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Computing α-robust equilibria in two integrated assessment models for climate change

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Abstract: In this paper we show how to robustify the computation of equilibria in two integrated assessment models for climate change. Both models deal with the optimal timing of a transition to a 'clean' economy where a technology with low emissions but high energy cost can be used in the production process. The game represents the competition between industrialized and developing countries. A cost-benefit approach is implemented with an economic loss factor that represents the damages due to climate change. In the first model one assumes that both technologies, 'dirty' and 'clean' are available, but the economic loss factor is very uncertain. In the second model one assumes that the 'clean' technology is not yet available and some R&D investment must be made to get the technology breakthrough permitting its penetration. In this second model, formulated in continuous time, the jump rate of the controlled stochastic process describing the effect of R&D investment on the probability of breakthrough, is also considered as very uncertain. In both models we introduce a concept of α-robust equilibrium, where the robustification is achieved through the use of ambiguous probability distributions with a Kulbach-Leibler divergence cost structure for the worst case choice by Nature.

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1 Introduction

Integrated assessment models (IAMs), like e.g. [7, 8, 10, 14, 15, 19, 25, 26], combine a macroeconomic description of a production, investment, consumption, emission system for a group of countries with a schematized representation of the carbon cycle and resulting surface atmospheric temperature (SAT). An optimization approach, typically a deterministic optimal control one, is used to perform a cost-benefit analysis. The economic costs and benefits are evaluated through a utility function applied to consumption, whereas the cost due to climate change are evaluated through a damage function, linking the temperature change with a loss of economic output. These models are used to provide a "social value of carbon" and to give an indication of the optimal timing of abatement policies or of adaptation measures, as in [3]. Several controversies have developed (see [18, 27, 23]) about the validity of the scenarios and carbon values derived from these models, motivated by the sensitivity of the models to the assumed value of key parameters, which remain very uncertain. This is particularly the case for the pure time preference rate, used to discount future utilities, and for the climate sensitivity parameter. In a recent paper [21] Pindyck has concluded that these models could not be used to derive a correct social value for carbon. He has also advocated a cost effectiveness approach rather than a cost benefit one, to cope with the possibility of abrupt or catastrophic climate change. (Notice that Manne and Richels had already adopted a cost effectiveness approach with random long term limits on cumulative emissions to deal with an uncertain climate sensitivity [13].) In the same vein, Lempert and Collins have proposed a "robust approach" in the use of climate models [11]. Babonneau et al. [2] have implemented robust optimization technique to deal with the uncertainty in a meta-modeling approach to climate negotiations. In this game model cost effectiveness analysis was considered instead of the cost-benefit analysis usually applied when a Nash equilibrium was to be computed in an IAM as e.g. in [20].

In this paper we propose the concept of α-robust equilibrium to deal directly with the uncertainty plaguing multi-country noncooperative IAMs. We implement the concept with a Kullback-Leibler divergence to measure the uncertainty deviation, in order to robustify equilibrium solutions in a multi-country cost-benefit model. For our demonstration we use two versions of an IAM already presented in [4] and [5], respectively. In these models one distinguishes between a 'carbon' and a 'low-carbon' economy. The first model is deterministic and we robustify the damage function, whereas the second model is stochastic and we robustify the controlled jump rate which describes the access to the clean technology as a breakthrough influenced by R&D investment.

The paper is organized as follows: in Section 2 we define the concept of α-robust equilibrium for a game described in its normal form, in Section 3 we adapt an approach recently proposed in [12] to robustify the damage function that drives the environmental game model proposed in [4]; in Section 4 we show how to use the stochastic control method developed by Todorov [24] to robustify the controlled jump rate in the stochastic game model proposed in [5]; in Section 5 we conclude with an interpretation of the robust equilibrium solutions thus obtained. In this paper we focus on the technical game theoretic aspects of the approach. The significance of robust game theory models for climate change policy will be addressed in another paper.

2 Robust equilibrium concept

Consider an m-player game in normal form [16] defined by the following data:

$$
\begin{array}{rcl}
\text{Players} & & j = 1, \ldots, m \\
\text{Payoffs} & & \Psi_j(\mathbf{u}; \Theta), \quad j = 1, \ldots, m \\
\text{Controls} & & \mathbf{u} = (u_1, \ldots, u_m) \\
u_j & \in & U_j, \quad j = 1, \ldots, m.
\end{array}
$$

Θ is an uncertain parameter taking value in a space Ξ endowed with a pseudo-metric, or in the case where Θ is a measure of probability, a divergence,[1] denoted $\|\cdot\|$. This parameter has an a priori nominal value Θ^0.

[1] A divergence is a way to measure the distance between statistical distributions. Note that divergences need not satisfy the triangle inequality nor be symmetric.

Definition 1 An α-robust equilibrium $\mathbf{u}^\star$ satisfies:

$$\Phi_j(\mathbf{u}^\star) = \max_{u_j \in U_j} \Phi_j([u_j, \mathbf{u}^{\star -j}]), \quad j = 1, \ldots, m \tag{1}$$

where

$$\Phi_j(\mathbf{u}) = \Psi_j(\mathbf{u}; \Theta_j^\star) \tag{2}$$

and with

$$\Theta_j^\star = \operatorname*{argmin}_{\Theta \in \Xi} \{\Psi_j(\mathbf{u}; \Theta) + \alpha \|\Theta - \Theta^0\|\}, \quad j = 1, \ldots, m \tag{3}$$

We will consider in particular the case where Θ is a random variable and the pseudo-metric is given by the Kullback-Leibler divergence.

