Connectivity, Transitivity and Chromaticity: The Pioneering Work of Bernard Roy in Graph Theory

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Abstract

We review the work of B. Roy in graph theory and its posterity.

Keywords: Graph, connectivity, transitivity, chromaticity, review

Résumé

On passe en revue l'oeuvre de B. Poy en théorie des graphes et sa postérité.

Introduction

Before exploring in depth and breadth Decision Aid Methods, Bernard Roy devoted a few years to the study of graph theory. This led to a series of contributions including a large book (Roy 1969/70). Generated in a context of intensive research in graph theory, pioneering ideas in several of his papers induced long streams of results by many authors up to the present time.

In this chapter, we review the work of B. Roy on graph theory about forty years after its publication. We also outline its posterity by presenting a sample of the extensions of his seminal results and we describe the general context in which his research was carried out. We do not aim at exhaustivity, but rather, in a tutorial spirit, try to present to a large audience the main themes of this research.

We assume the reader is familiar with the basic concepts of graph theory and refer to the book of C. Berge, "Graphs and Hypergraphs" (Berge 1973) for definitions not given here.

1 Connectivity and transitivity

In his first paper (Roy 1958), B. Roy considers a graph $G = (X, \Gamma)$ where X denotes a set of vertices $x_1, x_2, \ldots, x_n$ and Γ a one-to-many application from X to X. He then introduces a network R defined as follows:

- its vertex set is the union of X and a copy Y of X plus a source vertex x_0 and a sink vertex z;
- its arc set contains arcs $x_i y_j$ if and only if $x_i x_j$ is an arc of G, plus arcs $x_0 x_i$ and $y_j z$ for each vertex x_i;
- nonzero capacities c_i are then associated with all arcs $x_0 x_i$ and $y_j z$; the remaining arcs have infinite capacities (see Fig. 1).

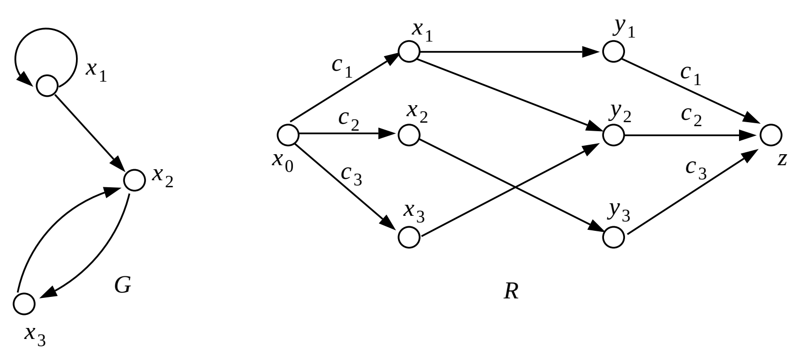


Figure 1: The construction of a network R associated to a graph G in which every vertex is contained in a circuit.

Then B. Roy obtains the following results:

Theorem 1 *The following statements are equivalent:*

(i) *there is no subset A of X strictly containing its image $\Gamma(A)$*

(ii) *each vertex belongs to at least one circuit of G*

(iii) *capacities c_i can be chosen in such a way that there is a flow in R which saturates the extremal arcs x_0x_i and y_jz.*

The graph G in Fig 1 satisfies the conditions (i), (ii), (iii), as can be verified.

Theorem 2 *A graph G is strongly connected if and only if there is no proper subset A of X containing its image $\Gamma(A)$.*

In the graph G in Fig. 1 the subset $A = \{x_2, x_3\}$ contains its image $\Gamma(A) = A$, hence G is not strongly connected.

D. Gale (1959) observes that this last result appears in a long paper of R. Rado (1943) on linear combinatorial topology and general measure.

P. Camion (1959) uses the proof technique of (Roy 1958) to derive a well-known result on Hamiltonian circuits:

Theorem 3 *(P. Camion 1959): A complete graph has a Hamiltonian circuit if and only if it is strongly connected.*

Fig. 2 shows a complete graph which is not strongly connected; it has however a Hamiltonian path (but no Hamiltonian circuit according to theorem 3).

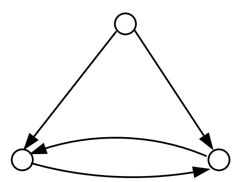


Figure 2: A complete graph which is not strongly connected.

The result of Camion has been in turn extended by M. Goldberg and J.W. Moon (1972). We recall that a graph is *k-strong* if and only if between any two vertices there are k arc-disjoint paths.

Theorem 4 *(Goldberg and Moon 1972): A k-strong tournament has at least k distinct Hamiltonian circuits.*

Many additional results on this topic have been obtained later by various authors, see e.g. (Thomassen 1980), (Bermond and Thomassen 1981), (Zhang and Song 1991) and (Bang-Jensen and Gutin 1988) for surveys.

