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On the difference of energies of a graph and its complement graph

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Abstract: The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of all eigenvalues of G. In this paper we study the difference of energies of a (regular) graph G and its complete graph $\overline{G}$, that is, $\mathcal{E}(G) - \mathcal{E}(\overline{G})$. In particular, we provide the answer to Problem 12 raised in Nikiforov (2016) [17]. Moreover, we give a lower bound for the energy of a regular graph in terms of the order and the clique cover number.

Keywords: Eigenvalues, graph energy, complement graph, regular graph

1 Introduction

The *energy* $\mathcal{E}(G)$ of a graph G, defined as the sum of the absolute values of its eigenvalues, certainly belongs to the most popular graph invariants in chemical graph theory. It originates from the π-electron energy in the Hückel molecular orbital model, but has also gained purely mathematical interest on its own right. Gutman introduced this definition of the energy of a simple graph in his paper "The energy of a graph". He notes that at first, very few mathematicians appeared to be attracted by the definition. In the past decade, interest in graph energy has increased and many different versions have been proposed.

The *adjacency matrix* $A(G)$ of G is defined by its entries $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of $A(G)$. The energy of the graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,, \tag{1}$$

where $\lambda_i,\ i = 1, 2, \ldots, n$ are the eigenvalues of graph G. For its basic properties, applications including various lower and upper bounds, see [4, 5, 6, 7, 9, 11, 13, 14, 16, 19].

For a given interval I, let $m_G(I)$ be the number of eigenvalues of G located in I. The *positive* (*negative*) inertia $n^+(G)$ ($n^-(G)$) of the graph G is the number of positive (negative) eigenvalues of G. The *nullity* $n^0(G)$ of G is the multiplicity of eigenvalue 0 in G. An equivalent definition for energy of the graph G is given as follows:

$$\mathcal{E}(G) = 2\sum_{i=1}^{n^+(G)} \lambda_i = 2\max_{1\leq j\leq n}\sum_{i=1}^{j} \lambda_i = -2\sum_{i=1}^{n^-(G)} \lambda_{n-i+1} = 2\max_{1\leq j\leq n}\sum_{i=1}^{j} -\lambda_{n-i+1}. \tag{2}$$

A k-regular graph is a graph with all vertex degrees equal to k. It seems that the studies on the energy of regular graphs are limited to only few papers [3, 10, 15, 17]. In [17] section 3, Nikiforov studied energy of a graph and its complement graph. In particular, he gave the following problem:

Problem 1 ([17]; Problem 12) Find the best possible upper bounds for $\mathcal{E}(G) - \mathcal{E}(\overline{G})$ for general and for regular graphs.

The aim of this paper is to provide the answer to Problem 1 in both cases general and regular graph G. Moreover a lower bound for $\mathcal{E}(G)$ of a regular graph G in terms of order and the clique cover number G is given.

The paper is organized as follows. In Section 2 we give a list of some necessary lemmas. In Section 3, we study the difference between energies of a general graph G and its complement graph $\overline{G}$, that is, $\mathcal{E}(G)-\mathcal{E}(\overline{G})$. In Section 4, we obtain the exact values of $\mathcal{E}(G) - \mathcal{E}(\overline{G})$ for regular graphs and utilizing this value, we give an upper bound for this difference of energies. In Section 5, we give a lower bound for energy $\mathcal{E}(G)$ of the regular graph G.

2 preliminaries

In this section, we list some previously known results that will be needed in the next three sections.

Lemma 1 [1] *Let G be a graph of order n with m edges. Then*

$$\lambda_1(G) \geq \frac{2m}{n}$$

with equality holding if and only if G is a regular graph.

