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G-2014-102

November 2014

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La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2014.

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The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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# The impact of product substitution and retail capacity on the timing and depth of price promotions: Theory and evidence

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**November 2014**

**Les Cahiers du GERAD  
G-2014-102**

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**Abstract:** We investigate the impact of store capacity and extent of inter-product substitution in a retailer's assortment on the optimal timing and depth of price promotions. We develop a stylized model of a monopolistic retailer selling two substitutable products over time, where demand for each product in each period is a function of the price of the product in that and earlier periods, the price of the other product in that and earlier periods, and the degree of substitution between the two periods. We present closed-form solutions to limiting cases of the model, and observe the following: When retailers optimize profits, (1) price promotions are relatively deeper in both absolute and relative terms at higher capacity stores than at low capacity stores, (2) price promotions for expensive products are relatively deeper (shallower) in both absolute and relative terms than price promotions for cheaper products if the degree of substitution is low (high) and (3) the products are sequentially promoted if the degree of substitution is low, and simultaneously promoted if the degree of substitution is high. To confirm that these insights from a simple stylized two-product model are relevant in practice, we survey price promotions within the shampoo and detergent assortments of four mass-market retailers, and observe behavior corresponding to the results from our stylized model.

**Key Words:** Promotions, pricing, retailing.

# 1 Introduction

Effective revenue management strategies for fast moving consumer goods often focus on price promotions. Indeed, price promotions account for 31% of marketing budgets on average (Gelb et al. (2007)). In settings with competing products, however, determining the optimal depth and timing of sales is complicated by potential inter-product substitution. Intuitively, issues of consumer loyalty and price sensitivity interact with operational issues like inventory or stocking levels of competing products. All of these should have a role to play in determining effective promotion plans, although the precise nature of that role is not immediately apparent. In this paper, we explore these issues, focusing on how the interplay of inter-temporal and inter-product substitution, price sensitivity, and available store capacity (shelf space, or maximum stocking level) impacts the timing and depth of price promotions.

Consider, for example, the shampoo assortment carried at two representative stores of large Canadian retailer chains, *Metro* and *Provigo* (note that these are two of the four chains represented in a data set with discuss later in this document). The stores vary in terms of the store capacity devoted to shampoo – *Metro* devotes 8.5 m<sup>2</sup> of aisle space to stocking its shampoo assortment, while *Provigo*, being relatively larger, devotes 33.8 m<sup>2</sup> of aisle space to stocking shampoos. *Metro* and *Provigo* also vary in the extent of potential inter-product substitution opportunities – *Metro* offers 44 unique shampoos, while *Provigo* offers 77, suggesting a higher degree of potential substitution at *Provigo*. In addition, both retailers offer both inexpensive and expensive shampoos within their assortments (For example, *Provigo* sells equal-sized bottles of *Sunsilk Thermashine* for \$3.95 and *Dove Go Fresh* for \$6.95.) Together, these retail characteristics raise a variety of challenging questions: Should *Metro* set relatively deeper or shallower price discounts on its shampoos, compared to the larger capacity *Provigo*? Do the higher inventory costs at the smaller capacity *Metro* require higher turnover and thus deeper promotions, or do the lower overall costs and higher margins at the larger scale *Provigo* lead to larger promotions? Should price promotions be relatively larger for expensive (e.g. *Dove*) or inexpensive (e.g. *Sunsilk*) products? How should the degree of inter-product substitution (potentially high at *Provigo*, lower at *Metro*) influence the depth of price promotions? Should *Metro* and *Provigo* promote *Dove* and *Sunsilk* shampoos simultaneously, or should they be promoted sequentially? How should the degree of inter-product substitution in their assortment (potentially high at *Provigo*, low at *Metro*), influence the timing of price promotions?

Indeed, the marketing literature provides compelling empirical evidence of the impact of inter-temporal and inter-product substitution effects on demand. Draganska and Jain (2006) provide evidence of inter-product demand interactions and shows that firms exploit these interactions to differentiate between customers with differing valuations. Pesendorfer (2002) analyzes price and sales data in a dynamic multi-product environment, and shows that demand for a particular product depends on both current and past prices of that product and its substitutes. For example, Pesendorfer (2002) shows that, holding other variables constant, a one percent increase in last week’s price for product (Heinz ketchup) will increase current-period demand for product A by 2.1 percent, and a one percent increase in last week’s price of product B (Hunts ketchup) increases current-period demand for A by one percent. These findings suggest that, depending on pricing decisions, customers may delay their purchases – *wait for the sale* – or change their preferences – *switch products*.

There is also significant empirical evidence that promotion and stocking decisions need to be coordinated. Corsten and Gruen (2003), and Taylor and Fawcett (2001), for instance, show that out-of-stock rates for promoted items are significantly higher than those for non-promoted items.<sup>1</sup> They also analyze the root causes of out-of-stock incidences, and identify lack of coordination between pricing and stocking decisions as the most significant source of the problem.

## 1.1 Research questions and summary of results

In this paper we address the following three research questions:

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<sup>1</sup>Corsten and Gruen (2003), in general, find a 2:1 ratio between promoted and non-promoted out-of-stock rates.

1. How should the space available for a product in a store (the capacity) influence the *depth* of the discount offered during a price promotion?
2. How should the potential for inter-product substitution within retail assortment influence the *depth* of discounts offered during price promotions, and should this effect be different for relatively cheap and relatively expensive products?
3. How should inter-product substitution within retail assortment influence the *timing* of price promotions for both expensive and inexpensive products?

To generate insight into these questions, we begin with a stylized discrete-time model of a monopolistic retailer selling two substitutable products over a finite number of time periods to customers who are heterogeneous in several dimensions. In each period, the firm coordinates pricing and inventory-ordering decisions for the products. Demand for each product in each period is a function of the price of that product in that period, the price of that product in previous periods, and price of the other product in that period and in previous periods.<sup>2</sup> Although even a relatively simple demand function with these characteristics leads to an intractable model, we are able to characterize optimal policies for several limiting cases, and observe the following: First, our analysis suggests that the optimal depth of price promotions for a product is increasing in the retail capacity allocated to that product. Second, we find that as the degree of product substitution increases, the retailer should offer deeper and deeper discounts for the cheaper products relative to discounts offered for more expensive products. Finally, we find that if the degree of product substitution is relatively high, the retailer should promote products simultaneously, whereas as the degree of substitution decreases, the retailer benefits more from promoting products sequentially.

Our model is highly stylized, and we analyze limiting cases of the model, so we turn to empirical analysis to confirm our observations. Specifically, we explore price promotion strategies within the shampoo and detergent categories at four mass-market retailers (Provigo, Metro, Loblaws, and IGA) located in Canada. Given the contents of our dataset, we assume that the number of unique products in a category carried by a store is a proxy for the degree of substitution in a category and the extent of aisle space allotted to a product category is a proxy for available store capacity for that category. Based on these assumptions, we make the following observations, which are consistent with our analytical results: Stores with *larger* capacity to stock products offer *deeper* discounts compared to stores with comparatively smaller retail capacity. Also, stores set deeper discounts on inexpensive products if the extent of inter-product substitution in their assortment is *low*. Finally, stores *sequentially* promote products within the same category if the extent of inter-product substitution available within their assortment is *low*, and simultaneously promote products with high substitution. To the best of our knowledge, our paper is the first to explore how store capacity and product substitution within retail assortment should correlate with the timing and depth of price promotions offered by retailers.

## 1.2 Related literature

There is a rich literature in operations management, often referred to as *revenue management*, that considers (i) time-based and (ii) product-based substitution effects on dynamic pricing problems. For instance, Ahn et al. (2007) and Ahn et al. (2009) consider a monopolistic firm selling a single product and analyze the impact of inter-temporal demand interactions on pricing and inventory-ordering decisions. Aviv and Pazgal (2008) and Elmaghraby et al. (2008) characterize optimal mark-down pricing policies with two or more price levels. Research in the second category models the impact of demand interactions among products on pricing decisions. In this stream of literature a seller either makes assortment decisions without changing prices (see, e.g., Cachon and Kok (2007) and Aydin and Hausman (2008)) or as in our model, a seller dynamically changes the prices for a fixed assortment (see, e.g., Kimes (1989), Harris and Pinder (1995), Aydin and Porteus (2008), and Dong et al. (2009)). Review papers by McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003) and Shen and Su (2007) provide comprehensive surveys of this literature. To the best of our knowledge, the models in this literature focus on either inter-temporal demand substitution in a single-product framework or inter-product demand substitution in a single/multi-period setting with no interactions between periods.

<sup>2</sup>Note that functions with these characteristics are consistent with empirical observations in Pesendorfer (2002).

Our analysis enriches this literature by modeling both inter-temporal demand and substitution effects in a capacitated retail environment, and analyzing the impact of capacity and substitution on promotion decisions. In addition to revenue management where inventory is not replenished after the starting of the selling season, researchers also consider joint dynamic pricing and inventory control under variety of assumptions, including no setup cost (Federgruen and Heching (1999)), non-zero setup cost (Chen and Simchi-Levi (2004), Huh and Janakiraman (2008)), lost sales (Chen et al. (2006)), and multiplicative demand uncertainty (Song et al. (2009)). A comprehensive survey of joint dynamic pricing and inventory control can be found in Chan et al. (2004). However, most of these models employ simple demand functions in which demand is affected only by current pricing decisions.

There is also significant prior research on price promotions in the marketing literature, including theoretical studies, empirical studies, and studies combining theory and empirical analysis. *Theoretical* articles on price promotions in the marketing literature have dealt with diverse issues such as firm and consumer asymmetry (Narasimhan 1988), dynamic competition (Lal 1990a), interaction between manufacturers and retailers (Lal and Villas-Boas 1998), and price-matching guarantees (Chen et al. 2001). *Empirical* articles on price promotions in the marketing literature have tested the implications of inter-temporal price dispersion models developed in Varian (1980) and Narasimhan (1988). In these models, the temporary price promotions are interpreted as mixed-strategy equilibria and caused by customer search costs. Extensive literature reviews are also available (see Villas-Boas 1995, Rao et al. (1995); Blattberg et al. (1995)). However, as far as we are aware, there has been limited effort to investigate how retail capacity and degree of product substitution in retail assortment influence the timing and depth of price promotions. Our paper attempts to fill this gap in the literature.

Past research related to our paper includes Lal (1990a), Freimer and Horsky (2008), and Silva-Risso et al. (1999). Lal (1990a) models the timing of price promotions by two national firms and one local firm competing in a market with switchers and loyal customers. He shows that the national firms should set price promotions sequentially, if there are a sufficiently large number of switchers in the market. Our paper enriches this line of thinking by analyzing how the degree of product substitution in a retailer’s assortment drives the timing of price promotions. Freimer and Horsky (2008) explain the effect of consumer learning on price promotions. They examine whether it is optimal for competing national brands to offer periodic price promotions when price changes allow consumers to learn by trying out a product. They find that brands periodically reduce their prices to induce purchases by non-buyers of the brands. These consumers may, after learning about the characteristics and fit of the brands, decide to continue purchasing them even if their prices are later increased. In related work, Silva-Risso et al. (1999) propose a decision-support system for an optimal sales promotion calendar for the frequency, depth and duration of deals and how these are affected by changes in market response, competitive activities and pass-through rates. Our paper enriches this literature by studying how retail assortment characteristics impact the optimal timing and depth of price promotions.

## 2 Model and analysis

To generate initial insight into the research questions outlined above, we develop a stylized model of a monopolistic retailer with heterogeneous customers.

**Retailer:** Consider a monopolistic retailer sells two substitutable products belonging to a common product category over a finite number of time periods. Let  $q$  denote the maximum store capacity devoted to the product category,  $\mathbf{M} = \{1, 2\}$  denote the set of product indices, and  $c^i$  denote the retailer’s unit cost of product  $i$ . Without loss of generality, let  $c^1 \leq c^2$ , so that product 1 is relatively cheaper than product 2. Let  $\mathbf{T} = \{1, \dots, T\}$  denote the set of time indices. At the beginning of each period  $t \in \mathbf{T}$ , the retailer sets the unit prices,  $p_t^i$ , for all products  $i \in \mathbf{M}$ , where prices are scaled to range between 0 and 1, so  $0 \leq p_t^i \leq 1$ . Let  $D_t^i(\mathbf{p})$  denote the demand for product  $i$  at period  $t$ , where  $\mathbf{p}$  represents the vector of prices across both products and all time periods. The model parameters are summarized in Table 1.

Table 1: Summary of model parameters.