The second paper of B. Roy, motivated by applications in sequencing and scheduling problems, states conditions of existence of systems of potentials: we are given a system of linear inequalities of the following form

$$t_j - t_i \geq a_{ij} \qquad \text{for } i, j \in K \tag{1}$$

where the t_i are unknown and the a_{ij} are real numbers given for each pair i, j in a given set K.

The pairs i, j in K can be associated with the arcs of a graph G. The problem so defined, after introduction of a linear objective function, is the dual of a minimum cost flow problem, and it can be transformed into the dual of a transportation problem.

Theorem 5 *(Roy 1959a): A necessary and sufficient condition for the existence of a solution of* (1) *is that the sum of the* a_{ij}*'s over all arcs* i, j *of any elementary circuit is non-positive.*

As an illustration, examine the graphs in Fig. 3: for the graph of Fig. 3 (a), there is no solution to (1); for the graph in Fig. 3 (b), there is a solution.

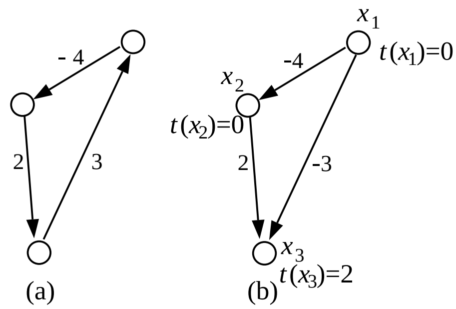


Figure 3: Graphs with and without systems of potentials.

After that, A. Ghouila-Houri (1960) presents in a group-theoretical wording a more general result formulated in terms of algebraic properties of the a_{ij}'s; this formulation contains the theorem of A. Hoffman (see (Berge 1973)) on the existence of circulations in capacitated networks and an early result of D. König (see (Berge 1973)) on the two-colorability of graphs (the vertices of a graph can be colored with two colors if and only if every cycle has an even number of edges).

The theory of tensions (systems of potentials), of which the problem considered by B. Roy is a special case, has been extensively studied in the book of C. Berge and A. Ghouila-Houri (1965).

Based on the study of B. Roy on systems of potentials, the so-called MPM method (méthode des potentiels METRA) was developed for solving sequencing problems. In opposition to the classical critical path method (CPM) used previously, MPM uses graph representations where the various tasks of a project are associated to the vertices of a graph (instead of the arcs). This formulation is extremely fruitful since it enables to model much more general constraint types than classical CPM.

The third paper of B. Roy (Roy 1959b), on transitivity and connectivity, presents a major result which has generated a number of interesting developments by many authors.

B. Roy considers the adjacency matrix $A = (a_{hk})$ of a graph $G = (X, \Gamma)$ (assuming $a_{kk} = 1$ for all k) and studies its transitive closure, i.e., the matrix $\hat{A} = (\hat{a}_{hk})$ such that

$\hat{a}_{hk} = 1$ if there is a path from vertex x_h to vertex x_k in G and $\hat{a}_{hk} = 0$ otherwise. The basic tool in this computation is the transformation T_i defined as follows: starting from a matrix $A = (a_{hk})$ we apply T_i and get $T_i \cdot A$ where

$$\text{for every } h \text{ with } a_{hi} = 1 \text{ we set } a_{hk} := \max(a_{hk}, a_{ik}).$$

Transformation T_i reproduces the ones of row i in any row containing a one in column i: all the vertices x_k which can be reached from vertex x_i can also be reached from any vertex x_h if there is an arc (x_h, x_i) or more generally if x_i can be reached from x_h.

The transformations T_i, T_j are commutative. Applying T_i to A does not change its transitive closure and $A = \hat{A}$ if and only if A is not modified by any transformation T_i.

Theorem 6 *(Roy 1959b): The matrix $T_n \cdot T_{n-1} \ldots T_1 \cdot A$ is equal to the transitive closure $\hat{A}$ of A.*

$\hat{A}$ can be seen as the vertex-vertex incidence matrix of a graph $\hat{G}$ which is called the transitive closure of $G : \hat{G}$ has an arc (x_i, x_j) if and only if there is a path in G from x_i to x_j.

This gives immediately a $O(n^3)$ algorithm for the transitive closure of a graph. We also see that a graph G is strongly connected if and only if all entries of $\hat{A}$ are equal to one. The same algorithm was discovered independently, but published three years later, by S. Warshall (1962).