Lemma 2 [2] *Let G be a k-regular graph of order n. If the eigenvalues of G are $\lambda_1 = k \geq \lambda_2 \geq \ldots \geq \lambda_n$, then the eigenvalues of $\overline{G}$ are*

$$\lambda_1(\overline{G}) = n - 1 - k, \quad \textit{and} \quad \lambda_i(\overline{G}) = -1 - \lambda_{n-i+2}, \quad \textit{for } 2 \leq i \leq n.$$

Let $A + B = C$, where A and B are the square Hermitian matrices of order n. The well-known Weyl's inequality states that if $j + k - n \geq i$, then $\lambda_j(A) + \lambda_k(B) \leq \lambda_i(C)$. Let $A = A(G)$ and $B = A(\overline{G})$. Then $C = J - I$, where J and I are respectively the all ones matrix and the identity matrix of order n. Now considering $j \in \{2, 3, \ldots, n\}$ and $k = n - j + 2$ and $i = 2$ in the Weyl's inequality, we arrive at the following result:

$$\lambda_j(G) + \lambda_{n-j+2}(\overline{G}) \leq -1, \tag{3}$$

for $j \in \{2, 3, \ldots, n\}$.

The regular graphs satisfy the equality in (3), by Lemma 2. In the following we show that the regular graphs are the only class of graphs satisfying the equality in (3).

Lemma 3 *The equality holds in (3) if and only if G is regular.*

Proof. If G is regular graph, then we have the equality in (3) by Lemma 2. This completes one direction of the proof.

Conversely, assume that the equality holds in (3), that is, $\lambda_j(G) + \lambda_{n-j+2}(\overline{G}) = -1$ for $j \in \{2, 3, \ldots, n\}$. From this with the fact

$$\sum_{i=1}^{n} (\lambda_i(G) + \lambda_i(\overline{G})) = 0,$$

we arrive at $\lambda_1(G) + \lambda_1(\overline{G}) = n - 1$. On the other hand, we have

$$\lambda_1(G) + \lambda_1(\overline{G}) \geq \frac{2m}{n} + \frac{2\overline{m}}{n} = n - 1,$$

which gives $\lambda_1(G) = \frac{2m}{n}$ and $\lambda_1(\overline{G}) = \frac{2\overline{m}}{n}$. This proves that G is regular, by Lemma 1. Hence the proof of the lemma is done. □

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The following lemma is given by Hoffman [12].

Lemma 4 (Hoffman [12]) *Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of a graph G and let $c = \chi(G)$ be the chromatic number of G ($c \geq 2$). Then $0 \leq \lambda_1(G) \leq -\sum_{i=1}^{c-1} \lambda_{n-i+1}$.*

Lemma 5 [18] *Let G be a graph on n vertices. Then $\chi(G) + \chi(\overline{G}) \leq n + 1$.*

Elphick et al. [8] studied the Nordhaus-Gaddum type results on the nullity and inertias of graphs. In particular they gave the following lower bound for $n^-(\overline{G}) + n^-(G)$.

Lemma 6 [8] *Let G be a graph of order n with the negative inertia $n^-(G)$ and let $\overline{G}$ be its complement graph. Then $n^-(\overline{G}) + n^-(G) \geq n - 1$. Equality holds if G is a strongly regular graph.*

Gutman et al. [10] showed that the energy of a regular graph is at least equal to the order n.

Lemma 7 [10] *Let G be a regular graph on n vertices. Then $\mathcal{E}(G) \geq n$.*

3 The gap between energies of a graph and its complement graph

In this section we study the gap between energies of a graph and its complement graph. We provide a answer for Problem 1 as well as we improve a result obtained by Nikiforov [17]. Nikiforov [17] obtained the following upper bounds on differences between energies of G and its complement $\overline{G}$:

$$\begin{aligned} \mathcal{E}(\overline{G}) - \mathcal{E}(G) &\leq 2\lambda_1(\overline{G}) \leq 2n - 2, \\ \mathcal{E}(G) - \mathcal{E}(\overline{G}) &\leq 2\lambda_1(G) \leq 2n - 2. \end{aligned} \tag{4}$$

Any upper bound in the above given for $\mathcal{E}(\overline{G})-\mathcal{E}(G)$ is corresponding to an upper bound for $\mathcal{E}(G)-\mathcal{E}(\overline{G})$ and vise versa. Due to this, we focus on finding an upper bound on either $\mathcal{E}(G)-\mathcal{E}(\overline{G})$ or $\mathcal{E}(\overline{G})-\mathcal{E}(G)$. The following result provides the answer for Problem 1, in the case G is a general graph.

