for each $(m_{1:T-1}^0, p_{1:T-1}^0)$ the consumers agree on a unique Nash equilibrium (equivalently, a unique mean trajectory $\bar{r}_{1:T}$), then the producer can anticipate the consumers' best responses to his decisions and act optimally. More precisely, the producer constructs a map from each strategy $(m_{1:T-1}^0, p_{1:T-1}^0)$ to the corresponding $\bar{r}_{1:T}$ and maximizes her utility, which depends only on $(m_{1:T-1}^0, p_{1:T-1}^0)$ when $\bar{r}_{1:T}$ is replaced by this map.

4.1 The consumers' game

We suppose that producer 0 announces his strategies $(m_{1:T-1}^0, p_{1:T-1}^0)$. As a consequence, by (1), the goodwill trajectory $G_{1:T}^0$ is known to all the agents. By the MFG methodology, we assume for the moment that the mean rating trajectory $\bar{r}_{1:T}$ of the consumers under their Nash strategies is given. We show later that this trajectory always exists and can be computed by knowing only the initial distribution $\pi(r)$ and the products' goodwill and prices. To maximize her total utility and compute her best response to the assumed $\bar{r}_{1:T}$, a generic consumer with decision variable c_t and rating r_t solves a dynamic program, where the per-step utility is given by (3). Accordingly, this optimization can be carried-out without loss of performance by observing at each time t the previous action c_{t-1} and current rating r_t instead of $c_{0:t-1}$ and $r_{1:t}$. Thus, we replace the generic consumer's information state $\mathcal{I}_t = \{c_{0:t-1}, r_{1:t}, \bar{r}_{1:t}, \pi, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$ by $\bar{\mathcal{I}}_t = \{c_{t-1}, r_t, \bar{r}_{1:T}, \pi, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$. It should be noted that the only random elements in $\bar{\mathcal{I}}_t$ are the previous choice c_{t-1} and current rating r_t . When $(c_{t-1} : r_t)$ takes a value $(j, r) \in \{0, 1\} \times [0, 1]$, we shall denote the corresponding realization of the information state by $\bar{\mathcal{I}}_t(j, r)$.

Given (3), the generic consumer computes her best response to $\bar{r}_{1:T}$ by solving the following backward dynamic program equation Bertsekas (1995):

$$V_t(\bar{\mathcal{I}}_t(j, r)) = \max_{c_t \in \{0, 1\}} \left(l_t(j, c_t, r, \bar{r}_t, p_t^0, p_t^1, G_t^0, G_t^1) + \int_0^1 V_{t+1}(\bar{\mathcal{I}}_{t+1}(c_t, x)) f_{(1-c_t)(1-G_t^0)+c_t G_t^1}(x) dx \right), \quad (6)$$

with $V_T = 0$, $r \in [0, 1]$, $j \in \{0, 1\}$, and l_t defined in (3). $V_t(\bar{\mathcal{I}}_t(j, r))$ is the optimal utility-to-go of the generic consumer at time t given that her information state $\bar{\mathcal{I}}_t = \bar{\mathcal{I}}_t(j, r)$. The consumer's best response to $\bar{r}_{1:T}$ is then a threshold policy

$$c_t^*(\bar{\mathcal{I}}_t(j, r)) = \begin{cases} 0 & \text{if } r \leq r_t^*(\bar{\mathcal{I}}_t(j, -)) \\ 1 & \text{if } r > r_t^*(\bar{\mathcal{I}}_t(j, -)) \end{cases}, \quad (7)$$

for $1 \leq t \leq T-1$, where

$$r_t^*(\bar{\mathcal{I}}_t(j, -)) = \frac{1 - 2\alpha_2 \bar{r}_t - \alpha_3(1 - G_t^0 + G_t^1) - \alpha_4(1 + p_t^0 - p_t^1) + (-1)^j \alpha_5}{2\alpha_1} + \frac{1}{2\alpha_1} \int_0^1 \left(V_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x)) f_{1-G_t^0}(x) - V_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x)) f_{G_t^1}(x) \right) dx, \quad (8)$$

and $\bar{\mathcal{I}}_t(j, -)$ denotes the set $\bar{\mathcal{I}}_t(j, r) \setminus \{r\}$. According to the best response (7), there exists at each step of time a threshold preference r_t^* above which the consumer chooses product 1. Here, the choice

between the two products is determined by the average rating of the consumers, the goodwill of the products, the prices, and the previous choice. For example, the closer $\bar{r}_t$ is to 1, the better the current personal experience r_t with product 0 needs to be in order to choose it.

Having characterized the consumers' best responses to their mean rating trajectory $\bar{r}_{1:T}$, we turn to the problem of determining this trajectory. Recall that $\bar{r}_{1:T}$ is the mean rating trajectory of the consumer population when all the consumers optimally respond to it. Thus, it must satisfy

$$\bar{r}_t = \int_0^1 r (g_t(\bar{\mathcal{I}}_t(0, r)) + g_t(\bar{\mathcal{I}}_t(1, r))) dr, \quad 1 \leq t \leq T, \quad (9)$$

where $g_t(\bar{\mathcal{I}}_t(j, r)) dr = \mathbb{P}(c_{t-1} = j, r_t \in [r, r + dr])$ is the probability joint distribution of the generic consumer's previous choice c_{t-1} and rating r_t under the optimal strategy (7). Recall that c_{t-1} and r_t are the only random elements in the information state. The flow of densities $g_{1:T}$ is the unique solution of the following forward CK equation Durrett (2010):

$$\begin{aligned} g_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x)) &= f_{1-G_t^0}(x) \left(\int_0^{r_t^*(\bar{\mathcal{I}}_t(0, -))} g_t(\bar{\mathcal{I}}_t(0, r)) dr + \int_0^{r_t^*(\bar{\mathcal{I}}_t(1, -))} g_t(\bar{\mathcal{I}}_t(1, r)) dr \right) \\ g_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x)) &= f_{G_t^1}(x) \left(\int_{r_t^*(\bar{\mathcal{I}}_t(0, -))}^1 g_t(\bar{\mathcal{I}}_t(0, r)) dr + \int_{r_t^*(\bar{\mathcal{I}}_t(1, -))}^1 g_t(\bar{\mathcal{I}}_t(1, r)) dr \right) \end{aligned} \quad (10)$$

for all $x \in [0, 1]$.

Following Assumption 1, the initial density $g_1(\bar{\mathcal{I}}_1(0, r)) = \pi(r)1_{[0, 0.5]}(r)$ and $g_1(\bar{\mathcal{I}}_1(1, r)) = \pi(r)1_{(0.5, 1]}(r)$, where the initial distribution of the preference state π is defined in Section 2.

In conclusion, the mean rating trajectory $\bar{r}_{1:T}$ is a fixed point of the map $\bar{r}_{1:T} \mapsto \mathcal{L}(\bar{r}_{1:T}, PG)$, where $PG = \{p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$, and $\mathcal{L} = \mathcal{L}_2 \circ \mathcal{L}_1$. Here, $\mathcal{L}_1$ maps any rating trajectory $\bar{r}_{1:T} \in [0, 1]^T$ to the sequence of thresholds $r_{1:T-1}^* \in (\mathbb{R}^2)^{T-1}$ defined by (6)–(8), and $\mathcal{L}_2$ maps $r_{1:T-1}^* \in (\mathbb{R}^2)^{T-1}$ to the rating trajectory $\bar{r}_{1:T} \in [0, 1]^T$ induced by the consumers' choices (7) with threshold trajectory $r_{1:T-1}^*$, i.e. the rating trajectory is the solution of (9)–(10). To guaranty the existence of a fixed point of this map $\mathcal{L}$, we make the following technical assumption.

Assumption 2 We assume that π and $\{f_G\}_{G \in [0, 1]}$ are uniformly bounded by a constant $K_0 > 0$.

The following lemma establishes the continuity of $\mathcal{L}$, which is sufficient to show the existence of a fixed point. In the following, we denote by $|x_s|_s = \max_s |x_s|$, where the maximum is taken over the domain of s . The proofs of the results are given in the Appendix.

Lemma 1 Suppose that Assumptions 1 and 2, hold. Then, for all $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$ in $[0, 1]^T$,

$$|\mathcal{L}(\bar{r}_{1:T}, PG)_t - \mathcal{L}(\bar{r}'_{1:T}, PG)_t|_t \leq \frac{\alpha_2}{\alpha_1} \max_{1 \leq t \leq T} \left(\frac{2^{T-t+1} K_0^2 (2^t K_0^t - 1)}{(2K_0 - 1)} \right) |\bar{r}'_t - \bar{r}_t|_t, \quad (11)$$

where α_1 and α_2 are defined in (3).

It should be noted that when $K_0 = 1/2$, then $(2^t K_0^t - 1)/(2K_0 - 1) = t - 1$. We conclude this section with the main result.

Theorem 1 Under Assumptions 1 and 2, the following statements hold:

1. For any $PG = \{p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$, the map $\bar{r}_{1:T} \mapsto \mathcal{L}(\bar{r}_{1:T}, PG)$ has at least one fixed point. Equivalently, the continuum of consumers game has always a Nash equilibrium given by (7) for a fixed point mean trajectory $\bar{r}_{1:T}$ of $\mathcal{L}$.
2. The mean field strategies (7), when applied by a finite number N of consumers, constitute an ϵ_N -Nash equilibrium (see Definition 1) with respect to the utility functions defined by (3), where $|\epsilon_N| \leq 2\alpha_2(T-1)(2/\sqrt{N} + 1/N)$.