Parameter	Meaning
$i \in \{1, 2\}$	Index denoting a product
$M = \{1, 2\}$	Set of product indices
$t \in \{1, 2, \dots, T\}$	Index denoting a time period
$\mathbf{T}$	Set of time indices
$p_t^i$	Price of product $i$ in period $t$
$\mathbf{p}$	price vector
$c^i$	Per unit cost of product $i$
$D_t^i(\mathbf{p})$	Demand of product $i$ in period $t$

The retailer's objective is to find an optimal pricing plan ( $\mathbf{p}^*$ ) that maximizes total profit (1a) subject to constraints on the store capacity<sup>3</sup> (1b) and bounds on feasible prices:

$$(\mathbf{P}_s) \quad \max_{\mathbf{p} \geq 0} \sum_{i=1,2} \sum_{t=1}^T (p_t^i - c^i) D_t^i(\mathbf{p}) \quad (1a)$$

$$\text{s. t.} \quad \sum_{i=1,2} D_t^i(\mathbf{p}) \leq q \quad \forall t \in \mathbf{T} \quad (1b)$$

$$0 \leq p_t^i \leq 1 \quad \forall t \in \mathbf{T} \text{ and } i \in \{1, 2\} \quad (1c)$$

To address the research questions raised in Section 1, we develop a simple linear<sup>4</sup> demand function  $D_t^i$  that has three components, one representing the demand for each product in that period due to the price of that product (the standard linear demand function, which we call *base* demand), one representing inter-temporal demand (the demand in this period because the price for a product is lower than the price of that product in the previous period), and one representing inter-product demand due to demand that was originally for other products, both in this period and in the previous period (substitution). We utilize the following function, which captures these characteristics in a reasonable but relatively tractable way, and discuss this function in more detail below:

$$D_t^i(\mathbf{p}) = \underbrace{1 - p_t^i}_{\text{Base Demand}} + (1 - \beta) \underbrace{\alpha [p_{t-1}^i - p_t^i]^+}_{\text{Product Specific Intertemporal Effect}} + \beta \left[ \underbrace{[p_t^j - p_t^i]^+}_{\text{Current Substitution}} + \underbrace{\alpha [\min(p_{t-1}^i, p_{t-1}^j) - p_t^i]^+}_{\text{Intertemporal Substitution}} \right]_{\text{Substitution Effects}}$$

where  $[x]^+ = \max(x, 0)$ ,  $i, j \in \{1, 2\}$  and  $i \neq j$ . In Appendix A, we introduce one possible customer behavior model that gives rise to this demand function.

The three components of the demand function are labeled in the equation above. *Base demand* for a product  $i$  represents the demand for product  $i$  in period  $t$  (as a fraction of the total potential period  $t$  market) that will result from pricing the product at  $p_t^i$  using a simple linear demand function  $1 - p_t^i$ . Observe that demand will range from 0 to 1, and recall that much of the operations/pricing literature focuses on this demand.

In this paper, we consider two additional terms. The increase over base demand for product  $i$  in period  $t$  if the price is lower in this period than in period  $t-1$  (which we conceptualize as a portion of the period  $t-1$

<sup>3</sup>In order to focus on promotion decisions, we do not consider inventory carry-over between periods in this paper. However, we have performed additional analysis on similar models that allow inventory carry-over between periods, and have shown that our results are not qualitatively impacted by this modeling extension.

<sup>4</sup>Linear demand functions have been commonly used in the operations management literature – see Lus and Muriel (2009) and references therein. Although these linear models are not based directly on individual consumer choice models, their aggregate linear form has both theoretical and empirical support. See Singh and Vives (1984) for an analytical derivation of linear demand functions from the maximization problem of a representative consumer with a quadratic concave utility function. Also, using the data collected from supermarket chains, Pesendorfer (2002) empirically validates the price-elasticity structure of such a linear demand function.

market that “waits for the period  $t$  sale) we call the *product-specific intertemporal effect*. We model this by multiplying  $p_{t-1}^i - p_t^i$ , the price difference between period  $t$  and the previous period, by a factor  $\alpha$ , where  $\alpha$  represents the degree of demand interaction across periods. This type of demand function is also considered in Ahn et al. (2007) to model inter-temporal substitution.

Finally, we model the increase over base demand for product  $i$  in period  $t$  due to *substitution effects* between product  $i$  and alternative product  $j$ , and this consists of two sub-effects. The first of these captures so-called *current substitution* that results in if  $p_t^i$ , the price for product  $i$  in period  $t$ , is less than  $p_t^j$ , the price of the alternative product. Specifically, if product  $i$  is less expensive than product  $j$  in period  $t$ , product  $i$  is substituted for product  $j$  by consumers with a lower reservation price. We model this for both products by using the price difference between product  $i$  and product  $j$  as a proxy for the degree of inter-product demand substitution. The second sub-component of this term captures demand for product  $i$  in period  $t$  if its price is also lower than that of product  $j$  in the previous period,  $t-1$ . Here, we use the price difference between the lowest price in period  $t-1$  and the price of product  $i$  in period  $t$  as a proxy for inter-temporal substitution effect and multiply it by the same factor  $\alpha$ .

We combine all of these terms to develop our overall demand function. Note that we weight the product specific and non-specific demand components by  $1 - \beta$  and  $\beta$ , respectively. Specifically, the inverse of  $\beta$ , i.e.,  $1 - \beta$ , measures the relative weight of product specific effects and substitution effects (and is thus represents product loyalty/substitutability).<sup>5</sup> This demand function has intuitively desirable characteristics: demand in period  $t$  is decreasing in price in period  $t$ , increasing in the price difference (that is, how much lower) the price in period  $t$  is than the price in the previous period, and increasing in the price difference (that is, how much lower) between this product and the alternative product in period  $t$  and the previous period. Note that to facilitate subsequent analysis, intertemporal effects are limited to a single period (so that demand in period  $t$  is impacted by pricing decisions in period  $t-1$ , but not earlier. Also, recall that intercept and slope of the current demand component are assumed to be 1 and  $-1$ , respectively, in our demand function. However, these assumptions are not restrictive. In fact, we have performed additional analysis<sup>6</sup> by using more complicated demand models where the intercept and slope of the current demand component are more general than 1 and  $-1$ , respectively, and product-specific and inter-temporal demand effects last for more than single period. It turns out that most of our qualitative theoretical insights are consistent with computational testing of these more complicated demand models, although not all of them can then be proved analytically.

We use this model to develop insights into optimal promotion strategies in the presence of inter-product substitution (measured by  $\beta$ ) and retail store capacity (measured by  $q$ ). However, there are two factors that complicate the complete characterization of optimal pricing strategies. First, the demand in any period depends on not only the prices in that period but also the previous period’s prices. Secondly, the capacity constraints in each period introduce additional dependencies between optimal pricing decisions in different periods. In order to simplify the analysis and to develop useful insights, we focus on the various limiting cases of  $\beta$  and  $q$ . Specifically, we first consider the uncapacitated version of our model (where  $q = \infty$ ), and analyze two limiting cases (namely,  $\beta = 0$  and  $\beta = 1$ ) in §3. This enables us to better understand the effects of interproduct substitution on the optimal promotion strategies, and particularly on the number of price offerings and depth of price promotions. Next, in §4, we consider capacitated version (where  $q$  is finite) and analyze the same two limiting cases,  $\beta = 0$  and  $\beta = 1$ , to investigate how retail capacity affects both depth and timing of price promotions.

### 3 The impact of product substitution under unbounded store capacity

Recall that in addition to base demand,  $D_t^i(\mathbf{p})$  has two additional components that model the product-specific and inter-product substitution effects, weighted by  $1 - \beta$  and  $\beta$ , respectively. In §3.1 we analyze a demand function exclusively consisting of product-specific effects, i.e.,  $\beta = 0$ . Similarly, in §3.2, we analyze a demand model exclusively consisting of inter-product substitution effects, i.e.,  $\beta = 1$ . This analysis gives insight

<sup>5</sup>This parameterization of product substitutability via  $\beta$  has been also commonly used in both economics and operations management literatures – see Röller and Tombak (1990, 1993), Goyal and Netessine (2007) and references therein.

<sup>6</sup>This additional analysis is available upon request from the authors.

into the impact of degree of substitution,  $\beta$ , on optimal pricing decisions – in §3.3 we use computational experiments to establish that the intermediate cases ( $0 < \beta < 1$ ) fall between the two limiting cases for which we have derived closed-form solutions.

### 3.1 When there is no inter-product substitution ( $\beta = 0$ )

Note that when  $\beta = 0$ , there is no interaction between the product, so model  $(\mathbf{P}_s)$  can be decomposed into 2 independent problems. In this case, we employ the analysis developed by Ahn et al. (2007) to characterize the optimal pricing plan as follows:<sup>7</sup>

**Proposition 1** *The retailer follows a high-low pricing strategy when there is no inter-product substitution ( $\beta = 0$ ) and store capacity is unbounded ( $q = \infty$ ). The optimal pricing strategy  $\forall i \in \{1, 2\}$  involves alternating between the following high- and low-prices:*

$$p_{hi}^i = \frac{1+c^i}{2} + \frac{1-c^i}{2} \left( \frac{\alpha(2+\alpha)}{4\alpha+4-\alpha^2} \right) \quad \text{and} \quad p_{lo}^i = \frac{1+c^i}{2} - \frac{1-c^i}{2} \left( \frac{\alpha(2-\alpha)}{4\alpha+4-\alpha^2} \right)$$

Proposition 1 indicates that model  $(\mathbf{P}_s)$  can be decomposed into 2 independent pricing problems, one for each product. Observe that in the context of this model, when there is no interaction between products, the full benefit of dynamic pricing strategy can be obtained by using only two price levels for each product. Chen et al. (2010) reached a similar conclusion using a stochastic model.

### 3.2 When there is inter-product substitution ( $\beta = 1$ )

Next, we explore Model  $(\mathbf{P}_s)$  setting  $q = \infty$  and  $\beta = 1$ , so that there are no capacity constraints and no product-specific intertemporal effects. In contrast to the previous section, the demand functions for products  $i$  and  $j$  are interrelated. Therefore, we cannot decompose the problem into two separate subproblems. However, it is always optimal for the firm (recall that the firm is a monopolist) to price products according to their marginal costs. In other words, we can show that since  $c^2 \geq c^1$ , we have  $p_t^2 \geq p_t^1$ . This implies that the firm uses pricing to segment customers, a result that is consistent with the price differentiation literature. For example, Draganska and Jain (2006) present empirical evidence that firms increase their profits by segmenting their customers based on multi-product pricing strategies. This observation allows us to express the price of product 2 in terms of the cheaper product 1, as follows:

**Lemma 1** *When  $\beta = 1$ , the optimal price of the expensive product  $p_t^2$  in terms of the price of the cheaper product  $p_t^1$ , given  $c^2 \geq c^1$ , is*

$$p_t^2 = \frac{1+p_t^1}{2} + \frac{c^2-c^1}{2}$$

Thus, we can transform the retailer's optimization problem into a single-product problem and ultimately characterize the optimal pricing strategy as follows:

**Proposition 2** *The retailer follows a high-low pricing strategy when  $\beta = 1$  and store capacity is unbounded ( $q = \infty$ ). The optimal pricing strategy involves alternating between the following high and low prices:*

$$p_{hi}^i = \begin{cases} c^1 + \left[ \frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2} \right] \frac{2(7/2+6\alpha)}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 1 \\ \frac{1+c^2}{2} + \left[ \frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2} \right] \frac{7/2+6\alpha}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 2. \end{cases} \quad (2)$$

$$p_{lo}^i = \begin{cases} c^1 + \left[ \frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2} \right] \frac{2(7/2+2\alpha)}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 1 \\ \frac{1+c^2}{2} + \left[ \frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2} \right] \frac{7/2+2\alpha}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 2. \end{cases} \quad (3)$$

<sup>7</sup>All the proofs are provided in the Appendix B.

Observe that the optimal price for each product outlined in Proposition 2 consists of a base component and an adjustment term that is larger in the high-priced period than in the low-priced period. The base price component for product 1 is equal to its marginal cost (i.e.  $c^1$ ), whereas for product 2, it is equal to the optimal non-interaction price<sup>8</sup> (i.e.,  $\frac{1+c^2}{2}$ ). Also, the the optimal depth of the markdown between two consecutive periods for product 1 is larger than that of product 2.

Propositions 1 and 2 imply that the effectiveness of a one period high/one period low pricing strategy is quite robust to the degree of interaction among products. In the next section, we restrict our attention to this specific pricing strategy, and explore how optimal high and low price levels change with respect to the degree of interaction between products,  $\beta$ .

### 3.3 Sensitivity analysis with respect to $\beta$

In this section, we study the effect of  $\beta$  on the depth of price promotions – the difference between high and low prices. Note that there are two alternative ways of measuring the difference: absolute difference, i.e.,  $\Delta_i^A = p_{hi}^i - p_{lo}^i$ , and relative difference, i.e.,  $\Delta_i^R = \frac{p_{hi}^i - p_{lo}^i}{p_{hi}^i}$ . In the next proposition, focusing on the two limiting cases  $\beta = 0$  and  $\beta = 1$  characterized by Propositions 1 and 2 respectively, we compare both absolute and relative differences for the less expensive product (product 1) to those for the more expensive one (product 2):

**Proposition 3** *Comparison of the depth of price promotions for the cheap product 1 and the expensive product 2:*

Loyalty ( $\beta$ )	Absolute Price Discount	Relative Price Discount
$\beta = 0$ ( <i>no substitution</i> )	$\Delta_1^A \leq \Delta_2^A$	$\Delta_1^R \leq \Delta_2^R$
$\beta = 1$ ( <i>full substitution</i> )	$\Delta_1^A \geq \Delta_2^A$	$\Delta_1^R \geq \Delta_2^R$

Proposition 3 indicates that substitution and product-specific effects have an opposite impact on pricing strategy. If there is no interaction between the two products (labeled *no substitution*), the retailer sets a shallower discount on the cheaper product (product 1) compared to the discount set on the expensive product 2. In contrast, if demands for two products are interrelated (labeled *full substitution*), the decision is the opposite – the retailer sets a deeper discount on the cheaper product.