Remark 1 The intuition on the use of α-robust equilibrium can be given as follows: The parameter Θ is uncertain, so each player will consider a 'worst case', as if 'Nature' is a malevolent player trying to minimize the player's payoff. However to moderate the detrimental choice made by 'Nature' one assumes that there is a 'cost' for 'Nature' as the $\Theta_j^\star$ differs from the a priori nominal value Θ^0. Changing the weight α given to this cost in the choice of 'Nature' one controls the risk aversion for the players.

Remark 2 This definition of a robust equilibrium differs slightly from the one proposed in [1], which is directly derived from robust optimization methods developed in [6]. Our definition is related to the approach of Hansen and Sargent [9] for robust macroeconomic modeling.

3 A model with two players, two technologies and uncertain damages

3.1 The deterministic model

The following indices and variables that enter in the description of the economic model are gathered in Table 1.

Table 1: Indices and variables

Symbol	Description
j	Index of each of the m regions
t	Running time (10 year periods)
$C(j,t)$	Total consumption in region j at time t, in trillions (10^{12}) of dollars
$I_i(j,t)$	Investment in capital i in region j at time t, in trillions of dollars
$K_i(j,t)$	Physical stock of productive capital i of region j at time t, in trillions of dollars
$L_i(j,t)$	Part of the (exogenously defined) labor force $L(j,t)$ of region j allocated at time t to economy i, in millions (10^6) of persons
$E_i(j,t)$	Yearly emissions of GHG (in Gt–10^9 tons–carbon equivalent) in the economy i of region j at time t
$Y_i(j,t)$	Economic output in the economy i of region j at time t, in trillions of dollars
$M(t)$	Atmospheric concentration of GHG at time t, in GtC equivalent
ELF(t)	Economic loss factor due to climate changes at time t, in %
WRG(j)	Discounted welfare of region j
W	Total discounted welfare

The economic description of each region j involves the following equations. First, a social planner is assumed to maximize social welfare (WRG), given by the sum over T 10-year periods of a discounted utility from available per capita consumption with the discount rate dr:

$$\mathrm{WRG(j)} = \sum_{\mathrm{t=0}}^{\mathrm{T-1}} 10\,\mathrm{dr(j,t)}\,\mathrm{L(j,t)}\ln\left[\frac{\mathrm{ELF(t)\,C(j,t)}}{\mathrm{L(j,t)}}\right]. \tag{4}$$

Remark 3 Notice that we apply the economic loss factor ELF directly to consumption C and not, as it is usually done, to the output Y. This is motivated by the fact that we will soon consider this factor as a random variable, with ambiguous distribution. Applying the economic loss on consumption allows us to keep an open-loop formalism. Otherwise we would have been forced to introduce feedbacks as player strategies and develop a stochastic game formalism, with all the difficulties for implementing a numerical solution.

Total labor (L) is divided between labor allocated to the carbon economy (L_1) and labor allocated to the low-carbon economy (L_2):

$$L(j,t) = L_1(j,t) + L_2(j,t). \tag{5}$$

Capital stock (K_i) evolves through investment (I_i) and a depreciation rate δ_K as follows:

$$K_i(j,t+1) = 10\, I_i(j,t) + (1-\delta_K(j))^{10}\, K_i(j,t) \qquad i = 1,2. \tag{6}$$

Economic output (Y) occurs in the two economies according to an extended Cobb-Douglas production function in three inputs, capital (K), labor (L) and energy (which use is measured through emission level E):

$$\begin{aligned} Y(j,t) = {} & A_1(j,t)\, K_1(j,t)^{\alpha_1(j)}\, (\phi_1(j,t)\, E_1(j,t))^{\theta_1(j,t)}\, L_1(j,t)^{1-\alpha_1(j)-\theta_1(j,t)} \\ & + A_2(j,t)\, K_2(j,t)^{\alpha_2(j)}\, (\phi_2(j,t)\, E_2(j,t))^{\theta_2(j,t)}\, L_2(j,t)^{1-\alpha_2(j)-\theta_2(j,t)}, \end{aligned} \tag{7}$$

where A_i is the total factor productivity in the carbon (resp. low-carbon) economy when $i = 1$ (resp. $i = 2$), α_i is the elasticity of output with respect to capital K_i, ϕ_i is the energy conversion factor for emissions E_i and θ_i is the elasticity of output with respect to emissions E_i. Finally, economic output is used for consumption (C), investment (I) and the payment of energy costs:

$$Y(j,t) = C(j,t) + I_1(j,t) + I_2(j,t) + \pi_1(j,t)\, \phi_1(j,t)\, E_1(j,t) + \pi_2(j,t)\, \phi_2(j,t)\, E_2(j,t), \tag{8}$$

where π_i is the energy price in the carbon (resp. low-carbon) economy (when $i = 1$, resp. $i = 2$).