The map $\mathcal{L}$ may have multiple fixed point trajectories $\bar{r}_{1:T}$, which can be computed by knowing only the initial distribution π and PG . Indeed, $\mathcal{L}$ depends on PG and on π through the initial distribution g_1 defined below (10). Thus, the MFG methodology allows a generic consumer to anticipate the mean trajectory and reduce his information state at time t from $\bar{\mathcal{I}}_t = \{c_{t-1}, r_t, \bar{r}_{1:T}, \pi, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1\}$ to $\bar{\mathcal{I}}_t \setminus \{\bar{r}_{1:T}\}$. When used by the consumers in a finite population, the reduced information state induces a loss performance, but this loss is bounded by item 2) of Theorem 1. In other words, if the consumers in an infinite population rely on the reduced information state to compute their strategies (7), the resulting strategy profile constitutes a Nash equilibrium. In a finite population, however, this information state breaks the equilibrium by allowing the consumers to profit from a unilateral deviant behavior, but this profit is negligible when their number N is large enough compared to the fixed time horizon T .

4.2 The producer's game

Given a producer 0's strategy $(m_{1:T-1}^0, p_{1:T-1}^0)$, there exists at least one consumers' Nash equilibrium. The equilibria are totally determined by the fixed points of the map $\mathcal{L}$. If for every $(m_{1:T-1}^0, p_{1:T-1}^0)$ there exists a unique Nash equilibrium, then producer 0 is able to anticipate the behavior of the consumers and act optimally. In general, however, multiple equilibria may exist. We assume that the consumers select an equilibrium according to some well-defined deterministic mechanism known by producer 0. For example, for each $(m_{1:T-1}^0, p_{1:T-1}^0)$, they compute an equilibrium according to some algorithm. This means that a fixed point $\bar{r}_{1:T}$ is the output of an algorithm $A(m_{1:T-1}^0, p_{1:T-1}^0)$, where $A : \mathcal{U}_0 \mapsto [0, 1]^T$. In this case, an optimal strategy of producer 0 (if it exists) is a solution of the following optimization problem:

$$\begin{aligned} \max_{\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0} \quad & \sum_{t=1}^{T-1} (\alpha_0 (1 - \bar{c}_t(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0, G_{1:T}^1, p_{1:T-1}^1)) p_t^0 - \beta_0(m_t^0)) \\ \text{s.t.} \quad & (m_{1:T-1}^0, p_{1:T-1}^0) \in \mathcal{U}_0 \\ & G_{t+1}^0 = h_t(G_t^0, m_t^0, p_t^0) \\ & \bar{r}_{1:T} = \mathcal{L}(\bar{r}_{1:T}, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1) \\ & \bar{r}_{1:T} = A(m_{1:T-1}^0, p_{1:T-1}^0). \end{aligned} \tag{12}$$

Here, $1 - \bar{c}_t(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0, G_{1:T}^1, p_{1:T-1}^1)$ is the fraction of consumers who choose product 0 given $(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0, G_{1:T}^1, p_{1:T-1}^1)$. Following the Law of Large Numbers Durrett (2010) and the independence of the consumers' initial ratings, this fraction is the probability that a generic consumer optimally responding to $(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0, G_{1:T}^1, p_{1:T-1}^1)$ chooses at time t product 0.

Assumption 3 We assume that $\lim_{G \rightarrow G_0} \int_0^1 |f_G(x) - f_{G_0}(x)| dx = 0$.

Assumption 4 We assume that the map A in (12) is continuous.

Under Assumption 2, Assumption 3 is satisfied, for example, when $f_G(x)$ converges to $f_{G_0}(x)$ for almost every $x \in [0, 1]$.

Theorem 2 Under Assumptions 1, 2, 3 and 4, the optimization problem (12) has an optimal solution.

Theorem 2 asserts that there exists at least one optimal pricing and marketing strategy for producer 0. In Section 5, we discuss how to compute an optimal solution.

5 Numerical scheme

In this section, we give an example of algorithms that compute a fixed of $\mathcal{L}$. Moreover, we propose a numerical scheme to compute a producer's optimal solution. We need the following assumption.

Assumption 5 We assume that $K_1 := \frac{\alpha_2}{\alpha_1} \max_{1 \leq t \leq T} \left(\frac{2^{T-t+1} K_0^2 (2^t K_0^t - 1)}{(2K_0 - 1)} \right) < 1$.

Assumption 5 is satisfied for example when the individual current rating has much more influence on the individual decisions than the average rating, i.e., when α_2 is small compared to α_1 . Under Assumption 5, and following Lemma 1, the map $\bar{r}_{1:T} \mapsto \mathcal{L}(\bar{r}_{1:T}, PG)$ is a contraction and hence by Banach fixed point Theorem Berinde (2007) has a unique fixed point $\bar{r}_{1:T}$ for every PG . This fixed point can be computed as the limit of the iterations

$$\bar{r}_{1:T}^{(n+1)} = \mathcal{L}(\bar{r}_{1:T}^{(n)}, PG), \quad (13)$$

with $\bar{r}_{1:T}^{(0)} \in [0, T]^T$. The map A in (12) sends $(m_{1:T-1}^0, p_{1:T-1}^0)$ to this unique fixed point. As per Theorem 2, A needs to be continuous, which is shown in the following theorem.

Theorem 3 Under Assumptions 1, 2, 3, and 5, the map A is continuous.

In the following, we propose an algorithm to compute an optimal solution $(m_{1:T-1}^0, p_{1:T-1}^0)$ of (12), where the function A is defined by the limit of the iterations (13). To simplify the presentation, we restrict our-self to the scalar marketing activities case, i.e. $d = 1$. The algorithm includes two nested loops. The external one is the projected gradient descent method Bertsekas (1999) that solves for a maximizer $((m_{1:T-1}^0)^*, (p_{1:T-1}^0)^*)$ of the producer's utility, $U = \sum_{t=1}^{T-1} (\alpha_0 (1 - \bar{c}_t(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0, G_{1:T}^1, p_{1:T-1}^1)) p_t^0 -$ with $\bar{r}_t = A(m_{1:T-1}^0, p_{1:T-1}^0)_t$. The iterations of the external loop are as follows:

$$\begin{aligned} (m_t^0)^{(n+1)} &= \pi_{\mathcal{U}_0} \left((m_t^0)^{(n)} + \xi \frac{\partial U}{\partial m_t^0} \left((m_t^0)^{(n)}, (p_t^0)^{(n)} \right) \right) \\ (p_t^0)^{(n+1)} &= \pi_{\mathcal{U}_0} \left((p_t^0)^{(n)} + \xi \frac{\partial U}{\partial p_t^0} \left((m_t^0)^{(n)}, (p_t^0)^{(n)} \right) \right), \end{aligned} \quad (14)$$

for $1 \leq t \leq T-1$, where $\xi > 0$ and $\pi_{\mathcal{U}_0}$ is the Euclidean projection on the set $\mathcal{U}_0$. Recall that the Euclidean projection of a point x on a convex set C is $\pi_C(x) = \operatorname{argmin}_{y \in C} \|y - x\|$. In particular, if $B_1 = B_2 = 0$ in (5), then $\pi_{\mathcal{U}_0}(x_1, \dots, x_{2(T-1)}) = (\min(1, \max(0, x_1)), \dots, \min(1, \max(0, x_{2(T-1)})))$ is the Euclidean projection on the cube $[0, 1]^{(m+1)(T-1)}$. The partial derivatives are computed using the finite difference formulas. This involves computing the value of U at different $(m_{1:T-1}^0, p_{1:T-1}^0)$, and more specifically, the probabilities $\bar{c}_t$ of consumers choosing product 1. The value of $\bar{c}$ at $(m_{1:T-1}^0, p_{1:T-1}^0)$ is computed by the internal loop as follows.

Computation of $A(m_{1:T-1}^0, p_{1:T-1}^0)$: Suppose that the output of the internal loop's $(k-1)^{th}$ iteration is $\bar{r}_{1:T}^{(k-1)}$. In the k^{th} iteration, we start by computing V the solution the dynamic programming equation (6) for $(m_{1:T-1}^0, p_{1:T-1}^0)$ and the mean trajectory $\bar{r}_{1:T} = \bar{r}_{1:T}^{(k-1)}$. Next, we compute the corresponding threshold $r_{1:T-1}^*$ (8), density functions $g_{1:T}$ (10) and mean trajectory $\bar{r}_{1:T}^{(k)}$ (9), which will be the output of the internal loop's k^{th} iteration. We repeat this procedure until the i^{th} iteration where $\bar{r}_{1:T}^{(i-1)}$ becomes very close to $\bar{r}_{1:T}^{(i)}$. The output of the internal loop is then $A(m_{1:T-1}^0, p_{1:T-1}^0) = \bar{r}_{1:T}^{(i)}$. This mean trajectory is then used to compute the probabilities $\bar{c}_{1:T}$ at $(m_{1:T-1}^0, p_{1:T-1}^0)$ as follows.

Computation of $\bar{c}_t$: As in the previous step, we compute V , $r_{1:T}^*$ and $g_{1:T}$, but this time for the mean trajectory $A(m_{1:T-1}^0, p_{1:T-1}^0) = \bar{r}_{1:T}^{(i)}$ obtained from the internal loop. $\bar{c}_t$ is then equal to $\sum_{j=0}^1 \int_{r_t^*}^1 g_t(r, j) dr$.