The intuition behind Proposition 3 is as follows: Price promotions enable a retailer to inter-temporally differentiate high-valuation consumers from low-valuation ones for both products. However, as  $\beta$  increases, a greater number of both high and low-valuation consumers switch from buying product 2 (expensive) to product 1 (cheaper). This increases the incentive for the retailer to price discriminate for the cheaper product, and thus to increase the discount offered for this product.

Although we are only able to analytically characterize optimal prices for the two limiting cases of  $\beta$ , our numerical study shows that as we might expect, when  $0 < \beta < 1$ , the direction of discounts falls between these two extremes. For example, in Figures 1(A) and 1(B), we present absolute and relative differences between optimal high and low prices as  $\beta$  changes between 0 and 1. Figure 1 illustrates that when  $\beta$  is large (interaction between the two products is strong), the lower priced product has a larger price decrease than the higher-priced product. However, as the degree of interaction decreases ( $\beta$  approaches 0), the higher-priced product's price drop increases relative to that of the lower-priced product.

## 4 The impact of bounded store capacity

In §3, we discussed the impact of the degree of inter-product substitution,  $\beta$ , on price promotions under unbounded retail capacity. There, we observed that the relative timing of price promotions is unaffected by  $\beta$ , and hence, we focused only on the depth of promotions. However, with finite retail capacity, both timing

<sup>8</sup>Note that  $\frac{1+c^2}{2}$  is the optimal price for product 2 if the monopolist ignores all interactions between products and time periods, i.e., both  $\alpha$  and  $\beta$  are equal to zero.

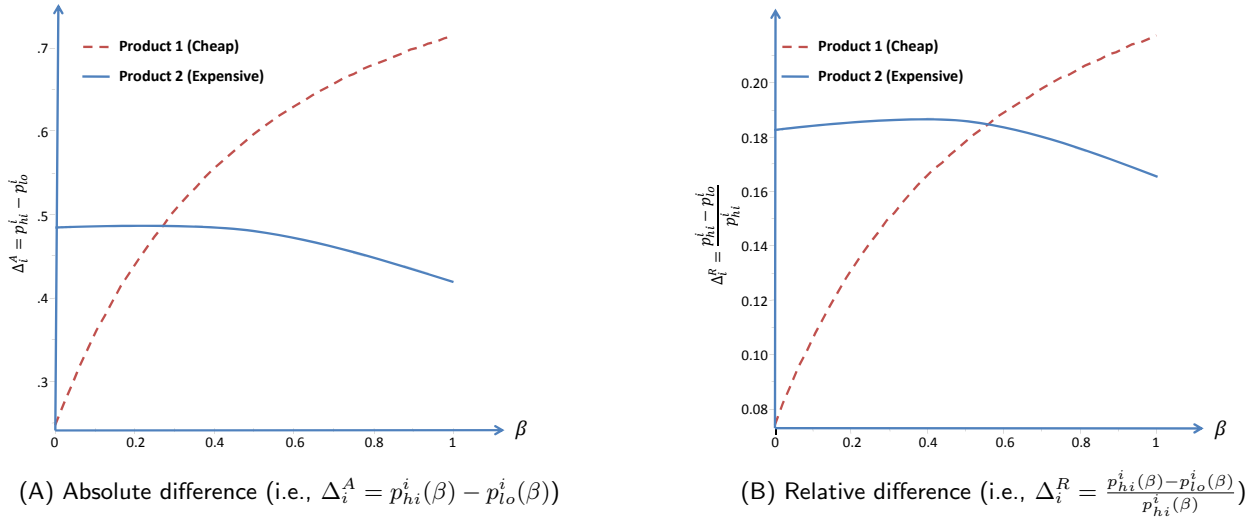


Figure 1: The change in absolute and relative differences between price promotions set for the relatively cheap product (Product 1) and expensive product (Product 2) with respect to  $\beta$ . In this example, we assume that  $c^1 = .1$ ,  $c^2 = .15$ , and  $\alpha = 1$ .

and depth of price promotion will be affected by  $\beta$ , due to the relationship between capacity utilization and pricing decisions. Specifically, capacity utilization can enter into the problem in two ways: First, when capacity is limited, it may be optimal to reduce the depth of price promotion in order to more effectively utilize limited capacity. Second, when capacity is limited, the relative timing of promotions can crucially impact capacity utilization.

For example, consider the case when  $\beta = 0$ . The alternating one period high/one period low pricing strategy can be implemented in two different ways: (i) the high and low prices for both products can be offered at the same time (so that one period both products have high prices, and the next period both products have low prices – we call this the *simultaneous* strategy), or (ii) the high and low prices can be offered at different times (so that in any period, one product has a high price and the other has the low price – we call this the *alternating* strategy). In our model, both strategies lead to exactly the same total demand realization for the firm, but when capacity constraints are binding they are very different in terms of capacity utilization. The *simultaneous* strategy requires that the firm leave some of the capacity unused when the price is high (because of low demand). In contrast, the *alternating* strategy enables the firm to shift the allocation of capacity in each period between the two products, and hence, leads to more effective capacity utilization.

Thus, regardless of demand interaction, prices for the two products are linked due to this “capacity interaction”. Furthermore, both price levels and specific timing of relative price offerings need to be explicitly determined in conjunction with capacity utilization decisions. A general analysis of pricing and timing of price promotions is analytically intractable, so in order to better understand the impact of capacity constraints on the timing of price changes for different products, we restrict our attention to simplified setting, infinite-horizon one period high/one period low pricing strategies, assuming that two products have identical marginal costs (i.e.,  $c^i = c$  for  $i = \{1, 2\}$ ). As in the previous section, we consider the  $\beta = 0$  and  $\beta = 1$  cases, separately.

We can decompose the problem into an infinite series of the following two-period pricing problem where optimal prices for periods  $t = \{1, 2\}$ , i.e.,  $\mathbf{p} = (p_1^1, p_1^2, p_2^1, p_2^2)$  can be computed by solving the following mathematical program:

$$\begin{aligned}
 & \max \sum_{i \in \{1,2\}} \sum_{t \in \{1,2\}} D_t^i(\mathbf{p})(p_t^i - c) \\
 & \text{s. t.} \quad \sum_{i \in \{1,2\}} D_t^i(\mathbf{p}) \leq q, \quad \forall t \in \{1,2\} \\
 & \quad \mathbf{p} = (p_1^1, p_1^2, p_2^1, p_2^2) \in \{\mathbf{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2), \mathbf{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)\} \\
 & \quad 0 \leq p_{lo}^i \leq p_{hi}^i \leq 1, \quad \forall i \in \{1,2\}
 \end{aligned} \tag{4}$$

The terms in the objective function account for the total profit, the first constraint models store capacity, and the second constraint specifies either a *simultaneous* (i.e.,  $\mathbf{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2)$ ) or an *alternating* strategy (i.e.,  $\mathbf{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)$ ). The final constraints specify non-negativity of the decision variables. We characterize the optimal solution to this problem as follows:

**Proposition 4** *The timing of price promotions, given an infinite-horizon, stationary, one-period-high, one-period-low pricing model with identical marginal costs (4), is as follows:*

	$\beta = 0$		$\beta = 1$	
Pricing Strategy	Alternating $\mathbf{p}^* = \mathbf{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)$		Simultaneous $\mathbf{p}^* = \mathbf{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2)$	
Price of Product 1	$p_{hi}^1$	$1 - \frac{1}{4} \min(q, \bar{q})(2 - \alpha)$	$1 - \frac{\min(q, q_1)}{2} + \frac{1}{2} [\min(q, q_2) - q_1]^+ \frac{3\alpha + 4\alpha^2}{7 + 14\alpha + 6\alpha^2}$	
	$p_{lo}^1$	$1 - \frac{1}{4} \min(q, \bar{q}) \frac{2 + 3\alpha - \alpha^2}{1 + \alpha}$	$1 - \frac{\min(q, q_1)}{2} - \frac{1}{2} [\min(q, q_2) - q_1]^+ \frac{7 + 7\alpha - 4\alpha^2}{7 + 14\alpha + 6\alpha^2}$	
Price of Product 2	$p_{hi}^2$	$1 - \frac{1}{4} \min(q, \bar{q})(2 - \alpha)$	$1 - \frac{\min(q, q_1)}{4} + \frac{1}{4} [\min(q, q_2) - q_1]^+ \frac{3\alpha + 4\alpha^2}{7 + 14\alpha + 6\alpha^2}$	
	$p_{lo}^2$	$1 - \frac{1}{4} \min(q, \bar{q}) \frac{2 + 3\alpha - \alpha^2}{1 + \alpha}$	$1 - \frac{\min(q, q_1)}{4} - \frac{1}{4} [\min(q, q_2) - q_1]^+ \frac{7 + 7\alpha - 4\alpha^2}{7 + 14\alpha + 6\alpha^2}$	

where the closed-form expressions for  $\bar{q}$ ,  $q_1$ , and  $q_2$  are provided in the Appendix.

Similarly, for the limiting cases  $\beta = 0$  and  $\beta = 1$ , we can characterize the sensitivity of the depth of price promotions to changes in capacity availability.

**Proposition 5** *The effect of increasing store capacity ( $q$ ) on the pricing and the depth of price promotions in both absolute and relative terms in an infinite-horizon, stationary, one-period-high, one-period-low pricing model (4), is as follows:*

Optimal Values	If degree of substitution is low ( $\beta = 0$ )	If degree of substitution is high ( $\beta = 1$ )
High Price ( $p_{hi}^i$ )	decreases	first decreases and then increases
Low Price ( $p_{lo}^i$ )	decreases	decreases
Absolute depth of promotion (i.e., $\Delta_i^A = p_{hi}^i - p_{lo}^i$ )	increases	increases
Relative depth of promotion (i.e., $\Delta_i^R = \frac{p_{hi}^i - p_{lo}^i}{p_{hi}^i}$ )	increases	increases

Observe that depth of price promotion increases in store capacity irrespective of the degree of inter-product substitution. Indeed, when capacity is tight, the retailer focuses on newly arriving demand, and so does not value the ability of deep promotions to segment customers.

### 4.1 Summary of analytical results

Before proceeding further, we summarize our analytical results of §3 and §4, which will subsequently be validated empirically in the next section:

- Analytical result 1: Price promotions at high capacity stores are deeper than those at relatively low capacity stores (Proposition 5).

- Analytical result 2: Price promotions offered on expensive products are *deeper* than those offered on cheaper products if the degree of substitution is *low*, and *shallower* than those offered on cheaper products if the degree of substitution is *high* (Proposition 3).
- Analytical result 3: Price promotions are scheduled *sequentially* if the degree of substitution is *low*, and scheduled *simultaneously* if the degree of substitution is *high* (Proposition 4).

## 5 Empirical survey

To validate our analytical observations in a more general and realistic setting, we conduct a survey of price promotions offered by four Canadian retailers (IGA, Loblaws, Metro, and Provigo within two popular CPG product categories (detergents and shampoos). We study the correlation between the depth and timing of price promotions offered by these retailers with their aisle capacity and the degree of inter-product substitution in their assortments. This survey supports the analytical results in the previous sections of this paper, subject to the limitations of our data set, which we detail in subsequent sections.

### 5.1 Data, measures and limitations

Our dataset is based on four Canadian retailers (IGA, Loblaws, Metro, and Provigo indexed by  $j \in \{1, 2, 3, 4\}$ ) within two popular CPG product categories (detergents and shampoos indexed by  $k \in \{1, 2\}$ ). It includes the brand, package size, and weekly price for each SKU sold by store  $j$  within category  $k$  over a period of  $T = 8$  weeks in June-August 2011. The summary statistics for the retailers and product categories under consideration are provided in the Appendix. We construct the following measures. (Tables 5-7 in Appendix C give the summary statistics and correlations between the measures.)

**Depth of price promotion:** We assume that the mode of the prices over the  $T$  weeks represents the price of a product when it is *not* under promotion. If the price in a given week is less than the mode, this indicates a price promotion or sale. Let  $p_{ijkt}$  denote the price of SKU  $i$  of the product category  $k$  at store  $j$  in week  $t$  and  $p_{ijkm} = \underset{t \in T}{\text{Mode}}(p_{ijkt})$  denote the mode of the price of SKU  $i$  over  $T = 8$  weeks. As in Mathur and Sinitsyn (2011), we measure the depth of a price promotion as the relative difference between the mode and the promoted price. The relative discount offered on SKU  $i$ , at store  $j$ , within category  $k$ , during week  $t$  is defined as  $R_{ijkt} = \frac{(p_{ijkm} - p_{ijkt})}{p_{ijkm}}$  if  $p_{ijkt} \leq p_{ijkm}$  and  $R_{ijkt} = 0$  if  $p_{ijkt} > p_{ijkm}$ .<sup>9</sup> Summary statistics for  $R_{ijkt}$  are given in the Appendix.