The simplified 'climate module' boils down here to computing the accumulation M of GHG in the atmosphere:

$$M(t+1) = 10\, \beta \sum_{j=1}^{m} (E_1(j,t) + E_2(j,t)) + (1-\delta_M)\, M(t) + \delta_M\, M_p, \tag{9}$$

where β is the marginal atmospheric retention rate (the fraction of emissions that remains in the atmosphere in the short run), δ_M is the natural atmospheric elimination rate (the rate of transfer from the atmosphere to the oceans) and M_p is the preindustrial level of atmospheric concentration.

3.2 Introducing uncertainty in the climate change damages

Damages due to climate change are summarized by the economic loss factor parameter ELF. These damages are uncertain. We shall assume that ELF is a function of a random parameter, with an ambiguous probability distribution for this random variable. More precisely:

Definition 2 ELF is defined by:

$$\mathrm{ELF}(t) = e^{-\bar{M}(t)\Gamma(t)}, \tag{10}$$

where, $\bar{M}(t) = M(t) - M(1990)$ and $\Gamma(t)$ is a random parameter that is a priori distributed according to an exponential law with probability density function (pdf) $\pi(\gamma) = \lambda e^{-\lambda\gamma}$.

Remark 4 In most IAMs, the ELF is defined as a function of temperature change. However, since this function is highly uncertain and since the temperature change is induced by the change in concentration, it makes sense to consider that ELF depends also in a highly uncertain manner on the evolution of GHG concentration, $M(t)$.

Γ can be interpreted as a random intensity of damages. For a given concentration $\bar{M}$, when Γ increases, the ELF parameter decreases, so more losses are incurred. The expected reward at time t is given by:

$$L(j,t)\int_0^{\infty} \ln\left[e^{-\bar{M}(t)\gamma}\frac{C(j,t)}{L(j,t)}\right]\lambda e^{-\lambda\gamma}\,d\gamma = L(j,t)\ln\left[e^{-\frac{\bar{M}(t)}{\lambda}}\frac{C(j,t)}{L(j,t)}\right]. \tag{11}$$

At each time t, the true distribution of the damage parameter $\Gamma(t)$ is ambiguous. Let $m(\gamma) = \frac{\hat{\pi}(\gamma)}{\pi(\gamma)}$ be the likelihood ratio of the worst distribution w.r.t. the a priori one. The K-L divergence between $\hat{\pi}(\gamma)$ and $\pi(\gamma)$, or relative entropy, is given by:

$$\Xi(\hat{\pi}(\gamma),\pi(\gamma)) = \int_{\Gamma}[m(\gamma)\ln[m(\gamma)]]\,\pi(\gamma)\,d\gamma, \tag{12}$$

where the integral is over the support of the random variable γ.

In the characterization of an α-robust equilibrium, we will assume that each player will consider the function $m(\cdot,t)$ as a control used by a malevolent player having the following criterion:

$$J(j) = \sum_{t=0}^{T-1} 10\,dr(j,t)\min_{m(\cdot,t)}\left\{L(j,t)\int_0^{\infty}\left(m(\gamma,t)\,\ln\left[e^{-\bar{M}(t)\gamma}\frac{C(j,t)}{L(j,t)}\right]\right.\right.$$
$$\left.\left. +\,\alpha\, m(\gamma,t)\ln[m(\gamma,t)]\right)\pi(\gamma)\,d\gamma\right\}, \tag{13}$$

where the likelihood ratio control of nature is subject to the constraints:

$$0 \le m(\gamma(t),t) \tag{14}$$
$$1 = \int m(\gamma(t),t)\,\pi(\gamma(t))\,d\gamma(t). \tag{15}$$

Writing the necessary optimality conditions (NOCs) for the minimizing $m(\cdot,t)$, we obtain:

$$0 = \int \ln\left[e^{-\bar{M}(t)\gamma(t)}\frac{C(j,t)}{L(j,t)}\right] + \big(\alpha(\ln[m(\gamma(t))]+1)+\kappa\big)\pi(\gamma(t))\,d\gamma(t), \tag{16}$$

where κ is a Lagrange multiplier associated with constraint (15). This yields to:

$$m^{\star}(\gamma(t)) = \frac{\exp\left(-\frac{\ln[e^{-\bar{M}(t)\gamma(t)}\frac{C(j,t)}{L(j,t)}]}{\alpha}\right)}{\int \exp\left(-\frac{\ln[e^{-\bar{M}(t)\gamma(t)}\frac{C(j,t)}{L(j,t)}]}{\alpha}\right)\pi(\gamma(t))\,d\gamma(t)} \tag{17}$$

as a solution to the NOCs for a minimizing likelihood ratio.