If this algorithm converges to a global maximum $((m_{1:T-1}^0)^*, (p_{1:T-1}^0)^*)$ of U , then $(m_{1:T-1}^0)^*$ and $(p_{1:T-1}^0)^*$ are respectively the optimal marketing and pricing policies of producer 0. In addition to the optimal strategies, producer 0 can anticipate the evolution of the average rating $\bar{r}_{1:T} = A((m_{1:T-1}^0)^*, (p_{1:T-1}^0)^*)$ and the corresponding market share $1 - \bar{c}_{1:T}$.

Remark 2 We don't have for the moment any formal result regarding the convergence of the numerical scheme to a local or global optimum. This requires a deeper understanding of the set of fixed points of $\mathcal{L}$, which we will consider in our future work. In order to get a “relatively good” approximate optimal solution for the numerical examples given in Section 6, we run our algorithm for several initial guesses $((m_{1:T-1}^0)^{(0)}, (p_{1:T-1}^0)^{(0)})$ chosen randomly and compare the resulting utilities.

6 Illustrative examples and sensitivity analysis

In this section, we illustrate our model with some numerical examples. Suppose that the producer's decision variables are the price p_t^0 and marketing (or advertising) expenditure m_t^0 at each time t . Assume that the product goodwill (or attraction) evolves according to the following difference equation:

$$G_{t+1}^0 = \min \left(1; \max \left(0; (1 - \delta) G_t^0 - \gamma p_t^0 + \gamma_1 m_t^0 \right) \right), \quad (15)$$

where $\delta \geq 0$ is the decay in goodwill due to forgetting and γ and γ_1 are positive parameters representing the marginal impact of price and advertising on the goodwill, respectively. We assume that the per-consumer advertising cost is $\beta_0(m_t^0)^2$, where β_0 is a positive parameter.

Our model has 17 parameters, namely:

Consumer's utility parameters	$\left\{ \begin{array}{ll} \alpha_1 & \text{Impact of own rating} \\ \alpha_2 & \text{Impact of average rating} \\ \alpha_3 & \text{Impact of goodwill} \\ \alpha_4 & \text{Impact of price} \\ \alpha_5 & \text{Impact of past purchase} \end{array} \right.$
Producer's parameters	$\left\{ \begin{array}{ll} \alpha_0 & \text{Revenue scaling} \\ \beta_0 & \text{Cost scaling} \\ \gamma & \text{Marginal impact of price on goodwill} \\ \gamma_1 & \text{Marginal impact of advertising on goodwill} \\ b & \text{Available budget for advertising} \\ p_l & \text{Lower bound on price} \\ G_0^0 & \text{Initial goodwill} \\ \delta & \text{Goodwill's decay rate} \end{array} \right.$
Competitor's parameters	$\left\{ \begin{array}{ll} G_t^1 & \text{Goodwill of product 1 at each time } t \\ p_t^1 & \text{Price of product 1 at each time } t \end{array} \right.$
Other parameters	$\left\{ \begin{array}{ll} \bar{r}_0 & \text{Initial average rating} \\ T & \text{Game horizon} \end{array} \right.$

As a base case, we retain the following parameter values:

$$\begin{array}{lllll}
 \alpha_1 = 0.2, & \alpha_2 = 0.2, & \alpha_3 = 0.285, & \alpha_4 = 0.285, & \alpha_5 = 0.03, \\
 \alpha_0 = 1, & \beta_0 = 3, & \gamma = 0.18, & \gamma_1 = 0.05, & G_1^0 = 0, \\
 \delta = 0.01, & p_l = 0, & b = \infty & G_t^1 = G^1 = 0.4, & p_t^1 = p^1 = 0.3, \\
 \bar{r}_1 = 0.505, & T = 20. & & &
 \end{array}$$

In this benchmark case, the impacts of own and average ratings on demand (or individual brand choice) are the same ($\alpha_1 = \alpha_2 = 0.2$). Also, price and goodwill are given the same weight ($\alpha_3 = \alpha_4 = 0.285$). The effect of past purchase is assumed to be negligible ($\alpha_5 = 0.03$). Further, we assume that the price of the competitor and its brand goodwill to remain constant over time. The initial goodwill is zero, which is meant to represent a new brand entry in the market. The planning horizon is 20 periods and we suppose there are no constraints on the price and the advertising budget. Finally, we assume triangular probability density functions f_G , with support $[0, 1]$ and maximum at G .

Remark 3 *With 17 parameters, we can obviously consider a very large number of cases, varying in many different ways the parameter values. We refrain from seeking to be exhaustive, an objective that anyway cannot be achieved, and focus rather on few examples to illustrate the type of insights that can be obtained from the model. The results can be provided upon request for any constellation of parameter values.*

Figure 1 gives the result for the base case. The firm adopts an oscillating high-low price sequence over time, while keeping its price generally lower than the competitor's. To give an initial boost to the goodwill, the firm advertises at a high level and decreases its advertising effort over time. The firm's market share mirrors the price, that is, it is high (low) when the price is low (high). It reaches 0.5 after few periods and goes down to around 0.4. Finally, we mention that the average rating of the firm's product remains in the neighborhood of 0.5 for the whole duration of the game, and that the firm's profit is 0.7353.

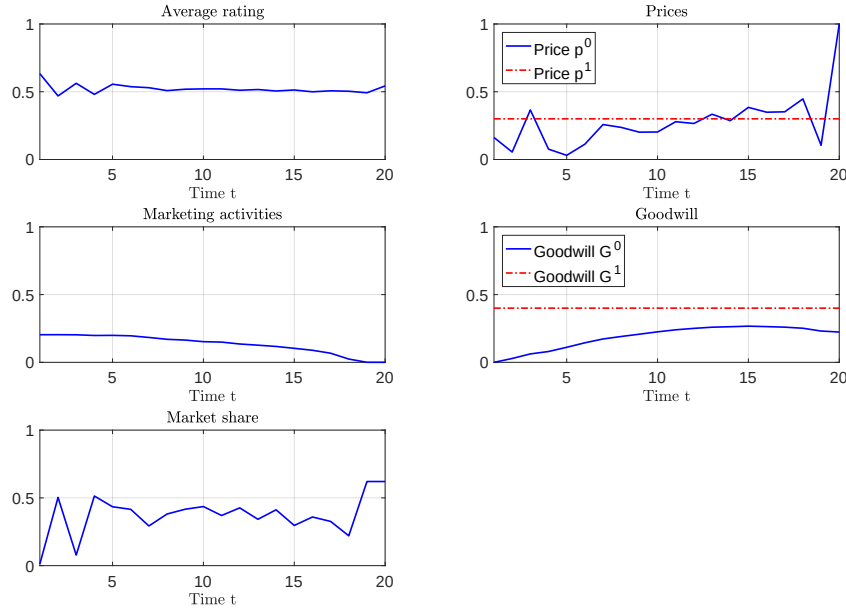
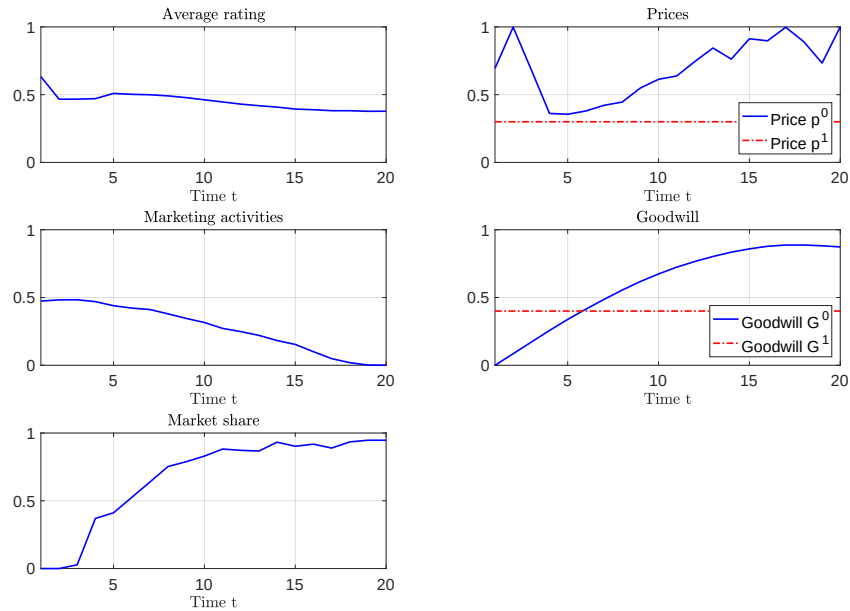


Figure 1: Base case

Remark 4 *The result that advertising and price take extreme values at terminal date T , that is, 0 and 1, respectively, is due to the end-of-horizon effect, which is typical in finite-horizon optimization problems, with no salvage value.*

In the seminal paper by Nerlove and Arrow (1962) where the (continuous-time) goodwill dynamics were introduced, and in the very large literature that followed (see, e.g., Huang et al. (2012) and Crettez et al. (2018)) the assumption was that only advertising effort affects the evolution of the goodwill. In our case, this assumption translates in setting $\gamma = 0$ in (15). Figure 2 provides the results in this case. Comparing the results to the benchmark case with $\gamma \neq 0$, we clearly see an upward shift of the price, marketing activities, goodwill and market share trajectories. This result is intuitive as price was contributing negatively to the goodwill, that is, rendering the product less attractive.