Theorem 2 *Let G be a graph of order n with the negative inertia $n^-(G)$ and let $\overline{G}$ be the complement of G. Then*

$$\mathcal{E}(G)-\mathcal{E}(\overline{G}) \leq 2\lambda_1(G)-2n+2n^-(G)+2. \tag{5}$$

The equality holds in (5) if and only if G is a regular graph with $n^-(G)+n^-(\overline{G})=n-1$.

Proof. For $G \cong K_n$, we have $\overline{G} \cong \overline{K_n}$, $\lambda_1(K_n)=n-1$ and $n^-(G)=n-1$. Then we have the equality in (5). Now we consider $G \ncong K_n$, and then $\lambda_2(G) \geq 0$, which implies $s(G)=n^+(G)+n^0(G)=n-n^-(G) \geq 2$, where n^+ and n^0 are the positive inertia and the nullity of G, respectively. Using this fact with considering non-increasing order for eigenvalues of G and $\overline{G}$ we have

$$\begin{aligned}
\mathcal{E}(G) &= \sum_{i=1}^{n}|\lambda_i(G)| = 2\sum_{i=1}^{s(G)}\lambda_i(G) \\
&= 2\lambda_1(G)+2\sum_{i=2}^{s(G)}\lambda_i(G) \\
&\leq 2\lambda_1(G)+2\sum_{i=2}^{s(G)}(-1-\lambda_{n-i+2}(\overline{G})) \quad \text{by (3)} \qquad (6)\\
&= 2\lambda_1(G)-2(s(G)-1)-2\sum_{i=2}^{s(G)}\lambda_{n-i+2}(\overline{G}) \\
&= 2\lambda_1(G)-2s(G)+2-2\sum_{i=1}^{s(G)-1}\lambda_{n-i+1}(\overline{G}) \\
&\leq 2\lambda_1(G)-2s(G)+2+2\max_{1\leq j\leq n}\sum_{i=1}^{j}-\lambda_{n-i+1}(\overline{G}) \qquad (7)\\
&= 2\lambda_1(G)-2n+2n^-(G)+2+\mathcal{E}(\overline{G}).
\end{aligned}$$

This completes the first part of the proof.

Now, suppose that we have equality in (5). Then all inequalities in the above must be equalities. The equality in (6) with Lemma 3 implies the graph G is regular. By definition of graph energy in (2), the equality in (7) holds if and only if

$$n^-(\overline{G}) \leq s(G)-1 \leq n^-(\overline{G})+n^0(\overline{G}),$$

that is,

$$n^-(\overline{G})+n^-(G) \geq n-1-n^0(\overline{G}), \quad \text{and} \quad n^-(\overline{G})+n^-(G) \leq n-1. \tag{8}$$

By Lemma 6 we have $n^-(\overline{G})+n^-(G) \geq n-1$ for any graph G. Then the left side inequality in (8) is obvious. Again applying Lemma 6 with the right side inequality in (8) implies G is a regular graph with $n^-(\overline{G})+n^-(G)=n-1$. This completes the second part of the proof. Hence the proof of the theorem is done. □

Remark 3 *Our result in Theorem 2 is obviously better than the Nikiforov's result in (4).*

Let G be a graph with the maximum degree $\Delta(G)$. It is a well-known fact that $\lambda_1(G) \leq \Delta(G)$ if and only if G is regular. This with Theorem 2 gives the following result.

Corollary 1 *Let G be a graph of order n with the maximum degree $\Delta(G)$ and the negative inertia $n^-(G)$ and let $\overline{G}$ be the complement of G. Then*

$$\mathcal{E}(G) - \mathcal{E}(\overline{G}) \leq 2\Delta(G) - 2n + 2n^-(G) + 2.$$

The equality holds in (5) if and only if G is a regular graph with $n^-(G) + n^-(\overline{G}) = n - 1$.

As we mentioned in the proof of Theorem 2 for $G \cong K_n$, we have $\mathcal{E}(K_n) - \mathcal{E}(\overline{K_n}) = 2n - 2$. The following result holds for non complete graphs.