Since:

$$\exp\left(-\frac{\ln[e^{-\bar{M}(t)\gamma(t)}\frac{C(j,t)}{L(j,t)}]}{\alpha}\right) = \left(e^{-\bar{M}(t)\gamma(t)}\frac{C(j,t)}{L(j,t)}\right)^{\frac{-1}{\alpha}}, \tag{18}$$

the second factor cancels when expressing $m^{\star}(t)$, which is thus given by:

$$m^{\star}(\gamma,t) = \frac{e^{\frac{\bar{M}(t)}{\alpha}\gamma}}{\int_0^{\infty} e^{(\frac{\bar{M}(t)}{\alpha}-\lambda)\gamma}\lambda\,d\gamma}. \tag{19}$$

We finally get:

$$\pi^{\star}(\gamma(t)) = \left(\lambda - \frac{\bar{M}(t)}{\alpha}\right)e^{-\left(\lambda-\frac{\bar{M}(t)\,L(j,t)}{\alpha}\right)\gamma(t)}. \tag{20}$$

So the worst distribution is also an exponential law with intensity $\lambda - \frac{\bar{M}(t)}{\alpha}$, which is well defined provided the parameter α has been be chosen so that the expression $\alpha\lambda - \bar{M}(t)$ remains positive, that is:

$$\alpha \geq \frac{\bar{M}(t)}{\lambda}. \tag{21}$$

Remark 5 When $\alpha \to \infty$, we keep the a priori distribution. When α decreases, the intensity tends also to decrease and the expected value for the intensity of damages Γ tends to increase.

3.3 Robustified payoffs and α-robust equilibrium

For a given α, we have characterized the worst distribution as an exponential law with intensity $\lambda - \frac{\bar{M}(t)}{\alpha}$. Therefore, in accordance with Definition 1, the α-robust equilibrium is the equilibrium computed with the robustified payoffs:

$$\int_0^\infty \ln\left[e^{-\bar{M}(t)\gamma}\frac{C(j,t)}{L(j,t)}\right]\pi^*(\gamma)\,d\gamma = \int_0^\infty \left(\ln[e^{-\bar{M}(t)\gamma}] + \ln\left[\frac{C(j,t)}{L(j,t)}\right]\right)\left(\lambda - \frac{\bar{M}(t)}{\alpha}\right)e^{-(\lambda-\frac{\bar{M}(t)}{\alpha})\gamma}\,d\gamma. \tag{22}$$

The first term is given by:

$$\int_0^\infty -\bar{M}(t)\gamma\left(\lambda - \frac{\bar{M}(t)}{\alpha}\right)e^{-(\lambda-\frac{\bar{M}(t)}{\alpha})\gamma}\,d\gamma = -\frac{\alpha\,\bar{M}(t)}{\alpha\lambda - \bar{M}(t)}, \tag{23}$$

while the second term is $\ln[\frac{C(j,t)}{L(j,t)}]$. So, the robust reward function is defined by

$$\ln\left[e^{-\frac{\alpha\,\bar{M}(t)}{\alpha\lambda-\bar{M}(t)}}\right] + \ln\left[\frac{C(j,t)}{L(j,t)}\right] \quad = \quad \ln\left[e^{-\frac{\alpha\,\bar{M}(t)}{\alpha\lambda-\bar{M}(t)}}\frac{C(j,t)}{L(j,t)}\right] \tag{24}$$

The expected robust reward for Player j is then given by:

$$J(j,\sigma;s^0) = \sum_{t=0}^{T-1} 10\,dr(j,t)\,L(j,t)\,\ln\left[e^{-\frac{\alpha\,\bar{M}(t)}{\alpha\lambda-\bar{M}(t)}}\frac{C(j,t)}{L(j,t)}\right], \tag{25}$$

whereas the minimized cost for nature is given by:

$$J(j,\sigma;s^0) = \sum_{t=0}^{T-1} 10\,dr(j,t)\,L(j,t)\left\{\ln\left[e^{-\frac{\alpha\,\bar{M}(t)}{\alpha\lambda-\bar{M}(t)}}\frac{C(j,t)}{L(j,t)}\right] + \alpha\int_0^\infty m^\star(\gamma,t)\ln[m^\star(\gamma,t)]\pi(\gamma)\,d\gamma\right\},$$

with

$$\alpha\int_0^\infty m^\star(\gamma,t)\ln[m^\star(\gamma,t)]\pi(\gamma)\,d\gamma = \alpha\left(\ln\left[1 - \frac{\bar{M}(t)\,L(j,t)}{\alpha\,\lambda}\right] + \frac{\bar{M}(t)}{\alpha\lambda - \bar{M}(t)}\right). \tag{26}$$

3.4 Numerical illustration

For this numerical illustration we use the same parameter values as in [4] to show the effect of robustification on the equilibrium solution. The world is divided into two regions: the first one (region $j = 1$) representing the developed countries, the second region ($j = 2$) representing the developing countries.

3.4.1 Numerical results – Nash

We first compare the Nash equilibrium solution and the α-robust equilibrium, looking more precisely at the accumulation of capital in the low-carbon economy (variable K_2).

Remark 6 We notice an important effect of the robustification on the equilibrium solution. Capital accumulation in the low-carbon economy occurs much earlier for both players.

Table 2: Nash solution vs α-robust Nash solution

	Nash		α-robust Nash	
Period	Region-1	Region-2	Region-1	Region-2
2005	0	0	0	0
2015	0	0	0	0
2025	0	0	0	117
2035	0	0	0	191
2045	0	190	0	251
2055	0	290	0	308
2065	0	363	101	366
2075	0	427	195	489
2085	0	492	223	554
2095	0	554	223	613

3.4.2 Numerical results – Pareto

Next, we do a similar comparison in the context of a Pareto equilibrium solution, obtained when optimizing a weighted sum of the regional welfares with equal weight given to each player. The next table reports again on the accumulation of capital in the low-carbon economy.