Remark 1 With a view on the process of formalizing proofs, P. Naur (1994) examines the papers (Warshall 1962) and (Floyd 1962), (which, as described below extends it to shortest paths), declares that the presentation of S. Warshall is “a complicated mixture of formal expression and informal prose”, and speculates on the importance or not of formalization in making a proof convincing. Unfortunately, he does not compare the paper of S. Warshall (1962) with the more formal presentation of B. Roy (1959b). □

The papers of Roy (Roy 1959b) and Warshall (Warshall 1962) led to a large stream of developments, up to the present day. These can be divided in two main categories.

On the one hand, increasingly large classes of problems which can be solved by similar matrix algorithms have been identified; on the other hand, refinements and improvements have been brought to the basic algorithm, including adaptations to parallel computing. We shall review them in turn.

Transformation T_i may be expressed in a condensed form by using the boolean sum $\dot{+}$ and product $\dot{\times}$:

$$a_{hk} := a_{hk} \dot{+} a_{hi} \dot{\times} a_{ik}.$$

A first extension of the transitive closure algorithm was to the computation of the matrix $\hat{L}$ of distances between all pairs of vertices of a graph G where each arc has a nonnegative length. First published as a twenty line paper by R.W. Floyd (1962) and referring to (Warshall 1962), this extension remained unnoticed for some time.

Let $L = (\ell_{hk})$ denote the matrix of arc lengths where the absence of an arc is expressed by an arbitrary large value in the corresponding entry. As before, let T_i be a transformation applied to L defined by $T_i \cdot L = (\ell_{hk})$ with

$$\ell_{hk} := \min(\ell_{hk}, \ell_{hi} + \ell_{ik}).$$

This means that the distance between x_h and x_k is at most equal to the minimum of the length ℓ_{hk} of a path from x_h to x_k and of the sum of the lengths ℓ_{hi} of a path from x_h to x_i and ℓ_{ik} of a path from x_i to x_k.

With some adaptation the proof of Theorem 6 shows that

$$T_n \cdot T_{n-1} \dots T_1 \cdot L = \hat{L}.$$

A further extension to the maximum capacity path can be made by starting from a matrix $C = (c_{hk})$ where c_{hk} is the capacity of arc (x_h, x_k) if it exists and 0 otherwise. The capacity of a path is defined as the smallest capacity of its arcs. T_i is defined by $T_i \cdot C = (c_{hk})$ with

$$c_{hk} := \max\big(c_{hk}, \min(c_{hi}, c_{ik})\big).$$

A similar formula gives the maximum reliability path by starting from $R = (r_{hk})$ where r_{hk} is the reliability of arc (x_h, x_k) if it exists and 0 otherwise. The reliability of a path is defined as the product of the reliabilities of its arcs. Then T_i is defined by $T_i \cdot R = (r_{hk})$ with

$$r_{hk} := \max(r_{hk}, r_{hi} \cdot r_{ik}).$$

The three cases presented above can be viewed as instances of a more general algorithm to solve a system of equations

$$a_{hk} := a_{hk} \oplus (a_{hi} \otimes a_{ik})$$

where $\oplus$ and $\otimes$ are the boolean sum and product for the transitive closure, the Min operator and the usual sum for shortest path, the Max and the Min operations for maximum capacity path and the Max operation and the usual product for reliability. This suggests there should be a general algebraic structure subsuming all these cases. It is indeed so. In a general setting it is a dioid (Gondran and Minoux 1984) (also called semi-ring by other authors), defined as a set S with two operations

(i) the operation $\oplus$ ("add") gives S a structure of commutative monoid (closure, commutativity, associativity) with neutral element ε.

(ii) the operation $\otimes$ ("multiply") gives S a structure of monoid (closure, associativity) with neutral element e (unit). Moreover, ε is *absorbing* ($a \otimes \varepsilon = \varepsilon$ for any a in S) and $\otimes$ is right and left distributive with respect to $\oplus$. Moreover,

(iii) the preorder relation $\geq$ (reflexivity, transitivity) induced by $\oplus$ (canonical preordering) and defined by $a \geq b$ if and only if there is a c in S such that $a = b \oplus c$ is a *partial order*, i.e., it satisfies $a \geq b$ and $b \geq a$ implies $a = b$ (antisymmetry).

Many authors have studied this structure (e.g. (Shimbel 1954), (Cunninghame-Green 1960, 1962, 1979), (Yoeli 1961), (Robert and Ferland 1968), (Tomescu 1968), (Gondran 1975) and (Wongseelashote 1976)). The reader is referred to (Gondran and Minoux 1984) for further references.

Properties of classical linear algebra, and in particular algorithms to solve systems of linear equations (Gauss, Gauss-Seidel, Jacobi, Jordan, etc.) can be adapted to dioids. Algorithms of (Dantzig 1967) and (Tabourier 1973) can be viewed in this light, as well of course and that of (Roy 1959b) which corresponds to a generalized Jordan method.