Next, we conduct a sensitivity analysis to assess the impact of varying some key parameter values on the average rating, price, advertising and goodwill. The analysis is divided into two parts. In the first part, we vary the consumer's utility parameter values ($\alpha_1, \dots, \alpha_5$) and in the second part, we vary the producer's parameter values. Given our normalization assumption that $\sum_i \alpha_i = 1$, varying any of these coefficients affects the others. Consequently, the analysis proceeds along the following three cases:

Figure 2: Second base case $\gamma = 0$

Brand loyalty: We increase the value of α_5 , while letting $\alpha_1 = \alpha_2 = 0.2/k$, $\alpha_3 = \alpha_4 = 0.285/k$, and $\alpha_5 = 0.06/k$ (resp. $\alpha_5 = 0.09/k$), where $k = 0.2 \times 2 + 0.285 \times 2 + 0.06$ (resp. $k = 0.2 \times 2 + 0.285 \times 2 + 0.09$) is a normalizing constant to force the constraint $\sum_{i=1}^5 \alpha_i = 1$. The rest of the parameters retain their base values and the results are shown in Figure 3. Increasing brand loyalty leads to higher investments in advertising and higher goodwill. This allows the firm to increase its price, while still being able to increase its market share with respect to the benchmark case. The results are consistent with the idea that loyal consumers are willing to pay a high price for their favorite product rather than switching to an alternative one. The average rating seems to be not very sensitive to the change in parameter values considered in this case. Brand loyalty has a significant impact of the firm's profit. Indeed, in case $\alpha_5 = 1.06/k$, the profit is 2.17, that is, almost three times the profit in the benchmark case. For $\alpha_5 = 1.09/k$, the profit jumps to 3.9, which is fivefold, roughly speaking, the base-case profit.

Social influence: We first consider the case where consumers give a higher value to the average rating than to their own, that is, $\alpha_1 < \alpha_2$. More specifically, we take $\alpha_1 = 0.15$ and $\alpha_2 = 0.25$ (resp. $\alpha_1 = 0.1$ and $\alpha_2 = 0.3$). Figure 4 gives the results. In a nutshell, a higher social influence, leads, qualitatively speaking, to the same results obtained with higher brand loyalty. Indeed, we observe that the firm increases its investments in advertising and decreases its prices in early periods, which yields a higher goodwill than in the base case. This allows it to increase its prices in later periods, while still securing a larger market share. The profits are 1.15 and 2.12, when $(\alpha_1, \alpha_2) = (0.15, 0.25)$ and $(\alpha_1, \alpha_2) = (0.1, 0.3)$, respectively.

Next, we consider the opposite case, i.e., lower social influence, and obtain unsurprisingly the opposite results for price, advertising and goodwill, than when social influence plays a higher role in consumer's brand choice. The results are given in Figure 5 for $(\alpha_1, \alpha_2) = (0.25, 0.15)$ and $(\alpha_1, \alpha_2) = (0.3, 0.1)$. The firm's profits are lower, that is, 1.2 in case $\alpha_1 = 0.25$ and 1.46 when $\alpha_1 = 0.3$.

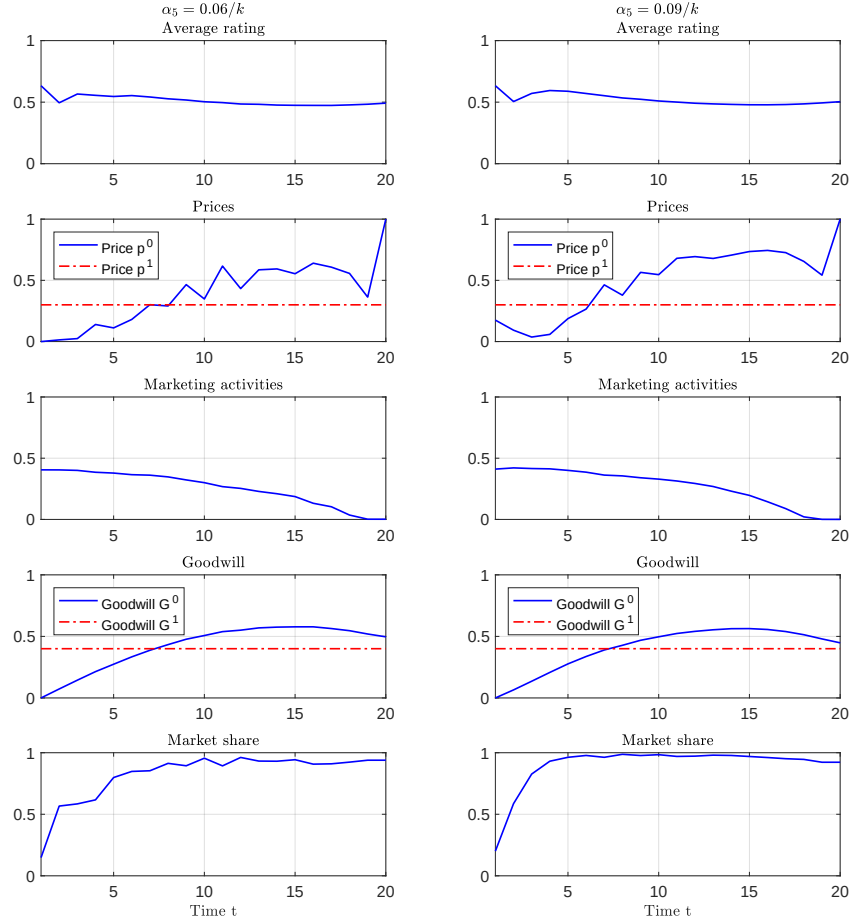


Figure 3: Varying brand loyalty

Price sensitivity: In the benchmark case, price (α_3) and goodwill (α_4) have the same impact, in absolute value, on consumer's choice. Here, we consider unequal weights, namely, $\alpha_3 > \alpha_4$ and $\alpha_3 < \alpha_4$. Figure 6 reports the results for two cases where the impact of price is lower than the impact of goodwill on her choice, that is, we let $\alpha_3 = 0.335$ and $\alpha_4 = 0.235$ (resp. $\alpha_3 = 0.385$ and $\alpha_4 = 0.185$), while keeping the other parameters unchanged. The firm's advertising policy is decreasing over time, while its price is increasing over time. The intuition is as follows: the firm boosts as much as possible the goodwill in the beginning, and surfs on its reputation for the rest of the planning horizon. Put differently, the high advertising expenditures and the low prices in early periods are to be seen as an investment in future market shares. The profits are 1.06, for $(\alpha_3, \alpha_4) = (0.335, 0.235)$, and 1.88 for $(\alpha_3, \alpha_4) = (0.385, 0.185)$.

In Figure 7, we provide the results for the mirror case where consumer is highly price sensitive and obtain lower prices and marketing activities than before, which is expected. Note, interestingly, that profits are not necessarily lower in this case.

Now, we turn to analyze the sensitivity of our model to changes in the producer's parameters. We consider the base case above. By dividing the producer's utility by α_0 , we do not affect the solution of the game. Therefore, we restrict our attention in the analysis to the parameters β_0 , δ , γ and γ_1 . We perturb one parameter at a time by $\pm 3\%$. Afterward, we compute the corresponding optimal marketing activities and prices, and the market share, and compare their values to the original ones, i.e., those obtained in the base case. For each set of parameters, our game might have multiple

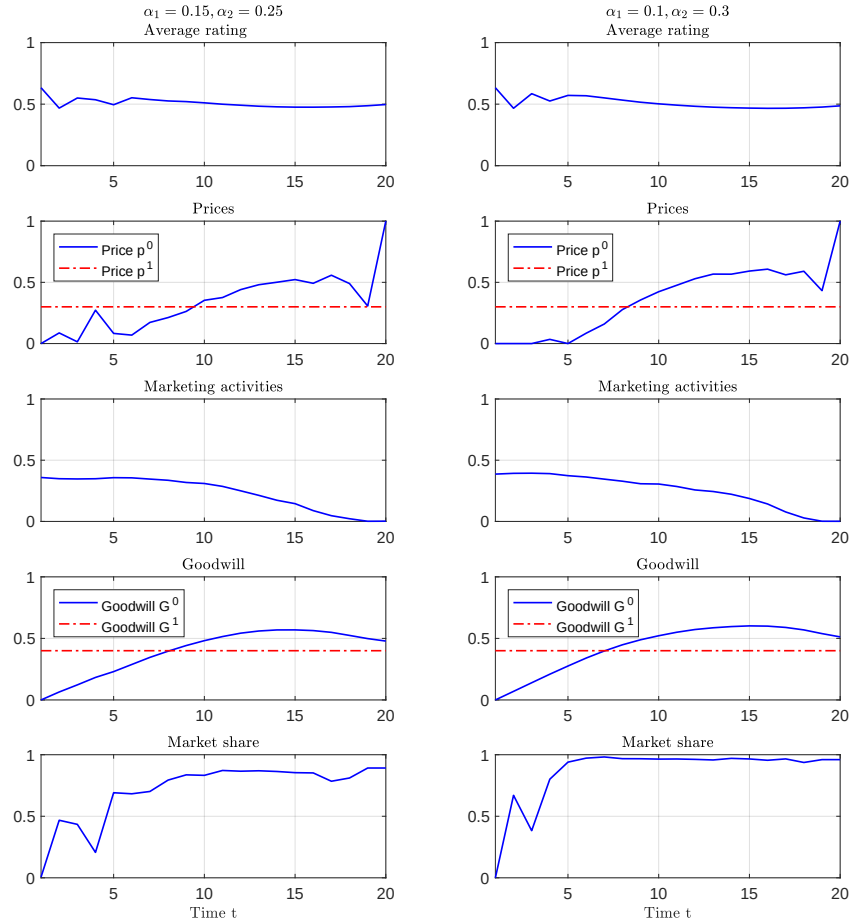


Figure 4: Varying social influence - $\alpha_1 < \alpha_2$

solutions. To make sure that the solution of the base case corresponds to the solution of the game for the set of perturbed parameters we adopt the following procedure. We solve the game for the base case using the numerical method in Section 5. We obtain an optimal producer's utility $U_{base} = 0.7353$, where the corresponding initial guess $((m_{1:T-1}^0)^{(0)}, (p_{1:T-1}^0)^{(0)})$ of iterations (14) is denoted by m_{base} . To compute the optimal strategies for the set of perturbed parameters, and which correspond to the base case, we repeat the numerical scheme (14) for different initial guesses chosen randomly in the neighborhood of m_{base} until obtaining a producer's utility greater than $0.9U_{base}$.