Remark 7 We notice a much lower effect of the robustification on the Pareto solution. Capital accumulation in the low-carbon economy is not much affected by the robustification. This is probably due to the fact that the Pareto solution has already reduced notably atmospheric GHG concentration and thus climate change damages.

Table 3: Pareto solution vs α-robust Pareto solution

	Pareto		α-robust Pareto	
Period	Region-1	Region-2	Region-1	Region-2
2005	0	0	0	0
2015	63	0	62	0
2025	95	0	95	116
2035	120	153	119	190
2045	143	239	142	249
2055	168	303	166	305
2065	194	363	192	362
2075	223	424	220	422
2085	253	287	250	484
2095	283	550	280	548

4 Optimal R&D investment in a stochastic continuous time model

In this section we introduce robustification in the stochastic game model proposed in [5]. The key uncertainty considered in this model is the controlled jump rate, which describes the probability of accessing to the clean, more efficient technology. We use again K-L divergence in a formulation of the nature minimization problem along the line of the general approach proposed by Todorov [24] to formulate and solve controlled Markov chains. The α-robust stochastic equilibrium is formulated and a numerical illustration is provided.

4.1 A continuous time framework

The dynamics is now represented by differential state equations:

$$\begin{aligned}\dot{K}_i(j,t) &= I_i(j,t) - \delta_i K_i(j,t), \quad i = 1,2,\ j = 1,\dots,m,\\ \dot{M} &= \sum_{j=1}^{m} (E_1(j,t) + E_2(j,t)) - \delta_M\,(M(t) - M_p),\end{aligned}$$

and the payoffs are given by random integrals

$$J(j) = \mathbb{E}\left[\int_0^\infty e^{-\rho(j)t} L(j,t) \ln\left[e^{-\frac{\tilde{M}(t)}{\lambda}} \frac{C(j,t)}{L(j,t)}\right] dt|\mathbf{K}(0), M(0)\right], \quad j = 1,\dots,m,$$

where the expectations are taken with respect to the probability measure associated with the stochastic jump process describing access to the clean technology.

Here again we consider a two-player game, *i.e.* we take $m = 2$.

4.2 Technological Breakthrough Dynamics

We assume that the technological breakthrough is driven by a stochastic jump process with an uncertain jump rate.

Variable: To describe the access to the cleaner productive system a binary variable $\xi \in \{0,1\}$ is introduced which indicates if the clean technology is available ($\xi = 1$) or not yet ($\xi = 0$).

Breakthrough dynamics: The initial value $\xi(0) = 0$ indicates that there is no access to the clean capital at the initial time. The switch to the value 1 occurs at a random time which is controlled through the global accumulation of R&D capital $\tilde{K}_2$ where:

$$\tilde{K}_2(t) = \sum_{j=1}^{2} K_2(j,t). \tag{27}$$

However this jump rate is uncertain. More precisely we assume that the elementary probability of a breakthrough is given by

$$P[\xi(t+dt) = 1|\xi(t) = 0, \tilde{K}_2(t)] = \Theta(t)\,dt + o(dt),$$

where

- $\Theta(t)$ is a positive jump rate control that can be chosen by Nature with a K-L running cost

$$\alpha\left(\Theta(t)\ln\left[\frac{\Theta(t)}{q_b(t,\tilde{K}_2(t))}\right] + q_b(t,\tilde{K}_2(t)) - \Theta(t)\right). \tag{28}$$

- where $q_b(t,\tilde{K}_2(t)) = \omega_b + \upsilon_b \tilde{K}_2(t)$ is the a priori nominal jump rate function, where the parameters take the following values

  ω_b **:** initial probability rate of discovery; $\omega_b = 0.05$;

  υ_b **:** slope w.r.t $\tilde{K}_2(t)$ of the probability rate of discovery; $\upsilon_b = 0.0019$.

Remark 8 One can easily verify that the running cost defined in (28) is always ≥ 0 and attains its minimum, equal to 0, when $\Theta(t) = q_b(t,\tilde{K}_2(t))$. It is directly related to the K-L divergence for the elementary jump probabilities.

- Assume that the breakthrough jump occurs at time τ. From time τ onwards, the carbon-free technology is available.

- At time τ, for a given state $s = (\mathbf{K}, M)$, a value function $V(j, \tau, \bar{x})$ is defined as the current-valued payoff to Player j in the equilibrium solution to the robust open-loop dynamic game defined as previously, but in a continuous time setting. Here $\bar{x}$ refers to the initial values of all state variables, K_1, K_2 for all the players and M.

The control problem that we assume to be solved by nature, is written as follows:

$$\min_{\Theta(\cdot)} \mathbb{E}\left[\int_0^\tau \alpha \left(\Theta(t) \ln\left[\frac{\Theta(t)}{q_b(t, \tilde{K}_2(t))}\right] + q_b(t, \tilde{K}_2(t)) - \Theta(t)\right) dt + V(j, \tau, \bar{x}(\tau)) - V^0(j, \tau, \bar{x}(\tau))\right] \tag{29}$$

s.t. state dynamics and jump stochastics. Here $V(j, \tau, \bar{x}(\tau))$ is the current valued expected cost to go (or player-j's payoff) from time τ onward, when the new technology is available (the jump has occurred). The function $V^0(j, \tau, \bar{x}(\tau))$ is current valued expected cost to go (or the player payoff) from time τ onward, if the new technology is never available (no jump can occur).