Corollary 2 *Let G be a non complete graph of order n. Then*

$$\mathcal{E}(G) - \mathcal{E}(\overline{G}) \leq 2\lambda_1(G) - 2, \tag{9}$$

with equality if and only if G is a disjoint union of complete graphs of the same order, i.e., $G \cong sK_{n/s}$ for $s \geq 2$.

Proof. Since $G \ncong K_n$, $n^-(G) \leq n - 2$. This fact with Theorem 2 gives the desired result in (9).

Now applying extremal graphs in Theorem 2 with $n^-(G) = n - 2$, we get that G is a regular graph with $n^-(\overline{G}) = 1$. It is a well-known fact that $n^-(H) = 1$ if and only if H is a complete multipartite graph. From the above with this fact $\overline{G}$ must be a complete multipartite graph with partitions of the same size, that is, the graph G is a disjoint union of complete graphs of the same order, that is, $G \cong sK_{n/s}$ for $s \geq 2$. Hence the proof is over. □

Corollary 3 *Let G be a graph of order n with the independence number $\alpha(G)$ and the negative inertia $n^-(G)$ and let $\overline{G}$ be the complement of G. Then*

$$\mathcal{E}(G) - \mathcal{E}(\overline{G}) \leq 2n^-(G) \leq 2\,(n - \alpha(G)). \tag{10}$$

Equality holds in (10) if and only if $G \cong K_n$.

Proof. Using the upper bound $\lambda_1(G) \leq n - 1$ in Theorem 2 we obtain the left side inequality in (10). Moreover, applying the well-known inequality $n^-(G) \leq n - \alpha(G)$ we arrive at the right side inequality in (10).

The second part of proof is obtain by the well-known fact that $\lambda_1(G) = n - 1$ if and only if G is the complete graph of order n. Hence the proof of the corollary is done. □

4 The gap between energies of a regular graph and its complement graph

In this section, we study the gap between energies of a regular graph and its complement graph by obtaining an exact value and an upper bound for $\mathcal{E}(\overline{G}) - \mathcal{E}(G)$.

Theorem 4 *Let G be a k-regular graph of order n and let $\overline{G}$ be the complement graph of G. Then*

$$\mathcal{E}(\overline{G}) - \mathcal{E}(G) = 2\Big(\overline{k} - n^-(G) - \sum_{\substack{2\leq i\leq n\\ \lambda_i\in(-1,0)}} (\lambda_i(G) + 1)\Big) \tag{11}$$

where $\overline{k}$ and $n^-(G)$ are respectively the valency of $\overline{G}$ and the negative inertia of G.

Proof. Let $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ be the eigenvalues of the graph G. From the definition of energy in (1) with Lemma 2 we have

$$\mathcal{E}(\overline{G}) = \lambda_1(\overline{G}) + \sum_{i=2}^{n} |\lambda_i(\overline{G})| = \overline{k} + \sum_{i=2}^{n} |\lambda_i(\overline{G})|$$

$$
\begin{aligned}
&=\overline{k}+\sum_{i=2}^{n}|-1-\lambda_i(G)| \\
&=\overline{k}+\sum_{\substack{2\le i\le n\\ \lambda_i\le -1}}(-1-\lambda_i(G))+\sum_{\substack{2\le i\le n\\ \lambda_i> -1}}(1+\lambda_i(G)) \\
&=\overline{k}-m_G[-k,\,-1]+\sum_{\substack{2\le i\le n\\ \lambda_i\le -1}}|\lambda_i(G)| \\
&\quad +m_G(-1,\,k)+\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}\lambda_i(G)+\sum_{\substack{2\le i\le n\\ \lambda_i\ge 0}}|\lambda_i(G)|, \qquad (12)
\end{aligned}
$$

where $m_G(I)$ is the number of eigenvalues of G inside of the interval I. On the other hand we have

$$\mathcal{E}(G)=k+\sum_{\substack{2\le i\le n\\ \lambda_i\le -1}}|\lambda_i(G)|+\sum_{\substack{2\le i\le n\\ \lambda_i\ge 0}}|\lambda_i(G)|+\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}|\lambda_i(G)|.$$