**Capacity:** We use the aisle area devoted by a retailer to a product category as a proxy for the store capacity assigned to a product category. Let  $A_{jk}$  denote the aisle area at store  $j$  devoted to product category  $k$ . We also control for the total aisle area available in the entire store, denoted by  $A_j$ . There are obvious limitations to using aisle space for measuring capacity. This approach implicitly assumes that the shelf space allocated to a category or brand is an exogenous and static decision, whereas in practice, it is possible that this is an endogenous, dynamic decision made by retailers. Nevertheless, given the limitations of our data set, this is our best possible measure of available capacity.

**Degree of substitution:** Let  $DS_{jk}$  denote the degree of substitution available at store  $j$ , within category  $k$ . We use the number of unique products carried by a store within a category as a measure for the degree of product substitution available to consumers.<sup>10</sup> Once again, we acknowledge the obvious limitations of this measure of product substitution. Depending on the product category, the number of unique products may be more reflective of the degree of segmentation of consumer needs within that category. However, it is

<sup>9</sup>Alternately, the depth of a price promotion can be measured with respect to the maximum price  $\underset{t \in T}{\text{Max}}(p_{ijkt})$ , as  $R'_{ijkt} =$

$\frac{\underset{t \in T}{\text{Max}}(p_{ijkt}) - p_{ijkt}}{\underset{t \in T}{\text{Max}}(p_{ijkt})}$  We repeated our analysis using this alternate measure and found the same qualitative results.

<sup>10</sup>An alternate measure for the degree of substitution available within a product category is the number of SKUs carried by store  $j$  within category  $k$ , denoted by  $M_{jk}$ . We repeated the empirical analysis using this alternate measure and find the same qualitative results. The number of unique products  $DS_{jk}$  is smaller than  $M_{jk}$  since the number of unique products was measured by counting the SKUs of different package sizes of the same brand as a single unique product.

reasonable to assume that the degree of product substitution available to consumers is likely to increase as the number of unique products stocked by a store increases, and given the limitations of our data set, this appears to be the best proxy available.

**Normalized prices:** We also need a measure of how expensive a product is relative to all products sold within its product category. We do this by comparing the normalized price of a product to the median normalized price of all products sold within its category. Let  $u_{ijkt}$  represent the normalized price of SKU  $i$  sold at store  $j$  in category  $k$  at time  $t$ , measured as the ratio of its price and package size. We define an indicator variable  $U_{ijkt}$  where  $U_{ijkt} = 1$ , if  $u_{ijkt} > \text{Median}_i[u_{ijkt}]$ ,  $U_{ijkt} = 0$  otherwise. Thus,  $U_{ijkt} = 1$  categorizes a product as relatively expensive, while  $U_{ijkt} = 0$  categorizes it as relatively cheap. Note that a binary classification of products is consistent with the theoretical model of two representative products that are analyzed in the previous section.

**Simultaneous price promotions:** We measure the extent to which price promotions within a product category at a store occur simultaneously. Define  $Y_t$  and  $Z_t$  to be indicator variables denoting whether SKUs  $y$  and  $z$  are under promotion during week  $t$ . We develop a weighting function  $F_t(Y_t, Z_t)$  that measures the extent to which SKUs  $y$  and  $z$  are simultaneously promoted over time  $t \in T$ . Given a pair of products, four distinct states are possible: If neither SKU is promoted at time  $t$ , then  $F_t(0, 0) = 0$ ; if the SKUs are simultaneously promoted at time  $t$ , then  $F_t(1, 1) = 1$ ; and if only one of the products is promoted at time  $t$ , then either  $F_t(0, 1) = -\frac{1}{2}$  or  $F_t(1, 0) = -\frac{1}{2}$ . Setting the weights in this way implies that  $E[F_t(Y_t, Z_t)] = 0$  if the four states of price promotion are equally likely. We define the function  $F_t(Y_t, Z_t) = 2Y_tZ_t - \frac{Y_t+Z_t}{2}$  to generate these weights. The sum  $S_{yz} = \sum_{t=1}^T F_t(Y_t, Z_t)$  characterizes the extent to which SKUs  $y$  and  $z$  are simultaneously promoted over the  $T$  period horizon, where  $(-\frac{T}{2} \leq S_{yz} < T)$ , and for ease of exposition, we assume that  $S_{yy} \equiv 0$ . We let  $\Delta_{yjk}$  be a measure of the average degree of simultaneous promotion between SKU  $y$  at store  $j$  in category  $k$  each of the other SKUs sold within the same category at the same store, and let  $M_{jk}$  denote the total number of SKUs at store  $j$  in category  $k$ . Then,  $\Delta_{yjk}$  is defined as follows.

$$\Delta_{yjk} = \frac{\sum_{z=1}^{M_{jk}} S_{yz}}{M_{jk} - 1} \quad (5)$$

Table 2 summarizes the notation used in the empirical study.

**Additional limitations:** Our empirical survey assumes that the primary incentive for retailers to offer price promotions is to improve their ability to segment high- and low-valuation consumers. However, there are other motives driving price promotions. For example, price promotions may be utilized to manage excess inventory, induce new-product trial, manage brand image. Our subsequent regression analysis does not control for these alternate explanations due to limitations in our dataset.

## 5.2 Model and estimation

We specify two models corresponding to the depth and timing of price promotions. First, we model the impact of retail capacity and extent of product substitution in retail assortment on the *depth* of price promotions, as follows.

$$R_{ijkt} = \beta_0 + \beta_1 A_{jk} + \beta_2 A_j + \beta_3 U_{ijkt} + \beta_4 DS_{jk} + \beta_5 U_{ijkt} DS_{jk} + \beta_6 C_k + \gamma_{i,t} + \varepsilon_{i,t} \quad (6)$$

In this model, we control for differences in price promotions across product categories. Suppose  $C_k$  is a dummy variable for the product category, where  $C_k = 1$  denotes a detergent, while  $C_k = 0$  denotes a shampoo. Secondly, we also control for differences across retailers by including the total aisle space available at store  $j$ ,  $A_j$ , as a regressor. Third, we control for the brand of each SKU in our data sample.

Next, recalling that  $\Delta_{ijk}$  is a measure of the extent to which SKU  $i$  at store  $j$  in category  $k$  is simultaneously promoted with respect to the other SKUs sold within the same category and store as SKU  $i$ , we specify

Table 2: Variables, descriptions and measurements.

Variable Name	Description / Measurement
$i, j, k$	Indices denoting a SKU, a retailer, a product category respectively
$t$	Index denoting a time-period
$T$	Total number of time-periods
$p_{ijkt}$	Price of SKU $i$ sold at retailer $j$ within category $k$ during time-period $t$
$p_{ijkm}$	Mode of Price of SKU $i$ sold at retailer $j$ within category $k$ during all time-periods $t \in T$
$R_{ijkt}$	Relative discount offered on SKU $i$ sold at retailer $j$ within category $k$ during time-period $t$
$A_{jk}$	Aisle area devoted by retailer $j$ towards stocking products of category $k$
$A_j$	Total aisle area available at retailer $j$
$DS_{jk}$	Degree of Substitution available in the retail assortment at store $j$ , within category $k$ , measured as the number of unique products carried.
$C_k$	Indicator variable representing category $k$ .
$R_j$	Indicator variable representing retailer $j$ .
$M_{jk}$	# SKU carried by store $j$ , within category $k$ .
$u_{ijkt}$	Normalized price of SKU $i$ sold at retailer $j$ within category $k$ during time-period $t$ , measured as the ratio of price and package size.
$U_{ijkt}$	Indicator variable denoting if SKU $i$ is categorized as expensive or inexpensive.
$Y_t, Z_t$	Indicator variables denoting whether SKU $y, z$ are respectively under promotion during week $t$ .
$F_t(Y_t, Z_t)$	Function measuring whether SKU $y, z$ are simultaneously promoted or not during week $t$ .
$\Delta_{yjk}$	Extent to which SKU $y$ at store $j$ in category $k$ is simultaneously promoted relative to all the other SKUs within its category

the following model to investigate how the extent of product substitution in retail assortment influences the *timing* of price promotions.

$$\Delta_{ijk} = \beta_7 + \beta_8 DS_{jk} + \beta_9 C_k + \sum_{j'=1}^3 \beta_{10,j'} R_{j't} + \gamma_{i,t} + \varepsilon_{i,t} \quad (7)$$

$DS_{jk}$  and  $C_k$  are as defined in the previous model.  $R_j$  represent dummy variables indicating that SKU  $i$  was sold at retailer  $j \in \{1, 2, 3\}$ .<sup>11</sup>

We use mixed-effect models to analyze our data. A key feature of mixed-models is that both random-effects and fixed-effects can be specified to extract multiple sources of variation. For instance, a given brand is often sold in different sizes and flavors. The price promotions offered on products belonging to the same parent brand (e.g. Pantene) but different flavors (e.g. Pantene Pro-V Classic Care, Pantene Pro-V Flat to Volume) may be more correlated than the price promotions offered across competing brands (e.g. Pantene vs. Sunsilk). Also, it is possible that trade promotions offered by manufacturers may influence the retail prices and therefore the resulting price promotions, and there may be other sources of variation. In an effort to control for such common variation within brands, we estimated the brand coefficients  $\gamma_{i,t}$  in both models shown above as a random effect, where  $\gamma_{i,t} = \lambda_i Brand_{i,t}$  and  $\lambda \sim N(0, \sigma_\lambda^2)$  is a vector of random parameters.

We estimate the mixed-effects models as follows. The models have the following general form:  $Y = X\beta + Z\lambda + \varepsilon$ , where  $\lambda \sim N(0, \Psi)$  and  $\varepsilon \sim N(0, \Lambda)$ . Here,  $X$  corresponds to the fixed-effects,  $\beta$  to the fixed-effect coefficients,  $Z$  to the random-effect (brand), and  $\lambda$  to the random-effect coefficients.  $\Psi$  is the covariance matrix of the random effects and  $\Lambda$  is the covariance matrix of the error term. We estimate the mixed-effect regression models using the restricted maximum likelihood (REML) estimation method. REML estimators are obtained by maximizing only the part of the likelihood function that is invariant to the fixed effects in the linear model, rather than the complete likelihood function. As a result, REML estimators take into account the loss of degrees of freedom in estimating the mean and therefore, produce unbiased estimators for the variance parameters (Smyth and Verbyla 1996).

We estimate three nested models – a model with intercept only, a model with fixed-effects only, and a model with random-effects only. Finally, we estimated the full model incorporating both fixed- and random-

<sup>11</sup>The retailer Provigo ( $j = 4$ ) is the reference group.

effects. The regression coefficients are summarized in Tables 3 and 4. The full model incorporating both fixed- and random-effects yields the best fit with the data, based on measures of log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC), demonstrating the value of estimating a mixed-model.

Table 3: Coefficient estimates for the model showing Result 1 and Result 2.

	Fixed Effects							
	(a) <sup>1</sup>		(b)		(c)		(d)	
	$\beta$	SE	$\beta$	SE	$\beta$	SE	$\beta$	SE
<i>Intercept</i>	.186***	.004	.169***	.032	.193***	.016	.139***	.038
<i>A<sub>jk</sub></i>			<b>.004**</b>	<b>.001</b>			<b>.004**</b>	<b>.001</b>
<i>A<sub>j</sub></i>			-.0003***	.000			-.0003***	.000
<i>U<sub>ijkt</sub></i>			.147***	.024			.195***	.026
<i>DS<sub>jk</sub></i>			.003***	.000			.004***	.000
<i>U<sub>ijkt</sub>DS<sub>jk</sub></i>			<b>-.002***</b>	<b>.000</b>			<b>-.002***</b>	<b>.000</b>
<i>C<sub>k</sub></i>			0.037	.015			.020	.032
Random Effects								
	$\gamma$	SE	$\gamma$	SE	$\gamma$	SE	$\gamma$	SE
Brand					.007**	.002	.008**	.003
-2 LL	-990		-1063.6		-1131.3		-1222.5	
AIC	-988		-1061.6		-1127.3		-1218.4	
BIC	-983.5		-1057.1		-1124.3		-1215.5	

\*  $p \leq 0.05$  \*\*  $p \leq 0.01$  \*\*\*  $p \leq 0.001$

<sup>1</sup> Description of nested models: (a) Intercept only, (b) Fixed-effects only, (c) Random-effects only, (d) Full Model: Fixed- and random-effects

Table 4: Coefficient estimates for the model showing Result 3.