Remark 9 This performance criterion used by Nature expresses the fact that the jump rate, used by the players in the computation of an equilibrium, will be determined in such a way that it reduces the payoff advantage given by an access to the new technology, while not diverging too much from the nominal jump rate function. This criterion for Nature is slightly different from the one used in Definition 1 when introducing the α-equilibrium concept; however the general idea is similar, Nature seeks to reduce the gains of the players.

4.3 Tenet of transition and optimal control of Nature

The control problem that we assume to be solved by Nature is defined in (29). Expliciting the probability density of the random jump time, and integrating by parts leads to the deterministic equivalent control problem

$$\begin{aligned}\min_{\Theta(\cdot)} \int_0^\infty e^{-\int_0^t \Theta(s)\, ds} \Big\{ &\Theta(t)(V(j, t, \bar{x}(t)) - V^0(j, t, \bar{x}(t))) \\ &+ \alpha \left(\Theta(t) \ln\left[\frac{\Theta(t)}{q_b(t, \tilde{K}_2(t))}\right] + q_b(t, \tilde{K}_2(t)) - \Theta(t)\right) \Big\} dt. \end{aligned} \tag{30}$$

Introduce the so-called desirability function $z(j, t, \bar{x}(t)) = e^{-\frac{V(j,t,\bar{x}(t)) - V^0(j,t,\bar{x}(t))}{\alpha}}$. Then Eq. (30) becomes

$$\min_{\Theta(\cdot)} \int_0^\infty e^{-\int_0^t \Theta(s)\, ds} \alpha \left(\Theta(t)\left(-\ln[z(j, t, \bar{x}(t))] + \ln[\frac{\Theta(t)}{q_b(t, \tilde{K}_2(t))}]\right) + q_b(t, \tilde{K}_2(t)) - \Theta(t)\right) dt \tag{31}$$

$$= \min_{\Theta(\cdot)} \int_0^\infty e^{-\int_0^t \Theta(s)\, ds} \alpha \left(\Theta(t) \ln[\frac{\Theta(t)}{q_b(t, \tilde{K}_2(t)) z(j, t, \bar{x}(t))}] + q_b(t, \tilde{K}_2(t)) - \Theta(t)\right) dt. \tag{32}$$

As noticed in Remark 8 above, the minimum of the expression

$$\Theta(t) \ln[\frac{\Theta(t)}{q_b(t, \tilde{K}_2(t)) z(j, t, \bar{x}(t))}] + q_b(t, \tilde{K}_2(t)) z(j, t, \bar{x}(t)) - \Theta(t)$$

is equal to 0 when

$$\Theta(t) = q_b(t, \tilde{K}_2(t)) z(j, t, \bar{x}(t)) = q_b(t, \tilde{K}_2(t))\, e^{-\frac{V(j,t,\bar{x}(t)) - V^0(j,t,\bar{x}(t))}{\alpha}}. \tag{33}$$

As the rest of the running cost does not depend on $\Theta(t)$ this defines the optimal control chosen by Nature.

Remark 10 Notice that the robustified jump rate is the a priori nominal jump rate multiplied by a factor which depends on the cost-to-go or value function computed for the after jump game equilibrium.

- When $\alpha \to \infty$ then $\Theta(t) \to q_b(t, \tilde{K}_2(t))$
- When $\alpha \to 0$ then $\Theta(t) \to 0$.

4.4 Computing an α-robust equilibrium

1. Solve, for an ensemble of initial states (space filling set) and times, the robustified open-loop DG, when the new technology is available;
2. Adjust an analytical form to the set of cost-to-go and obtain an approximation of $V(j,t,\bar{x}(t))$;
3. Then solve the open-loop DG defined by the modified payoffs

$$V^0(j,\bar{x}^0) = \max_{u(j,\cdot)} \int_0^\infty e^{-\left(\rho(j)t+\int_0^t q_b(t,\tilde{K}_2(s))\ \exp[-\frac{V(j,s,\bar{x}(t))-V^0(j,s,\bar{x}(t))}{\alpha}]\,ds\right)}$$
$$\Big\{L(j,t)\ln\Big[e^{-\bar{M}(t)\gamma(t)}\,\frac{C(j,t)}{L(j,t)}\Big] + q_b(t,\tilde{K}_2(t))\ \exp\Big[-\frac{V(j,t,\bar{x}(t))-V^0(j,t,\bar{x}(t))}{\alpha}\Big]V(j,t,\bar{x}(t))\Big\}dt \quad (34)$$

   s.t. the state equations.
4. Look at how the solution varies when α decreases.

Remark 11 By adapting the results of previous section to a continuous time setting, it is possible to combine both a robustification of ELF and a robustification of the technology breakthrough rate.