So the work of B. Roy on transitive closure is part of a vast stream which dates back to the middle fifties. This research program has been expanded further recently by M. Gondran (1996a, 1996b) and M. Gondran and M. Minoux (1997) who have shown how the use of dioids can extend nonlinear analysis.

Among possible applications of the transitive closure algorithm, we mention finding the transitive reduction of an oriented graph G without circuits, i.e., removing a maximal subset of arcs without changing the transitive closure (Gries, Martin, van de Snepscheut and Udding 1989). Another application is the efficient evaluation of single-rule Datalog programs with a slight generalization of the Floyd-Roy-Warshall algorithm (Papadimitriou and Sideri 1999).

P.L. Hammer and S. Nguyen (1977) consider the following logical problem, which generalizes the question of computing the transitive closure of a graph: we are given a set of binary relations between boolean variables y_j, i.e., relations of the form $y_h \leq y_k$ or $y_h \leq \bar{y}_k$ (which are equivalent to $\bar{y}_h \geq \bar{y}_k$ or $\bar{y}_h \geq y_k$, where $\bar{y}_k$ denotes the complement of y_k). We have to determine the logical closure of these relations, i.e., the set of all conclusions which can be of the following types:

a) a contradiction

b) some variable y_k takes only value 1 (or value 0)

c) some pairs of variables y_h, y_k may be identified

d) some pairs of variables y_h, y_k are such that $y_h = \bar{y}_k$.

P. Hansen (Hansen 1976/77) has extended the algorithm of B. Roy to this problem: observe that none, one or more of the four relations $y_h \leq y_k$, $y_h \geq y_k$, $y_h \leq \bar{y}_k$, $y_h \geq \bar{y}_k$ can hold for any pair of variables y_h, y_k. The subset of relations defines one among 16 states s_{hk}; then a table can be constructed for the product of states s_{ki} and s_{ih} and for the sum of states on the same pair of variables. Using these tables, the extension is straightforward.

Further extensions to path problems restricted in various ways were studied in (Klee and Larman 1979).

Turning to the computational improvements of the transitive closure algorithm of B. Roy, we note that several authors have proposed versions which reduce the number of computations to a half without however changing the worst case complexity. This is usually done by using in turn forward and backward processes; see (Hu 1967), (Bilde and Krarup 1969), (Hoffman and Winograd 1972), (Warren 1975), (Goldman and Tiwari 1986), (Farbey, Land and Murchland 1986). (Land and Stairs 1967) note that for weakly connected graphs a block structure of the matrix can be exploited. (Yuval 1975/76) observes that using Strassen's algorithm for matrix multiplication leads to a transitive closure algorithm with a complexity $O(n^{2.81})$. The many extensions of Strassen's algorithm can be transposed in a similar way.

As expected many adaptations to parallel computing have been presented for the transitive closure algorithm and for its generalizations to the all pair shortest path problems. For parallelization of the transitive closure algorithm, see e.g. (Rote 1985), (Zhu 1985) and (Poel and Zwiers 1993). A recent survey together with new results (also linked to complexity) is given by (Takaoka 1998).

B. Roy's results in graph theory together with many others are also presented in a large book, in two volumes (Roy 1969/70); a couple of papers (Roy 1962/69) survey shortest paths and connected graphs. Their material is included in the book. Algorithms are covered in detail with a rare wealth of applications, often based upon case studies done at METRA's scientific direction, headed by B. Roy for several years.

2 Paths and Colors

In a now classical paper (Roy 1967), B. Roy had the original idea of linking two seemingly different concepts of graph theory: the length of a path and the chromatic number. This result was also obtained independently by T. Gallai (1968).

In fact colors used in graphs are often replaced by integer numbers, so it may look more natural to examine the connections between colorings and orientations of graphs; if G is an (oriented) graph containing no circuits, then by associating to each vertex x a number $c(x)$ which is the number of vertices on the longest elementary path ending at x, we obtain a k-coloring of G where $k = s(P)$ is the number of vertices on the longest (elementary) path P of G.

So, if G is an (oriented) graph without circuits such that every (elementary) path P has $s(P) \leq k$ vertices, then G has a k-coloring (i.e., G is k-chromatic). Observe that in a graph without circuits, all paths are necessarily elementary. The result of B. Roy and T. Gallai is to extend this to oriented graphs (possibly with circuits) where one simply requires that $s(P) \leq k$ for any elementary path P.