	Fixed Effects							
	(a) <sup>1</sup>		(b)		(c)		(d)	
	$\beta$	SE	$\beta$	SE	$\beta$	SE	$\beta$	SE
<i>Intercept</i>	.557***	.013	.498***	.063	.560***	.033	.448***	.067
<i>DS<sub>jk</sub></i>			<b>.003***</b>	<b>.001</b>			<b>.003***</b>	<b>.001</b>
<i>C<sub>k</sub></i>			-.243***	.029			-.170***	.049
<i>R<sub>1</sub> (IGA)</i>			.041	.035			.072*	.034
<i>R<sub>2</sub> (Loblaws)</i>			-.093*	.042			-.098*	.040
<i>R<sub>3</sub> (Metro)</i>			-.112**	.038			-.105**	.037
Random Effects								
	$\gamma$	SE	$\gamma$	SE	$\gamma$	SE	$\gamma$	SE
Brand					.032***	.009	.015**	.005
-2 LL	427.9		258.1		236.0		201.9	
AIC	429.9		260.1		240.0		205.9	
BIC	434.4		264.5		243.5		209.4	

\*  $p \leq 0.05$  \*\*  $p \leq 0.01$  \*\*\*  $p \leq 0.001$

<sup>1</sup> Description of nested models: (a) Intercept only, (b) Fixed-effects only, (c) Random-effects only, (d) Full Model: Fixed- and random-effects

### 5.3 Results

We find that price promotions at high capacity stores are *deeper* than those at relatively low capacity stores, or ( $\beta_1 > 0$ ). Also, price promotions on expensive products are *deeper* than price promotions on cheaper products if the degree of substitution in the retail assortment is low, or ( $\beta_5 < 0$ ). Regarding the timing of price promotions, we find that retailers promote products simultaneously when the degree of substitution is relatively high, or ( $\beta_8 > 0$ ). Our results are summarized below.

- 
- 1) Price promotions at high capacity stores are deeper than those at relatively low capacity stores.
- 
- 2) Price promotions offered on expensive products are *deeper* than cheaper products if the degree of substitution is *low*.  
Price promotions offered on expensive products are *shallower* than cheaper products if the degree of substitution is *high*.
- 
- 3) Price promotions are scheduled *sequentially* if the degree of substitution is *low*.  
Price promotions are scheduled *simultaneously* if the degree of substitution is *high*.
- 

Thus the theoretical model analysis summarized in § 4.1 yields promotion strategies that are consistent with the three empirical patterns outlined above.

## 6 Conclusion

In this paper, we explore, both empirically and analytically, how the interplay of inter-temporal and inter-product substitution, price sensitivity, and store capacities impacts the timing and depth of sales promotions. Our analysis suggests that the additional value a firm can create by exploiting inter-temporal and substitution effects is limited by the amount of excess capacity it possesses, as well as the degree of substitution. Specifically, when there is a great deal of possible substitution between products, a firm is typically better off offering simultaneous sales (that is, low prices) for multiple products and offering larger price reductions for the less expensive products, whereas when there is little substitution, alternating sales and large price reductions for the more expensive products leads to higher profit. Also, the depth of price promotion depth should increase in available retail capacity.

In general, we see that promotion decisions generate significant benefit if they are aligned with the degree of substitution across products and time as well as with retail capacity. However, we did not explicitly consider “menu costs” associated with changing prices and disseminating promotion information, and these will ultimately play a role in the design and implementation of effective promotion strategies. Of course, even if potential gains from inter-temporal price differentiation dominate menu costs, retailers must design promotion strategies that minimize the cannibalization of demand for other products, and that account for capacity limitations.

In addition to omitting menu costs, our analysis has a variety of limitations. Our analytical model is clearly highly stylized. For example, we consider only two products, ignore competitive effects, and assume that demand functions are deterministic. Our empirical analysis similarly has limitations. Our analysis is based on data from only four firms and two product categories, and necessarily uses a potentially imprecise measures for degree of substitution and available capacity. Nevertheless, we believe that our analysis produces useful guidelines for promotion timing and level, and solid insights into why these guidelines make sense.

## Appendices

### A Consumer behavior model

In this section, we briefly discuss a stylized consumer behavior model that leads to the demand function  $D_i^j(\mathbf{p})$  employed in this paper. We assume that a total of  $2N$  new consumers arrive each period, where without loss of generality, we can normalize  $N = 1$ . The consumers are heterogeneous along four dimensions. First, consumers are horizontally differentiated, as follows. In each period, half of the new consumers prefer product 1 to product 2, while the remaining half of the new consumers prefer product 2 to product 1. Second, consumers are heterogeneous in their loyalty levels, as follows. They are divided into *loyals* and *switchers*. A *loyal* prefers a particular product (either product 1 or 2) and never considers the substitute product. In contrast, a *switcher* also prefers a particular product (either product 1 or 2), but may switch and purchase the substitute product if the preferred product is too expensive. A fraction  $\beta$  of consumers are *loyals*, and the remaining fraction  $1 - \beta$  are *switchers*. Thus, there are four segments of new consumers who arrive in

each period: (i)  $\beta$  product 1 *loyals*; (ii)  $(1 - \beta)$  product 1 switchers (that is, they prefer product 1 but may switch to product 2; (iii)  $\beta$  product 2 *loyals*; (iv)  $(1 - \beta)$  product 2 *switchers*.

Third, consumers are vertically differentiated. Each consumer has a valuation  $v$  that is uniformly distributed between 0 and 1. Fourth, consumers are heterogenous in the maximum time they stay in the market. An  $\alpha$  fraction of consumers stay in the market for up to two periods, while the remaining  $1 - \alpha$  fraction of consumers stay in the market for a single time period. If their valuation is not met during this time, they leave the market.

A newly arrived *switcher* makes a purchase decision as follows: (i) She buys her preferred product in the current period if her valuation exceeds product price; (ii) otherwise, she switches and purchases the substitute product in the current period if her valuation exceeds the price of that product; (iii) otherwise, with probability  $\alpha$  she waits for the next period and buys the less expensive product provided that her valuation exceeds the price, and with probability  $1 - \alpha$  leaves the market; (iv) otherwise, she leaves the market.

We refer to *switchers* who wait for the next period as *residual switchers*. If residual switchers remain in the market for two periods and purchase in the second period, they purchase the less expensive product. If the prices happen to be the same in the next period, they buy either product with equal probability. Let  $K_t^i(\mathbf{p})$  be the demand generated by switchers for product  $i$  during period  $t$ , expressed as follows.

$$K_t^i(\mathbf{p}) = \begin{cases} \beta(1 - p_t^i) + \beta [p_t^j - p_t^i]^+ + \beta\alpha [\min(p_{t-1}^i, p_{t-1}^j) - p_t^i]^+ & \text{if } p_t^i \leq p_t^j \\ \beta(1 - p_t^i) & \text{if } p_t^i > p_t^j \end{cases} \quad (8)$$

where  $i, j \in \{1, 2\}$ , and  $i \neq j$ .

In contrast to *switchers*, *loyals* never purchase the substitute product. A newly arrived *loyal* makes a purchase decision as follows: (i) She buys her preferred product in the current period if her valuation exceeds the product price; (ii) otherwise, she waits with probability  $\alpha$  for the next period and buys the product if her valuation exceeds the next periods price, and with probability  $1 - \alpha$  leaves the market; (iii) otherwise, she leaves the market. We refer to *loyals* who wait for the next period as *residual loyal*s. Let  $L_t^i(\mathbf{p})$  be the demand generated by *loyals* for product  $i$  during period  $t$ , expressed as follows.

$$L_t^i(\mathbf{p}) = (1 - \beta)(1 - p_t^i) + (1 - \beta)\alpha [p_{t-1}^i - p_t^i]^+ \quad (9)$$

The total demand at period  $t$  is therefore equal to  $D_t^i(\mathbf{p}) = K_t^i(\mathbf{p}) + L_t^i(\mathbf{p})$ , which can be rewritten as follows:

$$D_t^i(\mathbf{p}) = \underbrace{1 - p_t^i}_{\text{Current Demand}} + (1 - \beta) \underbrace{\alpha [p_{t-1}^i - p_t^i]^+}_{\text{Product Specific Intertemporal Effect}} + \beta \left[ \underbrace{[p_t^j - p_t^i]^+}_{\text{Current Substitution}} + \underbrace{\alpha [\min(p_{t-1}^i, p_{t-1}^j) - p_t^i]^+}_{\text{Intertemporal Substitution}} \right]_{\text{Substitution Effects}}$$

where  $[x]^+ = \max(x, 0)$ ,  $i, j \in \{1, 2\}$  and  $i \neq j$ .

## B Proofs of Propositions and Lemmas

### B.1 Proof of Proposition 1

Recall that when  $\beta = 0$ , there is no interaction between the products. Hence, we can decompose the problem into two independent pricing problems, one for each product  $i = 1, 2$  as follows:

$$\max_{\mathbf{p} \geq 0} \left[ (p_1^i - c^i)(1 - p_1^i) + \sum_{t=2}^T (p_t^i - c^i) \left( (1 - p_t^i) + \alpha [p_{t-1}^i - p_t^i]^+ \right) \right] \quad (10a)$$

The above problem is analyzed in Ahn et al. (2007). We refer the readers to Theorem 3 in Ahn et al. (2007) for the characterization of the optimal pricing decision for the above problem.  $\square$

## B.2 Proof of Lemma 1

There are two parts in this lemma. First, we show that optimal price for product 2 is greater than that for product 1 for all time periods. Suppose that it is not true, i.e., there exists a time period  $t$ , where  $p_t^1 \geq p_t^2$ . Note that replacing price of product 2 with the price of product 1, i.e.  $p_t^1 \leftarrow p_t^2$  and  $p_t^2 \leftarrow p_t^1$  will not change the total residual and substitute demands. Therefore, the exchange will always increase the total profit since the change in total demand and margin is positive. This implies that the retailer's profit function  $\pi_R(\mathbf{p})$  can be expressed as follows:

$$\pi_R(\mathbf{p}) = \left[ \sum_{t=1}^T (p_t^2 - c^2)(1 - p_t^2) + \sum_{t=1}^T (p_t^1 - c^1) \left( (1 - p_t^1) + (p_t^2 - p_t^1) \right) + \sum_{t=2}^T 2(p_t^1 - c^1) \alpha [p_{t-1}^1 - p_t^1]^+ \right]$$

Given product 1's price, we can express the optimal price of product 2 by equating the derivative of retailer's profit function with respect to  $p_t^2$  to zero and solving it for  $p_t^2$ :

$$\frac{\partial \pi_R(\mathbf{p})}{\partial p_t^2} = 0 \rightarrow p_t^2 = \frac{1 + p_t^1}{2} + \frac{c^2 - c^1}{2}$$

Next, we show that it is indeed never optimal for the retailer to charge same price for both products, i.e.,  $p_t^1 \neq p_t^2$  for all  $t$ . Suppose that it is not correct, i.e., there exists a pricing plan that has both products that have same price at time period  $t$ . We will show that there exists another pricing plan with higher profit, where the product prices are strictly different from each other. Without loss of generality, let's pick product 2. Now, we increase price of product 2 in period  $t$  by  $\epsilon$ . Note that increasing it by  $\epsilon$  would not change the total demand realization at period  $t$  since demand lost by product 1 will be recovered by the product 1. On the other hand, this strictly increases the total profit at period  $t$  since some non-zero fraction of the demand paid  $p_t^2$  previously and now pays  $p_t^2 + \epsilon$ . In order to complete the proof, we need to find a feasible ordering policy and show that this does not increase the cost. In order to do this, we just take the capacities that were originally used to satisfy demand for the product in 2 and shift them to product 1. Note that the assumption  $c_t^1 \leq c_t^2$  ensures that shifting capacity to product 1 always leads to a plan with lower cost.  $\square$

## B.3 Proofs of Proposition 2

Recall that using Lemma 1, we can transform the problem into single-product problem. However, this single-product pricing problem turns out to be quite complicated since the form of the objective function depends both on the level and the relative order of prices of product 1. Therefore, we impose an additional constraint, denoted by  $\pi_t$ , that would impose a particular order on the prices of product 1 over time.

$$p_{\pi_1}^1 \leq \dots \leq p_{\pi_t}^1 \leq \dots \leq p_{\pi_T}^1 \quad (11)$$

where  $\pi_t = \hat{t}$  if the minimum price at period  $\hat{t}$  is the  $t^{\text{th}}$  lowest price among all  $T$  product 1 prices. Imposing the constraints in (11), next, we show that retailer's profit function becomes concave in prices.

**Lemma 2** *When  $\beta = 1$ , model  $(\mathbf{P}_s)$ , together with the addition of linear constraints implied by  $\pi_t$  (i.e., constraints provided in (11)), is a concave optimization problem.*

**Proof of Lemma 2.** For notational simplicity, we define,  $p_{(t,k,i)} = \begin{cases} p_t^2, & \text{if } k = 0; \\ p_{t-1}^1, & \text{if } k > 0. \end{cases}$  and

$$I_{(t,k,i)} = \begin{cases} 1, & \text{if } k = 0 \text{ and } i = 1; \\ 1, & \text{if } k > 0; i = 1 \text{ and } p_t^1 \leq p_{t-1}^1; \\ 0, & \text{o.w.} \end{cases}$$

Also let  $(\underline{t}, k, i)$  and  $(\overline{t}, k, i)$  be the time and product indices of  $p_{(t,k,i)}$ . Finally, we let  $\alpha^0 = 1$  and combine the last two terms in  $D_t^i(\mathbf{p})$  by taking the summation on  $k$  from 0 to  $\min(1, t-1)$  as follows:

$$D_t^i(\mathbf{p}) = (1 - p_t^i) + \sum_{k=0}^{\min(1, t-1)} \alpha^k (p_{(t,k,i)} - p_t^i) I_{(t,k,i)}$$

Let  $f(\mathbf{p})$  be the retailer's revenue given a pricing plan  $\mathbf{p}$ , i.e.,

$$f(\mathbf{p}) = \sum_{i=1}^2 \sum_{t=1}^T D_t^i(\mathbf{p}) p_t^i$$

Let  $F_\pi$  be the set of prices that satisfy fixed ordering constraints  $\pi_t$ , i.e.,

$$F_\pi = \{\mathbf{p} | p_{\pi_1}^1 \leq \dots \leq p_{\pi_t}^1 \leq \dots \leq p_{\pi_T}^1\}$$

In order to show that  $f(\mathbf{p})$  is concave in  $p \in F_\pi$ , it suffices to prove that for any fixed  $\bar{\mathbf{p}} \in F_\pi$ , the tangent line at  $f(\bar{\mathbf{p}})$  always lies above  $f(\mathbf{p})$  for all  $\mathbf{p} \in F_\pi$ . In the next lemma, we prove this. Therefore,  $f(\mathbf{p})$  is concave over  $F_\pi$ , so that for any particular set of fixed ordering constraints, the corresponding problem to determine an optimal price plan becomes a concave maximization problem with the set of linear constraints.