4.5 Numerical illustration

4.5.1 Parameters

We have run the model with the following parameter values:

Energy prices in carbon economy:	$\pi_1(1) = 0.35 \quad \pi_1(2) = 0.3$;
Energy prices in carbon-free economy:	$\pi_2(1) = 0.75 \quad \pi_1(2) = 0.8$;
Rate of social time preference:	3 %;
Total factor productivity:	Initial values $A_i(j,t) \equiv 0.0302$, initial growth rate $gA_i(j,t) \equiv 0.1$, rate of decrease of growth rate $dgA_i(j,t) \equiv 0.005$
Capital elasticity in production function:	$\alpha_i(1) \equiv 0.3$, $\alpha_i(2) \equiv 0.35$;
Annual capital depreciation rate:	10N %;
World population levels (in million):	$L(1,0) = 2205$, $L(2,0) = 4205$, initial growth rate $gL(1,0) = 0.08$, $gL(2,0) = 0.08$, rate of decrease of growth rate $dgL(1,t) = 0.3$, $dgL(2,t) = 0.3$
Energy efficiency in carbon economy:	$\phi_1(1,0) = 1.1 \quad \phi_1(2,0) = 0.9$, initial growth rate $g\phi_1(1,0) = 0.15$, $gL(2,0) = 0.15$, rate of decrease of growth rate $dgL(1,t) = 0.2$, $dgL(2,t) = 0.2$;
Energy efficiency in carbon-free economy:	$\phi_2(1,0) = 3 \quad \phi_1(2,0) = 2.5$, initial growth rate $g\phi_1(1,0) = 0.15$, $gL(2,0) = 0.15$, rate of decrease of growth rate $dgL(1,t) = 0.2$, $dgL(2,t) = 0.2$;
Energy elasticity in production function:	$\theta_i(1,0) \equiv 0.05$, $\theta_i(2,0) \equiv 0.05$, initial growth rate $g\theta_i(1,0) = -0.012$, $g\theta_i = 0.012$, rate of decrease of growth rate $dg\theta_i(1,0) = -0.008$, $dg\theta_i = 0.008$, $i = 1,2$;
Carbon concentration in 1990:	750;
K-L parameter for damage function:	$\alpha_1 = 1$;
Expected value for random damage function parameter:	$\lambda = 1250$;
Initial values for capital stocks:	$K_1(j,0) \equiv 48.65$, $K_2(j,0) \equiv 0$,

4.5.2 Effect of robustification

We show below the accumulation paths for "dirty" $K_1(j,t)$ and "clean" $K_2(j,t)$ capital stocks when one varies the K-L coefficient α. Because of the orders of magnitude of the value functions and the K-L divergence cost, the α values must be quite large.

Table 4: $\alpha = 900'000$

	2005	2015	2025	2035	2045	2055	2065	2075	2085	2095
$K_1(\cdot,,\cdot)$										
Country 1	48.65	51.80	64.65	79.24	95.48	113.41	131.98	153.79	139.67	131.94
Country 2	48.65	68.20	78.76	97.14	119.92	146.40	176.26	208.80	243.37	166.53
$K_2(\cdot,,\cdot)$										
Country 1	0.00	8.04	0.51	0.27	0.22	0.14	0.15	0.12	0.10	0.00
Country 2	0.00	0.41	8.16	9.04	9.34	9.58	9.43	9.14	8.77	7.42

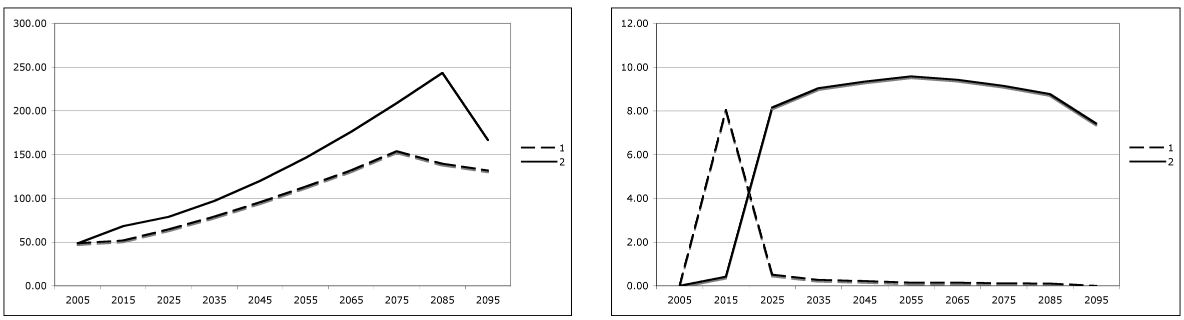


Figure 1: K_1 and K_2- $\alpha = 900'000$

Table 5: $\alpha = 9'000'000$

	2005	2015	2025	2035	2045	2055	2065	2075	2085	2095
$K_1(\cdot,,\cdot)$										
Country 1	48.65	56.19	67.96	82.45	92.36	98.37	107.70	118.18	115.60	120.84
Country 2	48.65	66.00	80.41	100.34	124.19	151.40	181.58	214.08	248.35	167.98
$K_2(\cdot,,\cdot)$										
Country 1	0.00	3.65	0.66	0.39	0.29	0.18	0.19	0.15	0.12	0.00
Country 2	0.00	2.61	5.74	6.09	6.22	6.36	6.35	6.39	6.41	6.38