Theorem 7 *(Roy 1967): If in a finite oriented graph G there is no elementary path P with $s(P) > k$, then G has a k-coloring.*

This result combined with the observation that the vertices of a graph G can always be colored with $\Delta(G) + 1$ colors (where $\Delta(G)$ denotes the maximum degree of G, i.e., the maximum number of arcs adjacent to a vertex) gives the following:

Corollary 1 *The edges of a graph G can always be oriented in such a way that the resulting graph has no circuit and each elementary path has at most $\Delta(G)$ arcs.*

This last result can be strengthened using a theorem of C. Szekeres and H.S. Wilf (1968): the maximum degree $\Delta(G)$ of G can be replaced by the maximum over all induced subgraphs H of G of the minimum degree of H.

S. Fajtlowicz (1988, 1999) designed the system Graffiti to obtain automatically conjectures in graph theory. He obtained and proved, in 1993 (see Fajtlowicz 1999), the following variant of B. Roy's theorem for undirected graphs:

Theorem 8 *(Graffiti's conjecture 748): Let G be a finite, connected and undirected graph. Then the chromatic number of G is not more than the minimum over all vertices v of G of the number of vertices in the longest elementary path beginning at v.*

This result was recently extended to directed graphs by Hao Li (1998).

An equivalent formulation of Theorem 7 due to (Berge 1982) is the following:

In every finite oriented graph G with chromatic number $\chi(G) = k$, there exists at least one path P with $s(P) = k$.

The second result of B. Roy in (Roy 1967) is related to the construction of a particular orientation of the edges of a graph G with $\chi(G) = k$:

Theorem 9 *(Roy 1967): For any graph G with $\chi(G) = k$, one may orient its edges in such a way that the resulting graph contains no circuit and has the following properties:*

(a) Let S_1 be the set of vertices without predecessors in G,
S_2 the set of vertices without predecessors in $G - S_1$,
S_3 the set of vertices without predecessors in $G - S_1 - S_2$, etc.
We thus define a partition of the vertex set into exactly k stable sets, i.e., a k-coloring of G.

(b) For any vertex x in S_{h+1}, the chromatic number of the subgraph $G_h(x)$ of G induced by $S_1, S_2, \ldots, S_h$ and x satisfies $\chi\big(G_h(x)\big) = h + 1$.

Fig. 4 (a) shows a 4-coloring of a graph G with $\chi(G) = 4$, in Fig. 2 (b) an orientation without circuits is given (it is derived in fact from the coloring in Fig. 2 (a)) with a 4-coloring having Property (a) of Theorem 9. Notice that it does not satisfy Property (b) since the subgraph generated by S_1, S_2 and x is 2-colorable.

The proof technique of B. Roy consists in starting from a $\chi(G)$-coloring of G with "colors" $1, 2, \ldots, \chi(G) = k$.

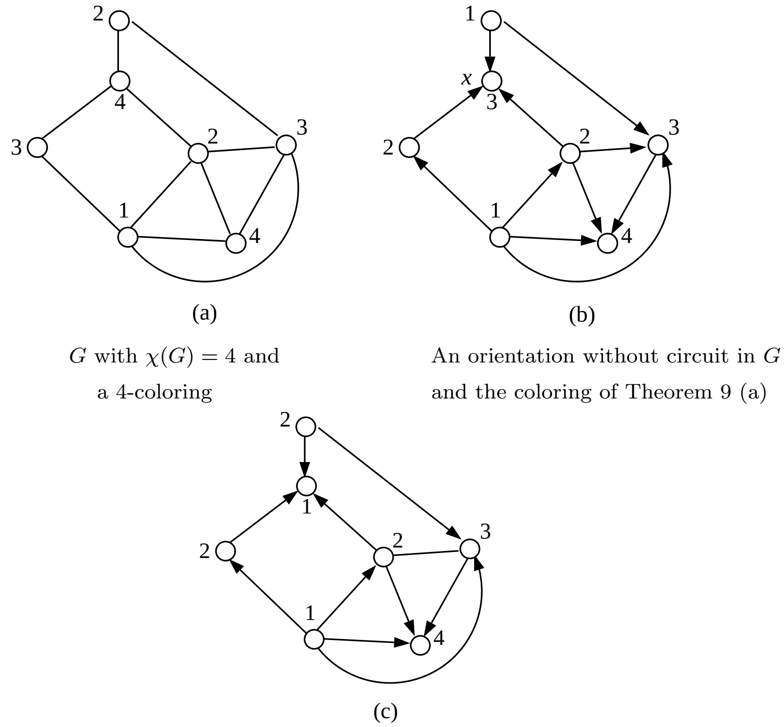
Scanning consecutively the vertices with colors $k, k-1, \ldots, 3, 2, 1$ one tries to assign to each vertex z a color $c(z)$ as small as possible. Then by orienting each edge $[x, y]$ from x to y if $c(x) < c(y)$ one obtains the orientation satisfying (a) and (b).