The equilibrium trajectories of marketing activities, price and market share for the different perturbed parameters are reported in Figures 8–11. In Figure 8, we see that the lower the marketing cost parameter β_0 , the higher the level of marketing activities, goodwill and the product's market share. Qualitatively speaking, this effect is expected. However, its order of magnitude is somewhat large as a result of the quadratic dependence on the marketing activities in the utility. Indeed, decreasing β_0 by 3% with respect its base-case value, leads to (roughly speaking) doubling the marketing activities, goodwill and market share. The effect is lesser, but still important, when we compare the trajectories obtained in the base case with those corresponding to an increase of 3% of β_0 . Further, increasing β_0 leads to lower prices. The reason is that when marketing activities become more expensive, it becomes harder to sustain a higher goodwill level, which exerts a downward pressure on the price. Recall that the goodwill is increasing in advertising and decreasing in price.

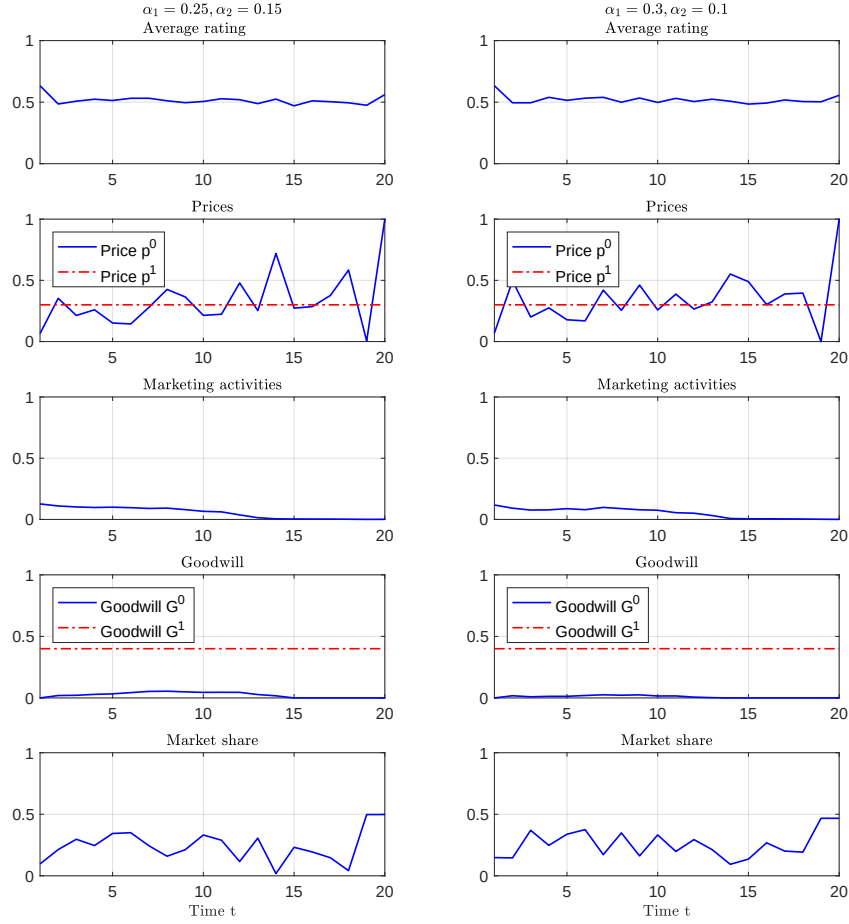


Figure 5: Varying social influence - $\alpha_1 > \alpha_2$

In Figure 9, we vary the decay (or forgetting) rate of goodwill δ . The lower this decay rate, the higher the investment in advertising, which is intuitive as its effects persist longer in time. Consequently, the goodwill is higher and so is the firm's market share. The impact on the price is harder to interpret. Whereas in the longer term, the impact is rather small, we see some very high peaks in the shorter term when the decay is larger.

In Figure 10, we provide the results of varying γ , while Figure 11 provides the results of varying γ_1 . The effect of both parameters is similar to that of δ , i.e. the marketing activities, market share and goodwill increase (resp. decrease) with γ (resp. γ_1).

To wrap up, concerning the numerical results, while most of them are intuitive, they give interesting insights into the equilibrium policies and how the latter may vary when a parameter value is changed.

7 Conclusion

We consider in this paper a consumer - producer model of advertising, pricing, rating and goodwill formation. The dynamics of the market share and average rating of the products are deduced from the behavior of a large number of strategic consumers, who are choosing, consuming and rating the products. The individual choices are shaped by the marketing decisions of the producers and the personal experience with the products. We analyze our model via the MFG methodology and propose a numerical scheme to generate the optimal policies.

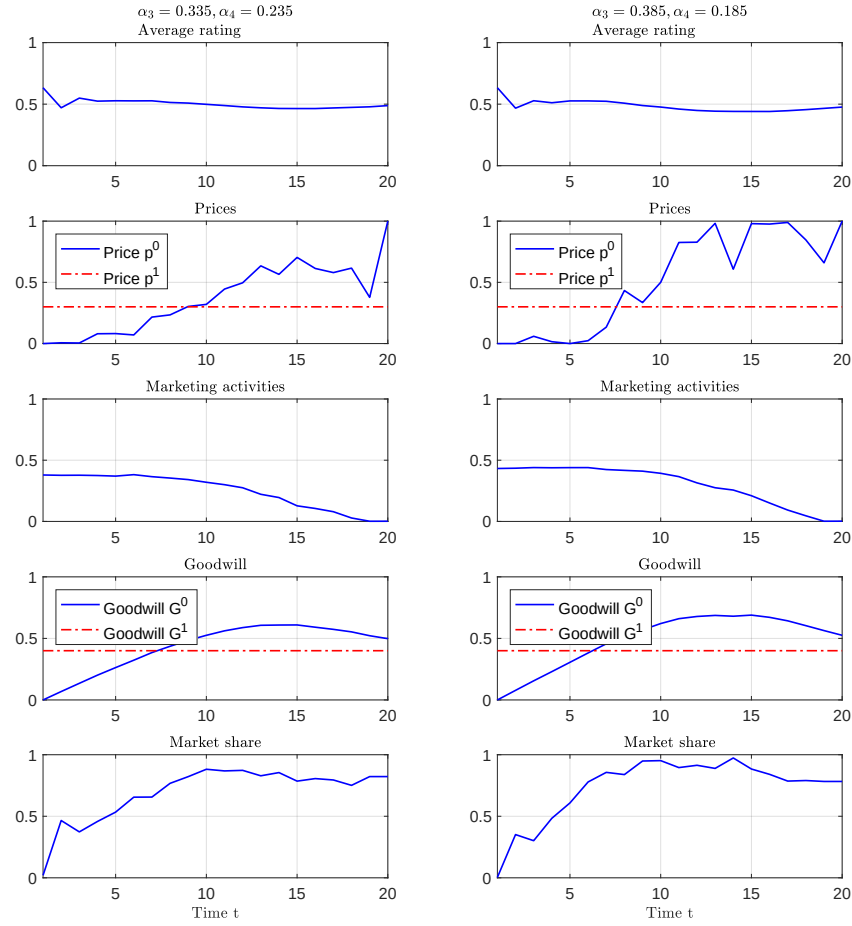


Figure 6: Varying price sensitivity - $\alpha_3 > \alpha_4$

We assume in the current formulation that the leader firm announces its strategy profile before the start of the game. The optimal pricing and marketing strategies are in this case open-loop Stackelberg. Feedback Stackelberg policies Simaan and Cruz (1973), where the consumers can only observe the current and past prices and goodwill, will be considered in our future work. This will allow us to compare the performance of the firm in case it announces its future pricing and marketing strategies with its performance when these future policies are not communicated to the consumers. We will also extend our current model to multiple active producers who compete à la Nash. Another interesting future extension would be to take into account the full distribution of the ratings, for example, on a five stars scale.