From this with (12) we get

$$
\begin{aligned}
\mathcal{E}(\overline{G}) =&\mathcal{E}(G)-\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}|\lambda_i(G)|-k+\overline{k}-m_G[-k,-1]+n-1-m_G[-k,\,-1]+\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}\lambda_i(G) \\
=&\mathcal{E}(G)+2\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}\lambda_i(G)+2\,\overline{k}-2\,m_G[-k,\,-1]. \qquad (13)
\end{aligned}
$$

Moreover, we have

$$m_G(0,\,k]+m_G(0)+m_G(-1,\,0)+m_G[-k,\,-1]=n,$$

that is,

$$n^+ + n^0 + m_G(-1,0)+m_G[-k,\,-1]=n,$$

that is,

$$m_G[-k,\,-1]=n-n^+-n^0-m_G(-1,0)=n^{-1}-m_G(-1,0).$$

From this with (13) and the fact

$$\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}(\lambda_i(G)+1)=\sum_{\substack{2\le i\le n\\ -1<\lambda_i<0}}\lambda_i(G)+m_G(-1,0)$$

we get the desired result in (11). □

The following result is obtain by considering the fact

$$\sum_{\substack{2\le i\le n\\ \lambda_i\in(-1,0)}}(\lambda_i(G)+1)\ge 0.$$

Corollary 4 *Let G be a k-regular graph of order n and let $\overline{G}$ be the complement of G. Then*

$$\mathcal{E}(\overline{G})-\mathcal{E}(G)\le 2\,\left(\overline{k}-n^-(G)\right) \qquad (14)$$

where $\overline{k}$ and $n^-(G)$ are respectively the valency of $\overline{G}$ and the negative inertia of G. Equality holds in (14) if and only if G has no eigenvalues in the interval $(-1,0)$.

Two graphs are said to be *equienergetic* if their energies are equal.

Corollary 5 *Let G be a k-regular graph and let $\overline{G}$ be the complement of G. Then G and $\overline{G}$ are equienergetic if and only if*

$$n^-(G) - \overline{k} = \sum_{\substack{2\le i\le n\\ \lambda_i\in(-1,0)}} (\lambda_i(G)+1).$$

In particular if G has no negative eigenvalue greater than -1, then G and $\overline{G}$ are equienergetic if and only if

$$n^-(G) = \overline{k}, \quad i.e., \quad n^+ + n^0 = k+1.$$

Corollary 6 *If the summation of all eigenvalues of G located in the interval $(-1,0)$ is not integer, then G and $\overline{G}$ are not equienergetic.*

In the following we give a sufficient condition for a graph to have no negative eigenvalue in the interval $(-1,0)$.

Theorem 5 *Let G be a graph with the clique number $\omega \ge 2$. If $n^-(G) = \omega - 1$ then G has no negative eigenvalue in the interval $(-1,0)$.*

Proof. For a graph with the clique number ω by interlacing theorem we have $\lambda_2(K_\omega) \ge \lambda_{n-\omega+2}$, which implies $n^-(G) \ge \omega - 1$. Thus our assumption $n^-(G) = \omega - 1$ grantees that there is no eigenvalues greater than -1. □

Proposition 6 *Among all regular complete multipartite graphs $G \cong K_{\underbrace{n/s,\ldots,n/s}_{s}}$ $(1 \le s \le n)$, only the graph with $n = s^2$ is equienergetic with its complement graph. Indeed, the complement graph of G is $\overline{G} = sK_{n/s}$ with the spectrum $\{(\frac{n}{s}-1)^s, -1^{n-s}\}$. The spectrum of G is obtained by Lemma 2 as follows: $\lambda_1(G) = n-1-\overline{k} = n-1-(\frac{n}{s}-1) = n-n/s$ and*

$$\lambda_i(G) = -1 - \lambda_{n-i+2}(\overline{G}), \quad \text{for } 2 \le i \le n.$$

That is, $\lambda_i(G) = 0$, for $2 \le i \le n-s+1$ and $\lambda_i(G) = -n/s$, for $n-s+2 \le i \le n$. Therefore, $n^-(G) = s-1$. On the other hand, we have $\omega(G) = \alpha(\overline{G}) = s$. This gives $n^-(G) = \omega(G) - 1$, and then by Theorem 5, G has no negative eigenvalue greater than -1. Of course, this fact can be also seen from the spectrum of G in the above. Now by Corollary 5 we have G and $\overline{G}$ are equienergetic if and only if $n^-(G) = \overline{k}$, that is, $s-1 = n/s-1$. This is true if and only if $n = s^2$, that is, $G \cong K_{\underbrace{s,\ldots,s}_{s}}$.