B. Roy gives additional properties of the orientation (Roy 1967):

(c) Any vertex x with $c(x) = h$ has a predecessor in each one of the sets $S_{h-1}, S_{h-2}, \ldots, S_1$.

(d) Every stable set S_h is (inclusion-wise) maximal in the subgraph of G induced by $S_h, S_{h+1}, \ldots, S_k$.

As an illustration, one may verify that the 4-coloring of Fig. 4 (c) satisfies the above properties. It is interesting to observe that such colorings are similar in spirit to some solutions of sequencing problems which are named "squeezed to the left" (Fr: "calé à gauche") which were also studied by B. Roy (1962). Here we may consider the graph G as associated to a *chromatic scheduling* problem as follows:



(a) G with $\chi(G) = 4$ and a 4-coloring

(b) An orientation without circuit in G and the coloring of Theorem 9 (a)

(c) A 4-coloring having Property (b) of Theorem 9 (b)

Figure 4: Some illustrations of Theorem 9

The nodes correspond to jobs with equal processing times, say 1. An arc (x, y) means that job x must precede job y. There is a one-to-one correspondence between k-colorings of G and feasible schedules in k time units where each job starts at some integer time.

The k-colorings satisfying (a)–(d) are precisely schedules squeezed to the left: according to (b), if a job x is scheduled at period $c(x) = h$, it is because all jobs in $S_1, S_2, \ldots, S_{h-1}$ need $h - 1$ periods and x cannot be scheduled in a period $i \leq h - 1$. It is thus a schedule where each job starts at its earliest date.

Remark 2 The connection between orientations and colorings is also illustrated in (Hansen, Kuplinsky and de Werra 1997) where mixed colorings are defined: a partially oriented graph G is given which contains edges and arcs. A *mixed k-coloring* is an assignment of colors $c(x) \in \{1, 2, \ldots, k\}$ to all nodes x of G in such a way that for each edge $[x, y]$ the colors $c(x)$ and $c(y)$ are different and for each arc (x, y), $c(x) < c(y)$. This model is introduced to take into account some scheduling problems where both precedence and disjunctive requirements occur. In (Hansen, Kuplinsky and de Werra 1997) bounds on a generalized chromatic number are derived and an algorithm is sketched for partially oriented trees.

At the end of his paper B. Roy mentions briefly the special case of perfect graphs G; these are the graphs in which each induced subgraph H of G satisfies $\chi(H) = \omega(H)$ where $\omega(H)$ is the maximum cardinality of a clique of H.

In such graphs, the colorings satisfying (a)–(d) have the following characteristics:
For each vertex $x \in S_h$ there exists a clique $K_h(x) \ni x$ with $K_h(x) \cap S_i \neq \emptyset$ for $i = h, h-1, \ldots, 1$.

Such colorings have been called *canonical* in (Preissmann and de Werra 1985) where *strongly canonical* colorings have also been defined for strongly perfect graphs (Berge 1984) (a graph G is *strongly perfect* if in every subgraph H of G there exists a stable set S such that $S \cap K \neq \emptyset$ for every inclusion-wise maximal clique K of H).

A coloring is strongly canonical if for any clique K of C there is a clique $C \supset K$ such that $C \cap S_i \neq \emptyset$ for $i = 1, 2, \ldots, \min\{\ell | S_\ell \cap K\}$.

As observed in (Preissmann and de Werra 1985), a graph G is strongly perfect if and only if every induced subgraph of G has a strongly perfect coloring. The graph G in Fig. 4 (a) is perfect (one may check that the 4-coloring in Fig. 4 (c) is canonical); it is also strongly perfect, as can be verified.

Coming back to the general case handled by B. Roy, one observes that the orientation constructed is such that there is a path P meeting (consecutively) $S_1, S_2, \ldots, S_k$; in fact, the coloring satisfies the following: every vertex z with $c(z) = h$ is on a path which meets (consecutively) $S_1, S_2, \ldots, S_h$.

Starting from this observation, Berge has obtained the following result:

Theorem 10 *(Berge 1982): Let k be the maximum number of vertices in a path of G. Then for every path P with k vertices, there exists a k-coloring $(S_1, \ldots, S_k)$ such that $|S_h \cap P| = 1$ for $h = 1, 2, \ldots, k$. Furthermore, this coloring is such that: for each $x \in S_h$ there is an arc from S_{h-1} to x.*

Notice that this does not imply that in any graph G, there exists a $\chi(G)$-coloring and a path P meeting every color exactly once.