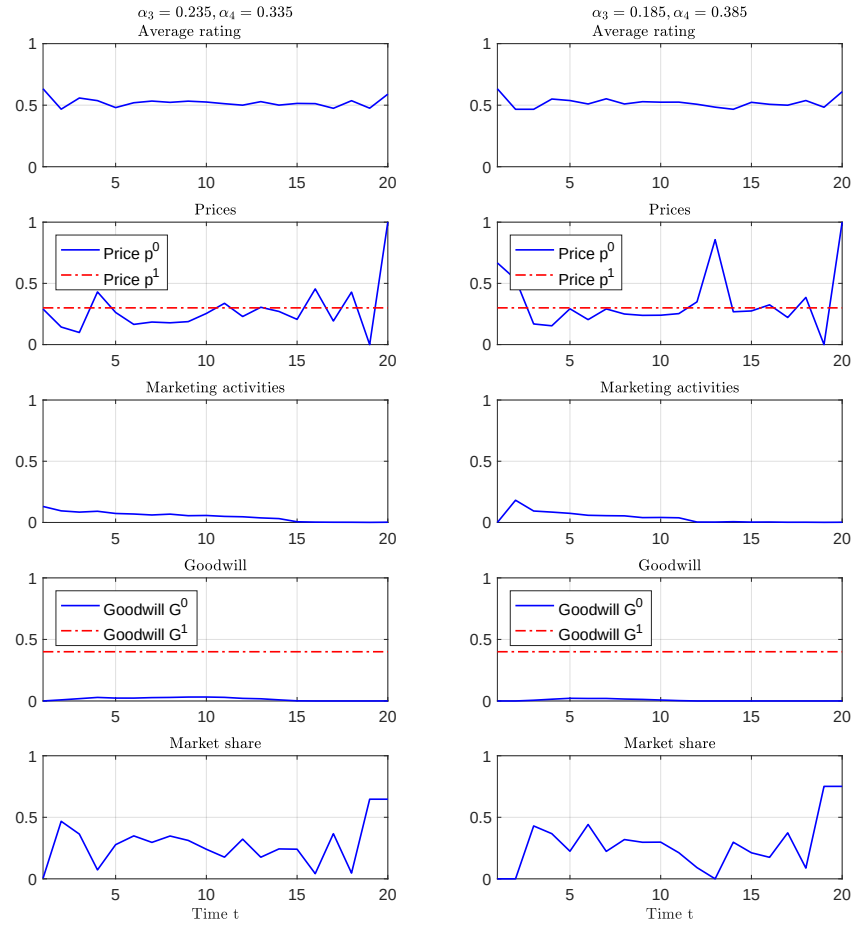


Figure 7: Varying price sensitivity - $\alpha_3 < \alpha_4$

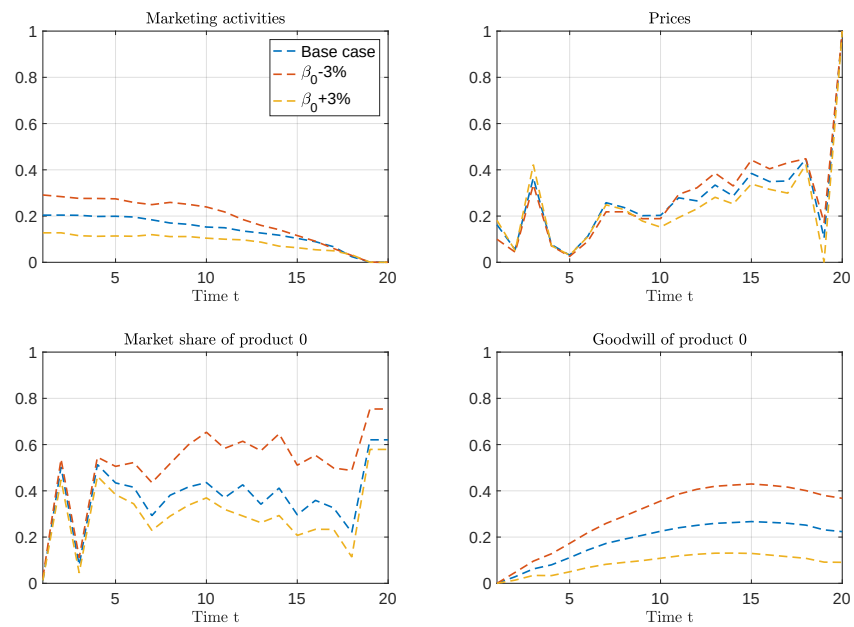
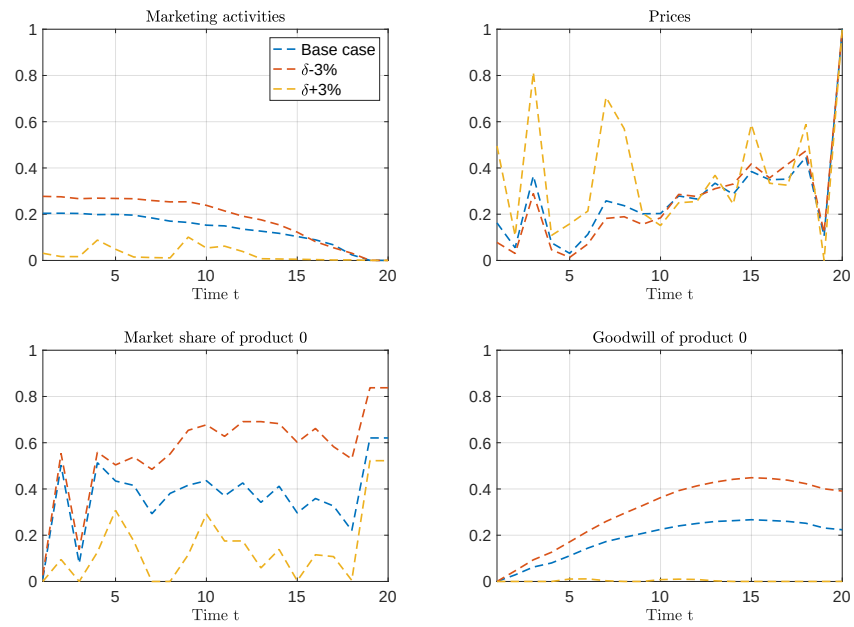
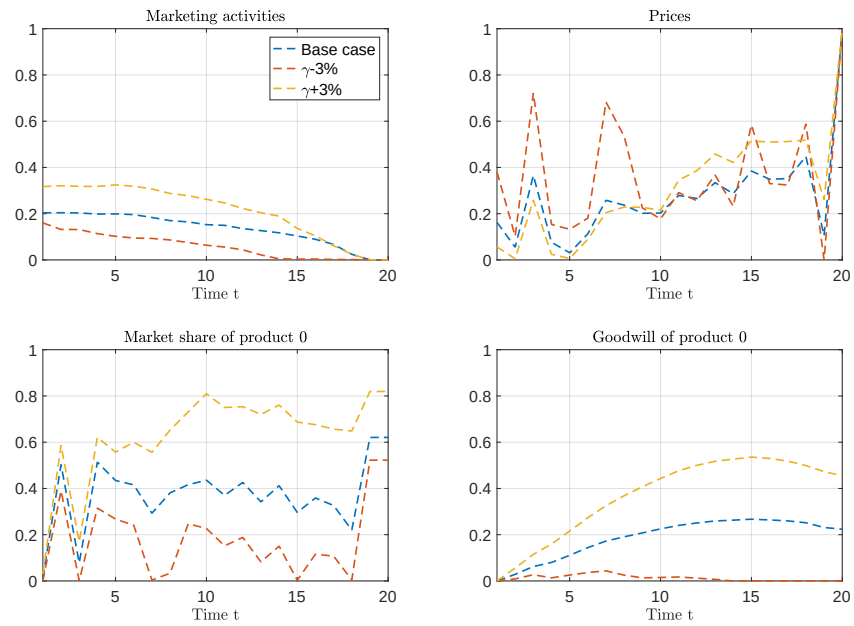
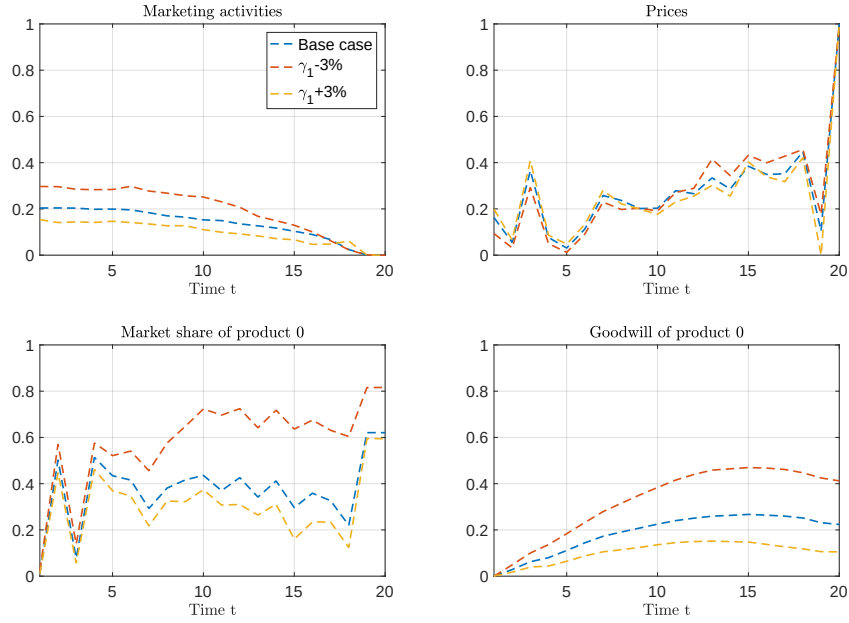


Figure 8: Sensitivity of β_0

Figure 9: Sensitivity of δ Figure 10: Sensitivity of γ

Figure 11: Sensitivity of γ_1

8 Appendix

8.1 Proofs of Lemma 1 and Theorem 1.

8.1.1 Preparatory material

Lemma 2 Under Assumptions 1 and 2, the following statements hold:

1. For all $\bar{r}_{1:T}$ in $[0, 1]^T$, $|V_t(\bar{\mathcal{I}}_t(j, r))|_{t,j,r} \leq T - 1$.
2. For all t in $\{1, \dots, T\}$, $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$ in $[0, 1]^T$,

$$|V_t(\bar{\mathcal{I}}_t(j, r)) - V_t(\bar{\mathcal{I}}'_t(j, r))|_{j,r} \leq (2^{T-t} - 1)\alpha_2 |\bar{r}_k - \bar{r}'_k|_k. \quad (16)$$

where $\bar{\mathcal{I}}_t(j, r)$ and $\bar{\mathcal{I}}'_t(j, r)$ correspond respectively to $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$, and α_2 is defined in (3).