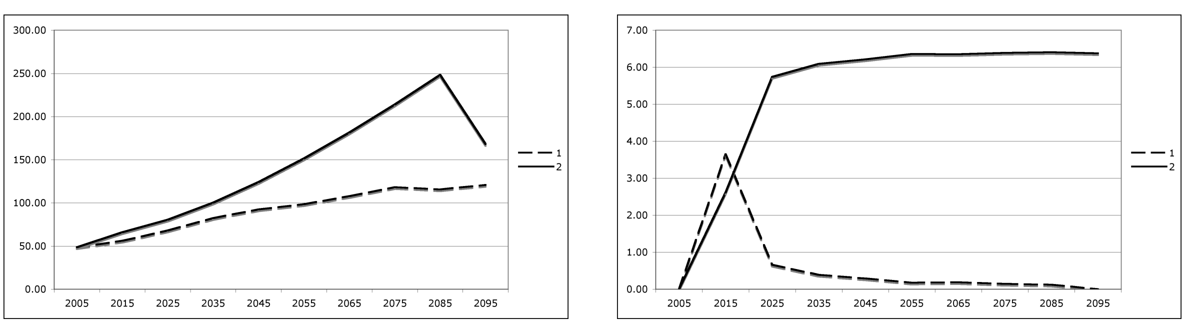
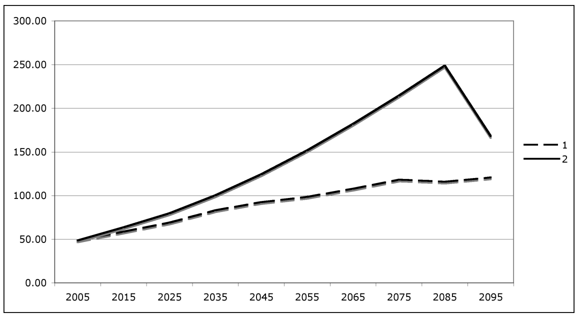


Figure 2: K_1 and K_2- $\alpha = 9'000'000$

Table 6: $\alpha > 90'000'000$

	2005	2015	2025	2035	2045	2055	2065	2075	2085	2095
$K_1(\cdot,,\cdot)$										
Country 1	48.65	58.64	69.20	83.13	92.52	98.57	107.84	118.08	115.72	120.65
Country 2	48.65	63.69	79.83	100.23	124.35	151.72	181.99	214.51	248.76	168.09
$K_2(\cdot,,\cdot)$										
Country 1	0.00	1.19	0.68	0.52	0.42	0.26	0.27	0.20	0.16	0.00
Country 2	0.00	4.92	5.54	5.74	5.85	6.03	6.04	6.12	6.19	6.29



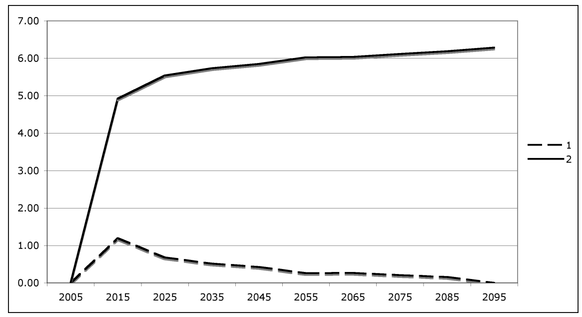


Figure 3: K_1 and K_2- $\alpha = 90'000'000$

4.5.3 Interpretation

A full exploitation of these numerical simulations will be the object of another paper. The present one is dedicated to the description of the method. In the case under study we have two regions in the world, with the same initial stock of "dirty" capital, i.e. a carbon economy. However the population size is much larger in region 2, which corresponds grossly to developing countries, whereas region 1 corresponds more or less to industrialized countries. The clean capital stock is essentially used as an R&D asset, which influences the probability of having access to a carbon-free economy. Recall that the jump rate for a technological breakthrough is $q_b(t, \tilde{K}_2(t)) = \omega_b + \upsilon_b \tilde{K}_2(t)$. In this numerical illustration we have chosen $\omega_b = 0$ and $\upsilon_b = 0.1$.

We see in Table 4 and Figure 1, that for a "low" value $\alpha = 900'000$, the pattern of investment in the R&D asset is higher than for the cases with "high" value $\alpha = 9'000'000$ or $\alpha > 90'00'000$. For these high values of α the solution tends to stabilize with a capital stock $K_2(2,t)$ staying around 6 trillion \$ for the developing countries and a much lower R&D capital stock $K_2(1,t)$ for industrialized countries, which, after an initial surge at 1.19 trillion \$ is left declining.

When $\alpha = 900'000$, i.e. for a low weight given to the K-L divergence cost, the possible perturbation due to nature choice of a detrimental jump rate is more important. Therefore in the α-robust equilibrium solution the R&D capital stock $K_2(2,t)$ stays around 9 trillion \$ for the developing countries and is still much lower for industrialized countries, since $K_2(1,t)$, after an initial surge at 8 trillion \$ is left declining.

5 Conclusion

In this paper we have shown how to introduce robustness in the computation of Nash equilibrium solution for multi-country IAMs. Using the K-L divergence as a cost to moderate the detrimental choice of a probability distribution by Nature, we have been able to obtain closed form solutions for the robust payoffs of the players in two versions of a multi-country IAM implementing cost-benefit analysis. The numerical illustration provided shows that the robustification of the game has an important influence on the solution. We advocate for the use of this robust game theory approach or the more direct one developed in [1] in future analyses performed on multi-country IAMs. The clear advantage of the α-robust equilibrium with K-L divergence is that we can obtain closed form solution for the minimization performed by Nature.

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