V. Chvátal (1972) has observed that Theorem 7 can be used to derive a consequence which is a generalization and a simplification of a result of (Busolini 1971):

Corollary 2 *(Chvátal 1972): Let $G = (X, U)$ be a finite oriented graph (without loops) where the arc set U is partitioned into $U_1, U_2, \ldots, U_k$. Assume $\chi(G) > m_1 m_2 \ldots m_k$ where $m_1, m_2, \ldots, m_k$ are positive integers. Then there exists an integer j with $1 \leq j \leq k$ such that $G_j = (X, U_j)$ contains an elementary path with $m_j + 1$ arcs.*

The proof consists in observing that we obtain a coloring of G by taking Cartesian products of colorings of the graphs G_j; so

$$m_1 m_2 \ldots m_k < \chi(G) \leq \chi(G_1)\chi(G_2) \ldots \chi(G_k)$$

and there is a G_j with $\chi(G_j) > m_j$, hence by Theorem 7, there is in G_j a path on m_j arcs.

The result of B. Roy can also be stated as follows:
In every finite oriented graph G, the maximum number $s(P)$ of vertices in a path P satisfies $s(P) \geq \chi(G)$.

J.A. Bondy has obtained a result linking the chromatic number to the length of a longest circuit:

Theorem 11 *(Bondy 1976): In a strongly connected graph G (with at least two vertices), the longest circuit has length at least $\chi(G)$.*

It is immediate to observe that Theorem 7 can be obtained from Theorem 11: it suffices to introduce into G a new vertex linked to every vertex of G in both directions.

Furthermore, one may also observe that Theorem 11 of J.A. Bondy implies Theorem 3 of P. Camion. Combining the concept of perfectness with Theorem 7, C. Berge has called *χ-diperfect* the graphs such that any subgraph H satisfies the following condition:
given any optimal k-coloring $(S_1, S_2, \ldots, S_k)$ of H with $k = \chi(H)$, there exists a path P with $|P \cap S_i| = 1$ for $i = 1, \ldots, k$.

He has shown that every perfect graph and every symmetric graph is χ-diperfect (Berge 1982). The graph G in Fig. 5 is not χ-diperfect: there exists a 3-coloring (S_1, S_2, S_3) for which no path P can be found with $|P \cap S_i| = 1$ for $i = 1, 2, 3$.

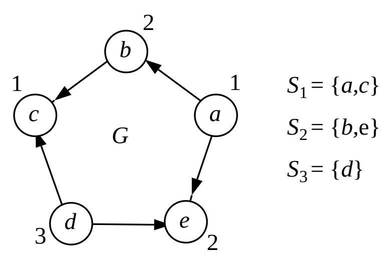


Figure 5: A graph G with a 3-coloring (S_1, S_2, S_3)

It is worth mentioning here a companion result of Theorem 7. We define a *path-partition* of an oriented graph G as a collection $M = (P_1, P_2, \ldots, P_k)$ of vertex-disjoint paths P_i which partitions the vertex set of G.

T. Gallai and A.N. Milgram have obtained the following:

Theorem 12 *(Gallai and Milgram 1960): If there is no path partition of G with less than k paths, then G contains a set of at least k non adjacent vertices.*

A set of non adjacent vertices is a *stable set* and the maximum cardinality of a stable set in G is denoted by $\alpha(G)$. Theorem 12 amounts to saying that $\min |M| \leq \alpha(G)$. The graph G in Fig. 5 has a path partition $M = \big(\{a, b, c\}, \{d, e\}\big)$ and $\alpha(G) = 2$.

C. Berge has defined in an analogous way *α-diperfect* graphs which are those in which every subgraph H has the following property:
given *any* maximum stable set S, there exists a path-partition $M = (P_1, P_2, \ldots, P_k)$ with $k = \alpha(H)$ and $|S \cap P_i| = 1$ for $i = 1, \ldots, k$.

Again perfect graphs and symmetric graphs are shown to be α-diperfect (Berge 1982). The graph G in Fig. 5 is not α-diperfect:

for $S = \{a, c\}$, the only path partition $\big(P_1 = \{a, b, c\},\ P_2 = \{d, e\}\big)$ in $\alpha(G) = 2$ paths is such that $P_2 \cap S = \emptyset$.

In general, for an arbitrary graph, the known proofs of Theorem 12 do not imply the existence of a maximum stable set S and of a path-partition $(P_1, \ldots, P_{|S|})$ with $|S \cap P_i| = 1$ for all i.

We would now like to recall the conjecture of C. Berge which would unify Theorems 7 and 12.

A path partition $M(P_1, \ldots, P_q)$ is called *k-optimal* if it minimizes the quantity

$$B_k(M) = \sum_{i=1}^{q} \min \big\{k, |P_i|\big\}.$$

Notice that a 1-optimal path partition M contains a minimum number $|M|$ of paths.