3. For all t in $\{0, \dots, T-1\}$, $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$ in $[0, 1]^T$,

$$|r_t^*(\bar{\mathcal{I}}_t(j, -)) - r_t^*(\bar{\mathcal{I}}'_t(j, -))|_j \leq \frac{2^{T-t-1}\alpha_2}{\alpha_1} |\bar{r}_k - \bar{r}'_k|_k.$$

where $\bar{\mathcal{I}}_t(j, -)$ and $\bar{\mathcal{I}}'_t(j, -)$ correspond respectively to $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$, and α_1 and α_2 are defined in (3).

Proof. First point: We prove by induction on t (backward) that $|V_t(\bar{\mathcal{I}}_t(j, r))|_{j,r} \leq T - t$, from which the first point follows. The result is true for $t = T$ since $V_T = 0$. Suppose that $|V_{t+1}(\bar{\mathcal{I}}_{t+1}(j, r))|_{j,r} \leq T - t - 1$. By (6) and $\sum_{i=1}^5 \alpha_i = 1$, we get $|V_t(\bar{\mathcal{I}}_t(j, r))|_{j,r} \leq 1 + T - t - 1 = T - t$. This proves the first point.

Second point: Using the equality $\max(a, b) = (|a - b| + a + b)/2$, we obtain that,

$$\begin{aligned} V_t(\bar{\mathcal{I}}_t(j, r)) = & \frac{1}{2} \left(\left| 1 - 2\alpha_1 r - 2\alpha_2 \bar{r}_t - \alpha_3(1 - G_t^0 + G_t^1) - \alpha_4(1 - p_t^1 + p_t^0) + (-1)^j \alpha_5 \right. \right. \\ & + \int_0^1 \left(V_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x)) f_{1-G_t^0}(x) - V_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x)) f_{G_t^1}(x) \right) dx \\ & \left. \left. + \int_0^1 \left(V_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x)) f_{1-G_t^0}(x) + V_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x)) f_{G_t^1}(x) \right) dx + K_t \right| \right), \end{aligned} \quad (17)$$

where $K_t = 1 - \alpha_3(1 - G_t^0 - G_t^1) + \alpha_4(1 - p_t^0 - p_t^1) - \alpha_5$. Hence,

$$|V_t(\bar{\mathcal{I}}_t(j, r)) - V_t(\bar{\mathcal{I}}'_t(j, r))|_{j,r} \leq \alpha_2 |\bar{r}'_k - \bar{r}_k|_k + 2 |V_{t+1}(\bar{\mathcal{I}}_{t+1}(j, r)) - V_{t+1}(\bar{\mathcal{I}}'_{t+1}(j, r))|_{j,r}.$$

Thus, by induction on t , we obtain (16).

Third point: The third point follows from the second point and (8). □

8.1.2 Proofs of the main results

Proof of Lemma 1. By (10) and Assumption 2, one can show by induction on t that

$$|g_t(\bar{\mathcal{I}}_t(0, r)) + g_t(\bar{\mathcal{I}}_t(1, r))|_{t,r} \leq K_0. \quad (18)$$

Moreover, we have

$$\begin{aligned} & |g_{t+1}(\bar{\mathcal{I}}'_{t+1}(0, x)) - g_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x))| + |g_{t+1}(\bar{\mathcal{I}}'_{t+1}(1, x)) - g_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x))| \leq \\ & K_0 \left(\sum_{k=0}^1 \sum_{j=0}^1 \left| \int_k^{r_t^*(\bar{\mathcal{I}}'_t(j, -))} (g_t(\bar{\mathcal{I}}'_t(j, r)) - g_t(\bar{\mathcal{I}}_t(j, r))) dr + \int_{r_t^*(\bar{\mathcal{I}}_t(j, -))}^{r_t^*(\bar{\mathcal{I}}'_t(j, -))} g_t(\bar{\mathcal{I}}_t(j, r)) dr \right| \right), \end{aligned}$$

where g is defined in (10), and $\bar{\mathcal{I}}_t$ and $\bar{\mathcal{I}}'_t$ correspond respectively to $\bar{r}_{1:T}$ and $\bar{r}'_{1:T}$. This implies

$$\begin{aligned} & \left| |g_{t+1}(\bar{\mathcal{I}}'_{t+1}(0, x)) - g_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x))| + |g_{t+1}(\bar{\mathcal{I}}'_{t+1}(1, x)) - g_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x))| \right|_x \leq \\ & K_0 \left| |g_t(\bar{\mathcal{I}}'_t(0, x)) - g_t(\bar{\mathcal{I}}_t(0, x))| + |g_t(\bar{\mathcal{I}}'_t(1, x)) - g_t(\bar{\mathcal{I}}_t(1, x))| \right|_x + 4K_0^2 |r_t^*(\bar{\mathcal{I}}'_t(j, -)) - r_t^*(\bar{\mathcal{I}}_t(j, -))|_j \\ & \leq K_0 \left| |g_t(\bar{\mathcal{I}}'_t(0, x)) - g_t(\bar{\mathcal{I}}_t(0, x))| + |g_t(\bar{\mathcal{I}}'_t(1, x)) - g_t(\bar{\mathcal{I}}_t(1, x))| \right|_x + \frac{K_0^2 2^{T-t+1} \alpha_2}{\alpha_1} |\bar{r}'_k - \bar{r}_k|_k, \end{aligned}$$

where the last inequality follows from the third point of Lemma 2. Thus, by induction on t we obtain that

$$\left| |g_t(\bar{\mathcal{I}}'_t(0, x)) - g_t(\bar{\mathcal{I}}_t(0, x))| + |g_t(\bar{\mathcal{I}}'_t(1, x)) - g_t(\bar{\mathcal{I}}_t(1, x))| \right|_x \leq \frac{2^{T-t+2} K_0^2 (2^t K_0^t - 1) \alpha_2}{\alpha_1 (2K_0 - 1)} |\bar{r}'_k - \bar{r}_k|_k. \quad (19)$$

The result follows from (9), (19) and $\int_0^1 x dx = 1/2$. □

Proof of Theorem 1. The first point is a direct consequence of the continuity of $\mathcal{L}$ proved in Lemma 1 and Brouwer's fixed point theorem (Conway, 1985, Section V.9). We now prove the second point. Fix $1 \leq i \leq N$, and let $c_{0:T-1}^i \in \{0, 1\}^T$ any strategy of consumer i and $r_{1:T}^i$ the corresponding state. Denote by $c_{1:T-1}^{j*}$, $1 \leq j \leq N$, the mean field based strategies applied by the N consumers for a fixed point trajectory $\bar{r}_{1:T}$ of $\mathcal{L}$. The corresponding states and average state at time t are denoted respectively

r_t^{j*} , $1 \leq j \leq N$, and $\bar{r}_t^{(N)} := (1/N) \sum_{j=1}^N r_t^{j*}$. We define $\bar{r}_t^{N,i} = (r_t^i/N) + (1/N) \sum_{j=1, j \neq i}^N r_t^{j*}$. The individual utility of consumer i is

$$J^i(c_{0:T-1}^i, \bar{r}_{1:T}, PG) = \mathbb{E} \sum_{t=1}^{T-1} l_t^i(c_{t-1:t}^i, r_t^i, \bar{r}_t, p_t^0, p_t^1, G_t^0, G_t^1), \quad (20)$$

where l_t^i is defined in (3). We have,

$$\begin{aligned} J^i(c_{0:T-1}^i, \bar{r}_{1:T}^{N,i}, PG) - J^i(c_{0:T-1}^{i*}, \bar{r}_{1:T}^N, PG) &= \underbrace{J^i(c_{0:T-1}^i, \bar{r}_{1:T}^{N,i}, PG) - J^i(c_{0:T-1}^i, \bar{r}_{1:T}, PG)}_{:=\xi_1} \\ &+ \underbrace{J^i(c_{0:T-1}^i, \bar{r}_{1:T}, PG) - J^i(c_{0:T-1}^{i*}, \bar{r}_{1:T}, PG)}_{:=\xi_2} + \underbrace{J^i(c_{0:T-1}^{i*}, \bar{r}_{1:T}, PG) - J^i(c_{0:T-1}^{i*}, \bar{r}_{1:T}^N, PG)}_{:=\xi_3}. \end{aligned}$$

By Cauchy-Schwarz inequality Rudin (1987) we get,

$$|\xi_1| = \alpha_2 \left| \mathbb{E} \sum_{t=1}^{T-1} (2c_t^i - 1) \left\{ (\bar{r}_t^N - \bar{r}_t) + \frac{1}{N} (r_t^i - r_t^{i*}) \right\} \right| \leq \alpha_2 \sum_{t=1}^{T-1} \left\{ (\mathbb{E} |\bar{r}_t^N - \bar{r}_t|^2)^{1/2} + \frac{2}{N} \right\}.$$