**Lemma 3** *Under any set of fixed ordering constraints  $\pi_t$ ,  $f(\mathbf{p}) - f(\bar{\mathbf{p}}) \leq \nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}})$  for all  $\mathbf{p} \in F_\pi$ .*

**Proof of Lemma 3.** Note that  $f(\mathbf{p}) - f(\bar{\mathbf{p}})$  can be expressed as follows:

$$\begin{aligned} f(\mathbf{p}) - f(\bar{\mathbf{p}}) &= \sum_{i=1}^2 \sum_{t=1}^T ([p_t^i - \bar{p}_t^i] - [(p_t^i)^2 - (\bar{p}_t^i)^2]) + \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [p_t^1 p_{(t,k,i)} - \bar{p}_t^1 \bar{p}_{(t,k,i)}] I_{(t,k,i)} \\ &\quad - \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [(p_t^1)^2 - (\bar{p}_t^1)^2] I_{(t,k,i)}. \end{aligned} \quad (12)$$

Noting that each residual revenue term,  $\alpha^k p_t^1 p_{(t,k,i)} I_{(t,k,i)}$ , in  $f(\mathbf{p})$  will contribute to  $\frac{\partial f}{\partial p_t^1} = \dots + \alpha^k p_{(t,k,i)} I_{(t,k,i)} + \dots$  and  $\frac{\partial f}{\partial p_{(t,k,i)}} = \dots + \alpha^k p_t^1 I_{(t,k,i)} + \dots$ . From this, we know that each term,  $\alpha^k p_t^1 p_{(t,k,i)} I_{(t,k,i)}$ , contributes two terms to  $\nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}})$  through  $\alpha^k \bar{p}_{(t,k,i)} [p_t^1 - \bar{p}_t^1] I_{(t,k,i)}$  and  $\alpha^k \bar{p}_t^1 [p_{(t,k,i)} - \bar{p}_{(t,k,i)}] I_{(t,k,i)}$ .

After some algebraic manipulation, we have

$$\begin{aligned} \nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}}) &= \sum_{i=1}^2 \sum_{t=1}^T ([p_t^i - \bar{p}_t^i] - [2\bar{p}_t^i p_t^i - 2(\bar{p}_t^i)^2]) - \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [2\bar{p}_t^1 p_t^1 - 2(\bar{p}_t^1)^2] I_{(t,k,i)} \\ &\quad + \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [\bar{p}_t^1 p_{(t,k,i)} + p_t^1 \bar{p}_{(t,k,i)} - 2\bar{p}_{(t,k,i)} \bar{p}_t^1] I_{(t,k,i)} \end{aligned} \quad (13)$$

When we subtract (13) from (12), cancel the appropriate terms and complete the squares, we obtain:

$$\begin{aligned} f(\mathbf{p}) - f(\bar{\mathbf{p}}) - \nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}}) &= - \sum_{i=1}^2 \sum_{t=1}^T [p_t^i - \bar{p}_t^i]^2 - \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [p_t^1 - \bar{p}_t^1]^2 I_{(t,k,i)} \\ &\quad + \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})] I_{(t,k,i)} \end{aligned}$$

We split the first and second terms into two halves as follows:

$$\begin{aligned} f(\mathbf{p}) - f(\bar{\mathbf{p}}) - \nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}}) &= -\frac{1}{2} \sum_{i=1}^2 \sum_{t=1}^T \left[ [p_t^i - \bar{p}_t^i]^2 - [p_t^i - \bar{p}_t^i]^2 - \sum_{k=0}^{\min(t-1,1)} \alpha^k [p_t^1 - \bar{p}_t^1]^2 I_{(t,k,i)} \right] \\ &\quad - \frac{1}{2} \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [p_t^1 - \bar{p}_t^1]^2 I_{(t,k,i)} + \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})] I_{(t,k,i)} \end{aligned}$$

We complete our proof by showing that each strictly positive term in the last summation is outweighed by two corresponding negative terms from the first three lines.

First consider terms with  $k = 0$  in the last line, i.e.  $\alpha^0 [(p_t^1 - \bar{p}_t^1)(p_{(t,0,i)} - \bar{p}_{(t,0,i)})] I_{(t,0,i)}$ . Note that  $\overline{(t, k, i)} = i$  when  $k = 0$ . Therefore, selecting  $-\frac{1}{2}\alpha^0 [p_t^1 - \bar{p}_t^1]$  and  $-\frac{1}{2}\alpha^0 [(p_t^i - \bar{p}_t^i)]$  from the first two lines, completing the squares, we obtain:  $-\frac{1}{2}\alpha^0 [p_t^1 - \bar{p}_t^1 - p_t^i + \bar{p}_t^i]^2 I_{(t,0,i)}$ . Now, consider the terms with  $k = 1$  in the last line. We pair  $\alpha[(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})] I_{(t,k,i)}$  from the last line with the corresponding  $-\frac{1}{2}\alpha [p_t^1 - \bar{p}_t^1]^2 I_{(t,k,i)}$  from the second line. Since  $\overline{(t, k, i)} = t - 1$ , the following term,  $-\frac{1}{2}\alpha [p_{(t,k,i)} - \bar{p}_{(t,k,i)}]^2 I_{(t,k,i)}$  can be selected from the second line. Completing the squares, we obtain:  $-\frac{1}{2}\alpha^0 [p_t^1 - \bar{p}_t^1 - (p_{(t,k,i)} - \bar{p}_{(t,k,i)})]^2 I_{(t,k,i)}$ .

Since every positive cross product term can be matched with two corresponding negative terms, a little algebra shows:

$$f(\mathbf{p}) - f(\bar{\mathbf{p}}) - \nabla f(\bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}}) \leq -\frac{1}{2} \sum_{i=1}^2 \sum_{t=1}^T \sum_{k=0}^{\min(t-1,1)} \alpha^k [p_t^1 - \bar{p}_t^1 - p_{(t,k,i)} + \bar{p}_{(t,k,i)}]^2 I_{(t,k,i)} \leq 0$$

□

Using the above result, we can generalize the “price cycle” concept introduced in Ahn et al. (2007) to the case when there is interaction between the demand for different products,  $\beta = 1$ . More specifically, we consider the sequence made up of the minimum price over all products in each period, and define a **minimum price cycle** as a decreasing subsequence of this “minimum price sequence”, which leads to a well defined unit of analysis for characterizing the optimal dynamic pricing strategy when  $\beta = 1$ . Indeed, for a price sequence of length  $n$ , we can fully characterize optimal prices as follows:

**Lemma 4** Consider model  $(\mathbf{P}_s)$  with  $q = \infty$  and  $\beta = 1$ . Let  $\hat{p}^i = \frac{1}{2} + \frac{c^i}{2}$  for all  $i \in \{1, 2\}$ . Then, the optimal prices for a price cycle of length  $n$  are

$$p_t^i = \begin{cases} c^1 + \kappa_t [(3/2)(1 - c^1) + (1/2)(c^2 - c^1)], & \text{for } i = 1 \\ \hat{p}^2 + \frac{\kappa_t}{2} [(3/2)(1 - c^1) + (1/2)(c^2 - c^1)], & \text{for } i = 2, \end{cases} \quad (14)$$

where  $\kappa_t = \frac{2(1+\theta_t)}{7}$ ,  $\theta_t = \frac{\beta_{n-t}\theta_{t-1}-1}{\beta_{n-t+1}}$  for  $t = 1 \dots n$ ,  $\theta_0 = 1$ , and  $\{\beta_t\}_{t=1}^n$  is a sequence such that  $\beta_0 = 1$ ,  $\beta_1 = \frac{7}{4\alpha} + 2$ ,  $\beta_t = (\frac{7}{4\alpha} + 2)\beta_{t-1} - \beta_{t-2}$  for  $t = 2 \dots n-1$  and  $\beta_n = \frac{7}{4\alpha}\beta_{n-1} - \beta_{n-2}$ .

**Proof of Lemma 4.** Next, we characterize the optimal  $n$ -period non-increasing pricing sequence. Let  $\mathbf{p}$  be the pricing sequence of length  $n$ , i.e.  $p_1^i \geq p_2^i \geq \dots \geq p_n^i$  for all  $i \in \{1, 2\}$ , and  $p_t^1 \leq p_t^2$  for all  $t = 1, \dots, n$  and  $i \in \{1, 2\}$ . Then profit function can be written as follows:

$$f_n(\mathbf{p}) = \sum_{\substack{t=1 \dots n \\ i=1 \dots 2}} (p_t^i - c^i)(1 - p_t^i) + \sum_{t=1 \dots n} (p_t^1 - c^1)(p_t^2 - p_t^1) + \sum_{\substack{t=2 \dots n \\ i=1 \dots 2}} \alpha(p_t^1 - c^1)(p_{t-1}^1 - p_t^1)$$

Let  $\hat{p}^i$  denote the non-interaction optimal for product  $i$ , i.e.,  $\hat{p}^i = \frac{1+c^i}{2}$ . Then, by letting  $p_t^i = \hat{p}^i(1 + \delta_t^i)$ , substituting  $1 + c^i = 2\hat{p}^i$ , and subtracting the resulting expression from  $f_n(\hat{\mathbf{p}})$ , we obtain

$$\begin{aligned} f_n(\mathbf{p}) - f_n(\hat{\mathbf{p}}) &= \sum_{\substack{t=1 \dots n \\ i=1 \dots 2}} -(\hat{p}^i)^2 (\delta_t^i)^2 + \sum_{t=1 \dots n} [\hat{p}^1 \hat{p}^2 \delta_t^1 \delta_t^2 - (\hat{p}^1)^2 (\delta_t^1)^2] + \sum_{\substack{t=2 \dots n \\ i=1 \dots 2}} [\alpha (\hat{p}^1)^2 \delta_{t-1}^1 \delta_t^i - \alpha (\hat{p}^1)^2 (\delta_t^1)^2] \\ &+ \sum_{t=1 \dots n} [(\hat{p}^1)^2 m(r^2 - 1) \delta_t^1 + \hat{p}^1 \hat{p}^2 m \delta_t^2] + \sum_{i=1 \dots 2} [\alpha (\hat{p}^1)^2 m (\delta_1^1 - \delta_n^1)] \end{aligned}$$

Dividing by the sequence length,  $n$ , we get the average profit per period  $\Delta f_n$  when  $n$ -period pricing sequence is used. In matrix form, we can rewrite  $\Delta f_n$  as follows:

$$\Delta f_n = \frac{\sum_{i=1}^2 (\hat{p}^i)^2}{n} \left[ \frac{1}{2} \delta^T \cdot \Omega \cdot \delta - e^T \cdot \delta \right] \quad (15)$$

where  $\Omega$  is  $2n \times 2n$  matrix, and  $e$  is  $2n \times 1$  vector. Let  $\Omega_{ij}$  be  $n \times n$  dimensional sub-matrix in  $\Omega$ , where  $i, j = 1 \dots 2$ .  $\Omega_{ij}$  can be expressed as follows:  $\Omega_{11} = \begin{pmatrix} -2 & \alpha & 0 & \dots & 0 & 0 \\ \alpha & -2(1+\alpha) & \alpha & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & -2(1+\alpha) \end{pmatrix}$   $\Omega_{12} = \Omega_{21} = \frac{\hat{p}^2}{2\hat{p}^1} I_{n \times n}$ ,  $\Omega_{22} = -(\hat{p}^2/\hat{p}^1)^2 I_{n \times n}$ , where  $I_{n \times n}$  is  $n \times n$  dimensional unit matrix. Likewise,  $e_i$  can be expressed as follows:  $e_1 = \frac{m}{2}(1-r)I_{n \times 1} - m\alpha J_{n \times 1}$  and  $e_2 = -\frac{m}{2}\frac{\hat{p}^2}{\hat{p}^1} I_{n \times 1}$ , where  $I_{n \times 1} = (1, \dots, 1)^T$ ,  $J_{n \times 1} = (-1, 0, \dots, 0, 1)^T$ ,  $m = 1 - c^1/\hat{p}^1$ , and  $r = \frac{\hat{p}^2 - \hat{p}^1}{\hat{p}^1 - c^1}$ , where  $()^T$  is transpose operator. We can take the derivative of quadratic form and set it to zero to obtain the optimal  $\delta$  and  $\gamma$  in matrix notation:

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \cdot \begin{pmatrix} \delta^1 \\ \delta^2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad (16)$$

Solving  $\delta^2$  in terms of  $\delta^1$  in (16), we obtain:  $\delta^2 = -\Omega_{22}^{-1} \cdot \Omega_{21} \cdot \delta^1 + \Omega_{22}^{-1} \cdot e_2 \rightarrow \delta_t^2 = \frac{\delta_t^1 \hat{p}^1}{2\hat{p}^i} + \frac{m\hat{p}^1}{2\hat{p}^i}$ . Substituting it back to (16), we obtain the following equation for  $\delta^1$ :  $(\Omega_{11} + \Omega_{12} \frac{\hat{p}^1}{2\hat{p}^i}) \cdot \delta^1 = (e_1 - \Omega_{12} \frac{m\hat{p}^1}{2\hat{p}^i})$ . Dividing both sides by  $\alpha$ , we obtain the following matrix equation:

$$\begin{pmatrix} -u & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & -u-2 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -u-2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & -u-2 \end{pmatrix} \cdot \delta = \begin{pmatrix} 1-\frac{1}{z} \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1+\frac{1}{z} \end{pmatrix} zm \quad (17)$$

where  $\delta = (\delta_1, \dots, \delta_n)^T$ ,  $u = 2/\alpha - 1/(4\alpha) = \frac{7}{4\alpha}$  and  $z = 1/(4\alpha) - r/(2\alpha) = \frac{1}{\alpha} [\frac{1}{4} - \frac{r}{2}]$ . We can rewrite the right hand side as follows:

$$\begin{pmatrix} 1-\frac{1}{z} \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1+\frac{1}{z} \end{pmatrix} zm = \overbrace{\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}}^{p^2} (1 - \frac{z}{u})m + \overbrace{\begin{pmatrix} 1-\frac{1}{u} \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1+\frac{1}{u} \end{pmatrix}}^{p^2} zm \quad (18)$$

In Lemma 5 below, we find  $\gamma$  and  $\zeta$  such that  $\Omega\gamma m = p^1$  and  $\Omega\zeta m = p^2$ . Hence,  $\delta^1 = (\gamma + \zeta)m$  satisfies (17).

Thus,  $\delta_t^1 = (\gamma_t - \frac{z}{u})m$  is the optimal perturbation. Note that we can rewrite  $\gamma_t = \gamma_0\theta_t = \theta_t - \frac{z}{u}\theta_t$  where  $\theta_t = \frac{\beta_{n-t}\theta_{t-1}-1}{\beta_{n-t+1}}$  for  $t = 1 \dots n$ ,  $\theta_0 = 1$ . This implies that optimal price of product 1 (cheap product) is

$$p_t^1 = \hat{p}^1(1 + \delta_t^1) = c^1 + 2\kappa_t \left[ (\hat{p}^1 - c^1)\frac{3}{2} + \frac{1}{2}(\hat{p}^j - \hat{p}^1) \right] = c^1 + \kappa_t \left[ (1 - c^1)\frac{3}{2} + \frac{1}{2}(c^2 - c^1) \right]$$

where  $\kappa_t = \frac{2(1+\theta_t)}{7}$ . Recall that when  $i = 2$ ,  $\delta_t^i = \frac{\delta_t^1 \hat{p}^1}{2\hat{p}^i} + \frac{m\hat{p}^1}{2\hat{p}^i}$ , and this implies that optimal price for product 2 (expensive product) is

$$p_t^2 = \hat{p}^2(1 + \delta_t^2) = \hat{p}^2 + \kappa_t \left[ (\hat{p}^1 - c^1)\frac{3}{2} + \frac{1}{2}(\hat{p}^2 - \hat{p}^1) \right] = \frac{1+c^2}{2} + \frac{\kappa_t}{2} \left[ (1 - c^1)\frac{3}{2} + \frac{1}{2}(c^2 - c^1) \right] \quad \square$$

### Lemma 5

(i) Let  $\gamma = [\gamma_1 \ \gamma_2 \ \dots \ \gamma_{n-1} \ \gamma_n]^T$  where  $\gamma_t = \frac{\beta_{n-i}\gamma_{t-1}-\gamma_0}{\beta_{n-t+1}}$  for  $t = 1 \dots n$ ,  $\gamma_0 = (1 - \frac{z}{u})$ ,  $\frac{z}{u} = \frac{\frac{1}{2}-r}{7/2}$  and  $\{\beta_t\}_{t=1}^n$  is a sequence such that  $\beta_0 = 1$ ,  $\beta_1 = u + 2$ , and

$$\beta_t = \begin{cases} (u+2)\beta_{t-1} - \beta_{t-2} & \text{for } t = 2 \dots n-1 \\ u\beta_{t-1} - \beta_{t-2} & \text{for } t = n. \end{cases} \quad (19)$$

Then,  $\gamma$  satisfies  $\Omega\gamma m = p^1$

(ii)  $\zeta = [-\frac{z}{u} \ -\frac{z}{u} \ \dots \ -\frac{z}{u} \ -\frac{z}{u}]^T$  satisfies  $\Omega\zeta m = p^2$ .

**Proof of Lemma 5.** This result follows by substituting the given  $\gamma$  and  $\zeta$  into the equations.  $\square$

Note that since problem parameters are stationary, the total profit generated within a price cycle depends only on the cycle's length  $n$ . Let  $\Pi_n$  be the optimal total profit generated by a price cycle of length  $n$ . Then, it is straightforward to show that  $\Pi_n$  increases in  $n$ . In order to characterize the optimal length of price cycle, we define  $\bar{\Pi}_n = \Pi_n/n$ , the average optimal profit per period generated by a price cycle of length  $n$ . In the next lemma, we show that  $n^* = 2$  maximizes  $\bar{\Pi}_n$  for all  $\alpha$  and cost parameter values; in other words, a cyclic pricing policy with a cycle length of 2 periods maximizes the average profit:

**Lemma 6** Consider model  $(P_s)$  with  $q = \infty$  and  $\beta = 1$ . The average cycle profit,  $\bar{\Pi}_n$ , is maximized by a price cycle of length  $n = 2$  for all  $\alpha$  and cost parameters.

**Proof of Lemma 6.** If we substitute the optimal  $\delta = \Omega^{-1}e$  into quadratic form (15) to obtain the optimal average profit per period,  $\Delta f_n^*$ :

$$\begin{aligned} \Delta f_n^* &= \frac{\sum_{i=1}^2 (\hat{p}^1)^2}{n} \left[ \frac{1}{2} e^T \Omega^{-1} \Omega \Omega^{-1} e - e^T \Omega^{-1} e \right] = \frac{\sum_{i=1}^2 (\hat{p}^1)^2}{n} \left[ \frac{1}{2} e^T \Omega^{-1} e - e^T \Omega^{-1} e \right] \\ &= -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{n} \left[ \frac{1}{2} e^T \Omega^{-1} e \right] = -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2n} \sum_{i=1}^2 (e_i^T \delta^i) \end{aligned} \quad (20)$$

Substituting  $\delta^2 = -\Omega_{22}^{-1} \Omega_{21} \delta^1 + \Omega_{22}^{-1} e_2$ :

$$\begin{aligned} \Delta f_n^* &= -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2n} [e_1^T \delta^1 - e_2^T \Omega_{22}^{-1} \Omega_{21} \delta^1 + e_2^T \Omega_{22}^{-1} e_2] \\ &= -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2n} [(e_1^T - e_2^T \Omega_{22}^{-1} \Omega_{21}) \delta^1 + e_2^T \Omega_{22}^{-1} e_2] \end{aligned} \quad (21)$$

where

$$(e_1^T - e_2^T \Omega_{22}^{-1} \Omega_{21}) = m \begin{pmatrix} (1/4)(1/2-r) \\ \vdots \\ (1/4)(1/2-r) \end{pmatrix} + m \begin{pmatrix} -\alpha \\ \vdots \\ \alpha \end{pmatrix}, e_2^T \Omega_{22}^{-1} e_2 = \frac{-m^2 n}{2} (1/4)$$

Substituting these terms back into (21),

$$\Delta f_n^* = -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2n} \left[ m(1/4)(1/2-r) \sum_{t=1}^n \delta_t^1 - m\alpha(\delta_1^1 - \delta_n^1) - \frac{m^2 n}{2} (1/4) \right]$$

Note that  $\sum_{t=1}^n \delta_t^1 = \frac{1}{u}(\delta_1^1 - \delta_n^1) - nmz/u$  due to the the row sum of the matrix equation. Therefore,  $\Delta f_n^*$  can be expressed as follows:

$$\Delta f_n^* = -\frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2n} \left[ m \left( \frac{1}{u} \frac{1}{2} (1/2-r) - \alpha \right) (\delta_1^1 - \delta_n^1) - \frac{m^2 n z}{u} \frac{1}{2} (1/2-r) - \frac{m^2 n}{4} \right]$$

Recall that  $\delta_1^1 - \delta_n^1 = m(\gamma_1 - \gamma_n)$ . Substituting into  $\Delta f_n^*$  and collecting terms that depend on  $n$ , we obtain:

$$\begin{aligned} \Delta f_n^* &= \frac{\sum_{i=1}^2 (\hat{p}^1)^2}{2} \left[ m^2 \left( \alpha - \frac{1}{u} \frac{1}{2} (1/2-r) \right) (\gamma_1 - \gamma_n)/n + \frac{m^2}{4} + \frac{m^2 z}{2u} (1/2-r) \right] \\ &= A_1 (\gamma_1 - \gamma_n)/n + A_2 \end{aligned} \quad (22)$$

where  $A_1 = \frac{\alpha(\hat{p}^1 - c^1)^2}{7} \left( \frac{3}{2} + \frac{1}{2} \frac{\hat{p}^2 - \hat{p}^1}{\hat{p}^1 - c^1} \right)$  and  $A_2 = \frac{(\hat{p}^1 - c^1)^2}{7} \left( \frac{7}{16} + \left( \frac{1}{2} \left[ \frac{1}{2} - \frac{\hat{p}^2 - \hat{p}^1}{\hat{p}^1 - c^1} \right] \right)^2 \right)$ . Notice that the increase in the average profit under an optimal non-increasing price sequence of length  $n$  depends only on the difference between the price distortion of product 1 (from its noninteraction price,  $\hat{p}^1$ ) in the first period (i.e.,  $\gamma_1^1$ ) and the price distortion in the last period (i.e.,  $\gamma_n^1$ ). Both quantities depend on the length of the price sequence,  $n$ . In order to show that a price sequence of length 2 is optimal, we develop the upper bound on the average profit increase when a  $n$ -period optimal decreasing pricing sequence is used. Using Lemma 9, we characterize

asymptotic properties for both  $\gamma_1$  and  $\gamma_n$ , which will be useful when we prove the optimality of period 2. For any optimal decreasing pricing sequence of length  $n$ , the average profit improvement is bounded by:

$$\Delta f_n^* = A_1(\gamma_1 - \gamma_n)/n + A_2 \leq \Delta F_n^* = \frac{A_1}{n} \left[ \gamma_0 \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] + A_2 \quad (23)$$

It is trivial to show that  $\Delta F_n^*$  decreases in  $n$ , and therefore, following relational structure holds:

$$\begin{array}{rcl} \Delta f_2^* & \leq \Delta F_2^* & A_1(\gamma_1 - \gamma_2)/2 + A_2 \leq \frac{A_1}{2} \left[ \gamma_0 \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] + A_2 \\ & \downarrow & \\ \Delta f_3^* & \leq \Delta F_3^* & A_1(\gamma_1 - \gamma_3)/3 + A_2 \leq \frac{A_1}{3} \left[ \gamma_0 \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] + A_2 \\ \vdots & \vdots & \vdots \\ \Delta f_n^* & \leq \Delta F_n^* & A_1(\gamma_1 - \gamma_n)/n + A_2 \leq \frac{A_1}{n} \left[ \gamma_0 \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] + A_2 \\ \vdots & \vdots & \vdots \end{array} \quad (24)$$

If we can show that  $A_1(\gamma_1 - \gamma_2)/2 + A_2 \geq \frac{A_1}{n} \left[ \gamma_0 \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] + A_2$  for some  $\bar{n}$  then the relational diagram above implies that  $\Delta f_2^* \geq \Delta f_n^*$  where  $n \geq \bar{n}$ . After some algebra, we can show that for all  $\alpha \in [0, 1]$ :

$$\begin{aligned} \frac{\gamma_1 - \gamma_2}{2} &= \gamma_0 \frac{7\alpha}{(7/2)(7/2 + 4\alpha) - 4\alpha^2} \geq \gamma_0 \left[ \frac{1}{4} \left( \frac{1}{r_1 - r_\infty} + \frac{1}{r_0 - 1} \right) \right] = \gamma_0 \left[ \frac{1}{4} \left( \frac{2}{u - 2 + \sqrt{u^2 + 4u}} + \frac{1}{1 + u} \right) \right] \\ &= \gamma_0 \left[ \frac{1}{4} \left( \frac{3u + \sqrt{u^2 + 4u}}{(u - 2 + \sqrt{u^2 + 4u})(1 + u)} \right) \right] = \gamma_0 \left[ \frac{2\alpha}{4} \left( \frac{3(7/2) + \sqrt{(7/2)^2 + 8\alpha(7/2)}}{(7/2 - 4\alpha + \sqrt{(7/2)^2 + 8\alpha(7/2)})(7/2 + 2\alpha)} \right) \right] \end{aligned}$$

Hence  $\Delta f_2^* \geq \Delta F_4^* \geq \Delta f_n^*$  where  $n \geq 4$ . There remains to prove that  $\Delta f_2^* \geq \Delta f_3^*$ . In fact, we can easily show that for all  $\alpha \in [0, 1]$ :

$$\frac{\gamma_1 - \gamma_2}{2} = \gamma_0 \frac{\theta_1 - \theta_2}{2} = \gamma_0 \frac{2\alpha(7/2)}{(7/2)(7/2 + 4\alpha) - 4\alpha^2} \geq \gamma_0 \frac{\theta_1 - \theta_3}{3} = \gamma_0 \frac{(\frac{4}{3})\alpha(7/2)(7/2 + 6\alpha)}{(7/2)^2(7/2 + 8\alpha) + 8\alpha^2(7/2 - 2\alpha)}$$

which implies that  $\Delta f_2^* \geq \Delta f_3^*$ . This completes the proof.  $\square$

Lemma 7 and 8 are used in the proof of Lemma 9. Due to space constraints, we provide them without proof:<sup>12</sup>

**Lemma 7** Let  $r_0 = 2 + u$  For all  $n \geq 1$ ,  $\beta_{i-1}(n)\beta_i(n) \geq r_0^i$ ,  $i = 1, \dots, n$

**Lemma 8**  $\lim_{n \rightarrow \infty} \frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{1}{r_1 - r_\infty}$  where  $r_\infty = \frac{u+2-\sqrt{u^2+4u}}{2}$  and  $r_1 = u$ .