The Strong Path Partition Conjecture (SPPC) is formulated as follows:

For every k-optimal partition $M = (P_1, \ldots, P_q)$ of an oriented graph G there exists a k-coloring of a subgraph H of G such that the number of different colors on P_i is

$$\min \big\{k, |P_i|\big\} \qquad \text{for } i = 1, \ldots, q$$

Fig. 6 shows a 2-optimal path partition M and a 2-coloring of H spanned by $\{a, b, c, d\}$; one verifies that each P_i in M contains vertices of $\min \big\{2, |P_i|\big\}$ different colors. For $k = \max \big\{s(P) : P \text{ path of } G\big\}$, the SPPC is true: it is Theorem 10. For $k = 1$, it was proved by N. Linial (1978).

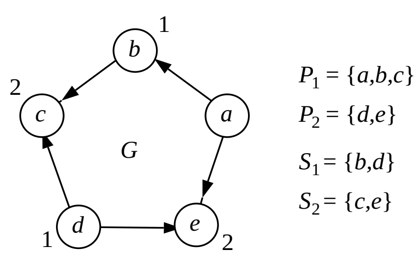


Figure 6: A 2-optimal path partition $M = (P_1, P_2)$ and a 2-coloring (S_1, S_2).

Theorem 13 *(Linial 1978): If $P = (P_1, \ldots, P_q)$ is an optimal path partition in an oriented graph G, there exists a stable set S with*

$$|S \cap P_i| = 1 \qquad \textit{for } i = 1, \ldots, q$$

For the graph G in Fig. 6, if we take $P_1 = \{a, b, c\}$, $P_2 = \{d, e\}$, we can choose $S = \{a, d\}$.

The SPPC has been shown to hold for special classes of graphs; it holds in particular for transitively oriented graphs without circuits, a result of (Greene and Kleitman 1976), for bipartite graphs as proved by (Berge 1984) and for oriented graphs where all cycles (and circuits) are vertex disjoint as shown in (Sridharan 1993).

As can be expected, for graphs without circuits, stronger results can be derived. We shall mention first the following statement due to K. Cameron (1986) and M. Saks (1986).

Theorem 14 *(Cameron 1982, Saks 1986): Let G be an oriented graph without circuits and k a positive integer. Then there exists a partial k-coloring $(S_1, S_2, \ldots, S_k)$ such that* for every k-optimal path partition $M = (P_1, \ldots, P_q)$ *every path P_i of M meets* $\min\{k, |P_i|\}$ *color classes.*

By interchanging the roles of paths and stable sets, we can obtain a "dual" result. The analogue of a partial k-coloring for paths is a family of at most k vertex disjoint paths $(P_1, \ldots, P_q)$. We may call it a *path k-packing.* Its cardinality is $|\cup_{i=1}^{q} P_i|$ and it is optimum if its cardinality is maximum. R. Aharoni, I. Ben-Arroyo Hartman and A.J. Hoffman (1985) have obtained the following:

Theorem 15 *(Aharoni, Ben-Arroyo Hartman and Hoffman 1985): Let G be an oriented graph without circuits and k a positive integer. Then there exists a coloring $C = (S_1, \ldots, S_p)$ such that for every optimum path k-packing $(P_1, \ldots, P_q)$ every color class S_i of C meets* $\min\{k, |S_i|\}$ *different paths of M.*

As a final observation we should indicate that the results of (Roy 1967), (Gallai 1968) and (Gallai and Milgram 1960) have been generalized and reformulated in terms of hypergraphs by H. Müller (1981).

Conclusion

We have presented in a condensed form the contributions of B. Roy to graph theory. Our discussion has shown that several of his results as well as some of the questions raised in his papers have undoubtedly had an impact on the work of numerous researchers.

Many generalizations and variations have followed; what is now called the "theorem of Roy-Gallai" has in particular been a source of inspiration for a number of researchers and we conjecture that it will continue for many years to come.

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Note added in proof:

It was brought to our attention by a referee that a striking allusion to Theorem 7 appears in the "Cymbalum Mathematicorum" of Chevalier Théo de Biroille (1534). We are pleased to quote an excerpt of this recently rediscovered manuscript.

Les longs chemins du Roy
Sont fort bien colores;
Il fault pour qu'on le voye
Points et lignes explorer:
Que soyent moult arcs donnes!
Pour toute direction
Qu'on veult bien ordonner
Une coloration
Peut etre ainsy trouvez
Au total de couleurs,
Comme cela est prouvez,
Egal a la longueur
Du plus long des chemins
Qu'on y peut parcourir
A pied ou a la main
A votre gloire, Messire

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