Since the initial state are independent and identically distributed, we obtain that

$$\mathbb{E} |\bar{r}_t^N - \bar{r}_t|^2 = \mathbb{E} \left| \frac{1}{N} \sum_{t=1}^N (r_t^i - \mathbb{E} r_t^i) \right|^2 = \frac{1}{N} \mathbb{E} |r_t^i - \mathbb{E} r_t^i|^2 \leq \frac{4}{N}.$$

Hence, $|\xi_1| \leq 2\alpha_2(T-1)(1/\sqrt{N} + 1/N)$. Similarly, $|\xi_3| \leq 2\alpha_2(T-1)/\sqrt{N}$. Since $c_{0:T-1}^{i*}$ is the optimizer of $J^i(c_{0:T-1}^i, \bar{r}_{1:T}, PG)$ for the deterministic fixed point path $\bar{r}_{1:T}$, then the term ξ_2 is negative. Hence, $J^i(c_{0:T-1}^i, \bar{r}_{1:T}^{N,i}, PG) \leq J^i(c_{0:T-1}^{i*}, \bar{r}_{1:T}^N, PG) + 2\alpha_2(T-1)(2/\sqrt{N} + 1/N)$. This proves the result. $\square$

8.2 Proofs of Theorems 2 and 3.

Proof of Theorem 2. If we show that $\bar{c}$ and $\mathcal{L}$ are continuous functions of $(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0)$, and $(\bar{r}_{1:T}, p_{1:T-1}^0, G_{1:T}^0)$, respectively, then the set

$$\left\{ (\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0) \in [0, 1]^T \times \mathcal{U}_0, \text{ such that } \bar{r}_{1:T} = \mathcal{L}(\bar{r}_{1:T}, p_{1:T-1}^0, p_{1:T-1}^1, G_{1:T}^0, G_{1:T}^1), \right. \\ \left. \bar{r}_{1:T} = A(m_{1:T-1}^0, p_{1:T}^0, G_{t+1}^0) = h_t(G_t^0, m_t^0, p_t^0) \right\}$$

is compact. Thus, the utility of producer 0 is continuous and has a maximum in this compact set, and problem (12) has an optimal solution. We show in the following that $\bar{c}_t$ and $\mathcal{L}$ are continuous. Following the continuity of h_t , $G_{1:T}^0$ defined by (1) is a continuous function of $(m_{1:T-1}^0, p_{1:T-1}^0)$. Following (17), we obtain that for all $(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0)$ and $(\bar{r}'_{1:T}, m'_{1:T-1}, p'_{1:T-1})$, (with $\bar{\mathcal{I}}_t(j, r)$ and $\bar{\mathcal{I}}'_t(j, r)$ the corresponding information states, respectively)

$$\begin{aligned} \Delta V_t(j, r) &:= |V_t(\bar{\mathcal{I}}'_t(j, r)) - V_t(\bar{\mathcal{I}}_t(j, r))| \leq \alpha_2 |\bar{r}'_k - \bar{r}_k|_k + \frac{\alpha_3}{2} |G_k'^0 - G_k^0|_k + \frac{\alpha_4}{2} |p_k'^0 - p_k^0|_k \\ &+ \underbrace{\left| \int_0^1 (V_{t+1}(\bar{\mathcal{I}}'_{t+1}(1, x)) f_{G_t'^0}(x) - V_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x)) f_{G_t^0}(x)) dx \right|}_{:=I_1} \\ &+ \underbrace{\left| \int_0^1 (V_{t+1}(\bar{\mathcal{I}}'_{t+1}(0, x)) f_{1-G_t'^0}(x) - V_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x)) f_{1-G_t^0}(x)) dx \right|}_{:=I_2}. \end{aligned}$$

We show by induction on t that $\forall r \in [0, 1], j = 0, 1$, ΔV_t converges to 0 as $(\bar{r}'_{1:T}, m_{1:T-1}^0, p'_{1:T-1})$ converges to $(\bar{r}_{1:T}, m_{1:T-1}^0, p_{1:T-1}^0)$. We have $V_T = 0$, hence the result is true for $t = T$. Assume that it is true for $t + 1$. By the first point of Lemma 2, we obtain

$$I_1 \leq T \int_0^1 |f_{(G'_t)}(x) - f_{G_t}(x)| dx + K_0 \int_0^1 |V_{t+1}(\bar{\mathcal{I}}'_{t+1}(1, x)) - V_{t+1}(\bar{\mathcal{I}}_{t+1}(1, x))| dx.$$

Under Assumption 3, the boundedness of V and Lebesgue's dominated convergence theorem Rudin (1987) imply that I_1 converges to zero. Similarly, I_2 converges to zero. This shows that ΔV_t converges to 0. Using this result, one can show that r_t^* defined in (8) is a continuous function of $(\bar{r}_{1:T}, p_{1:T-1}^0, G_{1:T}^0)$. We now show by induction on t that $\int_0^1 |g_t(\bar{\mathcal{I}}'_t(j, x)) - g_t(\bar{\mathcal{I}}_t(j, x))| dx$ converges to 0 as $(\bar{r}'_{1:T}, p'_{1:T-1}, G_{1:T}^0)$ converges to $(\bar{r}_{1:T}, p_{1:T-1}^0, G_{1:T}^0)$, for $j = 0, 1$, which with (9) imply the continuity of $\mathcal{L}$. Here, g is the probability distribution defined in (10). For $t = 0$, g_1 is given and does not depend on the strategies. Hence, the result is true for $t = 1$. Assume that this result is true for t . We have,

$$\begin{aligned} I_3 &:= \int_0^1 |g_{t+1}(\bar{\mathcal{I}}'_{t+1}(0, x)) - g_{t+1}(\bar{\mathcal{I}}_{t+1}(0, x))| dx \\ &\leq K_0 \left(\underbrace{\left| \int_0^{r_t^*(\bar{\mathcal{I}}'_t(0, -))} g_t(\bar{\mathcal{I}}'_t(0, x)) dx - \int_0^{r_t^*(\bar{\mathcal{I}}_t(0, -))} g_t(\bar{\mathcal{I}}_t(0, x)) dx \right|}_{:= I_4} \right. \\ &\quad \left. + \underbrace{\left| \int_0^{r_t^*(\bar{\mathcal{I}}'_t(1, -))} g_t(\bar{\mathcal{I}}'_t(1, x)) dx - \int_0^{r_t^*(\bar{\mathcal{I}}_t(1, -))} g_t(\bar{\mathcal{I}}_t(1, x)) dx \right|}_{:= I_5} \right). \end{aligned}$$

We have,

$$I_4 \leq \int_0^1 |g_t(\bar{\mathcal{I}}'_t(0, x)) - g_t(\bar{\mathcal{I}}_t(0, x))| dx + \left| \int_{r_t^*(\bar{\mathcal{I}}_t(0, -))}^{r_t^*(\bar{\mathcal{I}}'_t(0, -))} g_t(\bar{\mathcal{I}}_t(0, x)) dx \right|.$$

Thus, following the uniform boundedness of g_t (18), and the continuity of r_t^* , I_4 converges to 0. Similarly, I_5 converges to 0. Thus, I_3 converges to 0. Similarly, for case $j = 1$. This completes the induction. Finally, we show that $\bar{c}_t$ is continuous. Following (7), we get

$$\begin{aligned} \bar{c}_t &= \mathbb{P}(c_t = 1) = \mathbb{P}(r_t > r_t^*(\bar{\mathcal{I}}_t(0, -)), c_{t-1} = 0) + \mathbb{P}(r_t > r_t^*(\bar{\mathcal{I}}_t(1, -)), c_{t-1} = 1) \\ &= \sum_{j=0}^1 \int_{r_t^*(\bar{\mathcal{I}}_t(j, -))}^1 g_t(\bar{\mathcal{I}}_t(j, x)) dx. \end{aligned}$$

Following I_4 and I_5 , $\bar{c}_t$ is continuous. □

Proof of Theorem 3. We have for all $(m_{1:T-1}^0, p_{1:T-1}^0)$ and $(m'_{1:T-1}, p'_{1:T-1})$ in $\mathcal{U}_0$,

$$\begin{aligned} &\left| A(m_{1:T-1}^0, p_{1:T-1}^0)_t - A(m'_{1:T-1}, p'_{1:T-1})_t \right|_t \\ &= \left| \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t - \mathcal{L}(A(m'_{1:T-1}, p'_{1:T-1})_{1:T}, PG)_t \right|_t \\ &\leq \left| \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t - \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t \right|_t \\ &\quad + \left| \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t - \mathcal{L}(A(m'_{1:T-1}, p'_{1:T-1})_{1:T}, PG)_t \right|_t \\ &\leq K_1 \left| A(m_{1:T-1}^0, p_{1:T-1}^0)_t - A(m'_{1:T-1}, p'_{1:T-1})_t \right|_t \\ &\quad + \left| \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t - \mathcal{L}(A(m'_{1:T-1}, p'_{1:T-1})_{1:T}, PG)_t \right|_t. \end{aligned}$$

This implies that

$$\left| A(m_{1:T-1}^0, p_{1:T-1}^0)_t - A(m_{1:T-1}^0, p_{1:T-1}^0)_t \right| \leq \frac{1}{1 - K_1} \left| \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG')_t - \mathcal{L}(A(m_{1:T-1}^0, p_{1:T-1}^0)_{1:T}, PG)_t \right|.$$

The result follows from the continuity of $\mathcal{L}$. □

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