5 Lower bound for the energy of complement graph

In this section we give a lower bound for the energy of a graph in terms of different graph invariants such as order, and chromatic number. The main result of this section is as follows:

Theorem 7 *Let G be a regular graph with chromatic number $\chi(G)$. Then*

$$\mathcal{E}(\overline{G}) \ge 2(n - \chi(G)). \tag{15}$$

The equality holds for $G \cong K_{\underbrace{n/s,\ldots,n/s}_{s}}$, where $1 \le s \le n$.

Proof. Let G be a k-regular graph. By Lemmas 2 and 4 and considering $c = \chi(G)$, we obtain

$$\sum_{i=2}^{c} \lambda_i(\overline{G}) = \sum_{i=2}^{c} (-1 - \lambda_{n-i+2}(G))$$

$$
\begin{aligned}
&= \sum_{i=1}^{c-1} (-1 - \lambda_{n-i+1}(G)) \\
&\geq -c + 1 + \lambda_1(G) = k - c + 1. \qquad (16)
\end{aligned}
$$

Now, by definition of graph energy with (16), we obtain

$$
\begin{aligned}
\mathcal{E}(\overline{G}) = 2 \max_{1\leq j\leq n} \sum_{i=1}^{j} \lambda_i(\overline{G}) \geq & 2\sum_{i=1}^{c} \lambda_i(\overline{G}) \\
= & 2\lambda_1(\overline{G}) + 2\sum_{i=2}^{c} \lambda_i(\overline{G}) \\
= & 2(n-1-k) + 2(k-c+1) \\
= & 2(n-c).
\end{aligned}
$$

This completes the first part of the proof.

To complete the second part of the proof, suppose that $G \cong K_{\underbrace{n/s,\ldots,n/s}_{s}}$, where $1 \leq s \leq n$. By Proposition 6 we have

$$
\mathcal{E}(\overline{G}) = 2 \sum_{i=1}^{n^+(\overline{G})} \lambda_i(\overline{G}) = 2s(n/s - 1) = 2n - 2s.
$$

On the other hand, $\chi(G) = s$ and then we have the equality in (15). □

A *clique cover* of G is a set of cliques of G such that every vertex is in at least one of them. The *clique cover number* is the minimum size of a clique cover, and is denoted by $\theta_v(G)$. The clique cover number of G equals the chromatic number of its complement $\overline{G}$, that is,

$$
\theta_v(G) = \chi(\overline{G}). \qquad (17)
$$

Applying this relation, we restate Theorem 7 as follows:

Theorem 8 *Let G be a regular graph with the clique cover number $\theta_v(G)$. Then $\mathcal{E}(G) \geq 2(n - \theta_v(G))$.*

Remark 9 For a k-regular graph G, two simple lower bounds on $\mathcal{E}(\overline{G})$ are as follows:

$$
\mathcal{E}(\overline{G}) \geq 2\lambda_1(\overline{G}) \geq 2\overline{k} = 2(n-1-k) \qquad (18)
$$

and also

$$
\mathcal{E}(\overline{G}) \geq 2\lambda_1(\overline{G}) \geq 2(\chi(\overline{G}) - 1). \qquad (19)
$$

For k-regular graph G we have $\chi(G) \leq 1 + k$ by Brooks' theorem, that is, $2(n - \chi(G)) \geq 2(n-1-k)$, which proves that the lower bound given in Theorem 7 is better than (18). Moreover, by Lemma 5 we have $\chi(G) + \chi(\overline{G}) \leq n + 1$, that is, $2(n - \chi(G)) \geq 2(\chi(\overline{G}) - 1)$, which proves that the lower bound given in Theorem 7 is better than (19).

Remark 10 *Our result in Theorem 8 is better than the result given in Lemma 7 if and only if $\theta_v(G) \leq n/2$, i.e., $\chi(\overline{G}) \leq n/2$.*

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