**Lemma 9** (i)  $\gamma_1(n)$  is increasing in  $n$  and  $\lim_{n \rightarrow \infty} \gamma_1(n) = \frac{\gamma_0}{r_1 - r_\infty}$  and (ii)  $\liminf_{n \rightarrow \infty} \gamma_n(n) \geq \frac{-\gamma_0}{r_0 - 1}$  for all  $n \geq 1$  where  $r_\infty = \frac{u+2-\sqrt{u^2+4u}}{2}$ ,  $r_1 = u$ , and  $r_0 = 2 + u$ .

**Proof of Lemma 9.** From Lemma 8

$$\gamma_1(n) = \frac{\beta_{n-1}(n)\gamma_0 - \gamma_0}{\beta_n(n)}.$$

It can be shown that  $\gamma_1(n)$  increases as  $n$  increases using the fact that both  $\frac{\beta_{n-1}(n)}{\beta_n(n)}$  and  $\beta_n(n)$  are monotone increasing. Furthermore, the convergence of a sequence,  $\frac{\beta_{n-1}(n)}{\beta_n(n)}$  and the fact that  $\lim_{n \rightarrow \infty} \beta_n(n) = \infty$  imply the existence of the limit. Replacing  $\gamma_0$  with  $a$  as in Lemma 8, we have:  $\lim_{n \rightarrow \infty} \gamma_1(n) = \lim_{n \rightarrow \infty} \frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{\gamma_0}{r_1 - r_\infty}$ . To show the inequality, we consider the following system of difference equations:  $\gamma_1(n) = \frac{\beta_{n-1}(n)}{\beta_n(n)}\gamma_0 - \frac{1}{\beta_n(n)}\gamma_0$ ,  $\gamma_2(n) = \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)} \frac{\beta_{n-1}(n)}{\beta_n(n)}\gamma_0 - \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)} \frac{1}{\beta_n(n)}\gamma_0 - \frac{1}{\beta_{n-1}(n)}\gamma_0$ ,  $\dots$ ,  $\gamma_n(n) = \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0 \left( \frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \right.$

<sup>12</sup>Proofs of Lemma 7 and 8 are available from the authors upon request.

$\frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \dots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)}$ ). It is easy to see that  $\gamma_n(n)$  is decreasing in  $n$  since the positive term in the above expression decreases while the negative term grows in magnitude as  $n$  increases. To bound  $\gamma_n(n)$ , we utilize the  $\beta_i(n)\beta_{i-1}(n)$  terms. Lemma 7 implies that  $\beta_i(n)\beta_{i-1}(n)$  is bounded below by  $r^i$  and substituting the corresponding terms yields:

$$\begin{aligned} \gamma_n(n) &= \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0 \left( \frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \dots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)} \right) \\ &\geq \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0 \left( \frac{\beta_0(n)}{r_0^n} + \frac{\beta_0(n)}{r_0^{n-1}} + \dots + \frac{\beta_0(n)}{r_0^2} + \frac{\beta_0(n)}{r_0} \right) \geq -\gamma_0\beta_0(n) \left( \frac{1}{r_0^n} + \frac{1}{r_0^{n-1}} + \dots + \frac{1}{r_0^2} + \frac{1}{r_0} \right) \\ &\geq -\gamma_0\beta_0(n) \left( \frac{1}{r_0 - 1} \right) = \frac{-\gamma_0}{r_0 - 1} \end{aligned}$$

The second last inequality holds because  $\frac{\beta_0(n)}{\beta_n(n)}\gamma_0$  is nonnegative and the last equality results from  $\beta_0(n) = 1$ .  $\square$

In a finite horizon problem instance, repeating this one period high- one period low pricing strategy is optimal when  $T$  is even. On the other hand, such a strategy would leave the last period uncovered when  $T$  is odd; therefore, we fine-tune the optimal pricing strategy for the final three periods.

**Lemma 10** Consider model  $(P_s)$  with  $q = \infty$  and  $\beta = 1$ . Then, for all  $\alpha$  and cost parameter values,

1. If the planning horizon length,  $T$ , is an even number, then optimal product prices consist of 2-period price cycles and optimal total profit is equal to  $T/2 \times \bar{\Pi}_2$ .
2. On the other hand, if  $T$  is odd, then optimal product prices consist of 2-period price cycles for the first  $T - 3$  periods, and a 3-period price cycle for the last three periods, and optimal total profit is equal to  $(T - 1)/2 \times \bar{\Pi}_2 + \bar{\Pi}_3$ .

**Proof of Lemma 10.** We proved in Lemma 6 that the optimal profit increase results from repeating the 2-period optimal pricing sequence, which implies that if the planning period  $T$  is an even number then the pricing plan is involves repeating the 2-period optimal pricing sequence  $\frac{T}{2}$  times, where

$$p_t^i = \begin{cases} c^1 + \kappa_t [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 1 \\ \hat{p}^i + \frac{\kappa_t}{2} [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 2. \end{cases}$$

for  $t = hi, lo$  and  $\kappa_{hi} = \frac{2(7/2+6\alpha)}{7(7/2+4\alpha)-8\alpha^2}$ ,  $\kappa_{lo} = \frac{2(7/2+2\alpha)}{7(7/2+4\alpha)-8\alpha^2}$ .

However, when  $T$  is odd, a small modification of the pricing strategy is required for the final three periods. Note that there are two ways to construct a 3-period pricing plan. The first is to use  $(p_{hi}^i, p_{lo}^i)$  for the first two periods, solve a single period profit maximization problem that takes into account only the substitution effect, and use the optimal single period prices  $p_{single}^i$  for the last period. The second way is to use a 3-period decreasing pricing sequence  $(p_1^i, p_2^i, p_3^i)$  for the final three periods, where

$$p_t^i = \begin{cases} c^1 + \kappa_t [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 1 \\ \hat{p}^i + \frac{\kappa_t}{2} [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 2. \end{cases}$$

for all  $t = 1, 2, 3$  and  $\kappa_1 = \frac{2(3/2+6\alpha)(3/2+4\alpha)}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$ ,  $\kappa_2 = \frac{2((3/2+8\alpha)(3/2)+8\alpha^2)}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$ , and  $\kappa_3 = \frac{2((3/2)(3/2+6\alpha))}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$ .

The optimal pricing plan for the last three periods is determined by whichever of these two strategies generate the most profit. First, we calculate the profit of  $(p_{hi}^i, p_{lo}^i, p_{single}^i)$ . Recall that difference between base profit and profit generated by  $(p_{hi}^i, p_{lo}^i)$  is equal to  $2\Delta f_2 = A_1(\gamma_1 - \gamma_2) + 2A_2$ . To calculate the profit in the last period, we need to compute  $p_{single}^i$ . We provide the closed form expression for  $p_{single}^i$  below:

$$p_{single}^i = \begin{cases} c^1 + \frac{4}{7} [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 1 \\ \hat{p}^i + \frac{2}{7} [(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)] & \text{for } i = 2. \end{cases}$$

The difference between base profit and profit generated by  $p_{\text{single}}^i$ ,  $\Delta f_1$  is:

$$\Delta f_1 = \frac{(\hat{p}^1 - c^1)^2 2}{7} \left( \frac{7}{16} + \left( (1/2) \left[ \frac{1}{2} - \frac{\hat{p}^2 - \hat{p}^1}{\hat{p}^1 - c^1} \right] \right)^2 \right).$$

Note that  $\Delta f_1 = A_2$ . Hence total increase in the profit generated by  $(p_{\text{hi}}^i, p_{\text{lo}}^i, p_{\text{single}}^i)$  is equal to  $A_1(\gamma_1 - \gamma_2) + 3A_2$ . The second way of constructing the 3-period plan is  $(p_1^i, p_2^i, p_3^i)$ , which leads to a total increase in profit by  $A_1(\gamma_1 - \gamma_3) + 3A_2$ . Therefore, it is enough to show that  $A_1(\gamma_1(2) - \gamma_2(2)) + 3A_2 \leq A_1(\gamma_1(3) - \gamma_3(3)) + 3A_2$ . But this is true, since  $\gamma_1(n)$  increases in  $n$  and  $\gamma_n(n)$  decreases in  $n$  by Lemma 9.  $\square$

#### B.4 Proof of Proposition 3

In Propositions 1 and 2, we obtain closed form expressions for the optimal one-period high - one period low pricing strategy for model  $(\mathbf{P}_s)$  when  $\beta = 0$  and  $\beta = 1$ , respectively. Using these expressions, we can express the absolute difference between high and low prices for products  $i^*$  and  $i$  as follows:

$$\Delta_i^A(\beta = 0) = \begin{cases} (\hat{p}^1 - c^1) \frac{4\alpha}{4\alpha+4-\alpha^2} & \text{if } i = 1 \\ (\hat{p}^2 - c^2) \frac{4\alpha}{4\alpha+4-\alpha^2} & \text{if } i = 2 \end{cases}$$

and

$$\Delta_i^A(\beta = 1) = \begin{cases} \frac{8\alpha}{(7/2)(7/2+4\alpha)-4\alpha^2} [(\hat{p}^1 - c^1)(3/2) + (\hat{p}^2 - \hat{p}^1)(1/2)] & \text{if } i = 1 \\ \frac{4\alpha}{(7/2)(7/2+4\alpha)-4\alpha^2} [(\hat{p}^1 - c^1)(3/2) + (\hat{p}^2 - \hat{p}^1)(1/2)] & \text{if } i = 2 \end{cases}$$

where  $\hat{p}^i = \frac{1}{2} + \frac{c^i}{2}$ . Since  $8\alpha \geq 4\alpha$ , it is straightforward to show that

$$\Delta_1^A(\beta = 1) \geq \Delta_2^A(\beta = 1).$$

We can rewrite  $(\hat{p}^2 - c^2)$  as follows:

$$\hat{p}^i - c^i = \frac{1 + c^i}{2} - c^i = \frac{1 - c^i}{2}$$

Recalling that we assume  $c^1 \leq c^2$ , we can show that:

$$\hat{p}^1 - c^1 \leq \hat{p}^2 - c^2.$$

This implies that:

$$\Delta_1^A(\beta = 0) \leq \Delta_2^A(\beta = 0)$$

Now, we compare the relative difference between high and low prices for products 1 and 2. Relative difference for product 2 can be written as follows:

$$\Delta_i^R(\beta = 0) = \frac{(\hat{p}^i - c^i) \frac{4\alpha}{4\alpha+4-\alpha^2}}{\hat{p}^i + (\hat{p}^i - c^i) \frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{\hat{p}^i - c^i}{\hat{p}^i} \frac{4\alpha}{4\alpha+4-\alpha^2}}{1 + \frac{\hat{p}^i - c^i}{\hat{p}^i} \frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{1-c^i}{1+c^i} \frac{4\alpha}{4\alpha+4-\alpha^2}}{1 + \frac{1-c^i}{1+c^i} \frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{4\alpha}{\alpha(\alpha+2)} \frac{1-c^i}{1+c^i}}{\frac{4\alpha+4-\alpha^2}{\alpha(\alpha+2)} + \frac{1-c^i}{1+c^i}}$$

Also, using the fact that  $\frac{4\alpha+4-\alpha^2}{\alpha(\alpha+2)} > 1$  for all  $0 \leq \alpha \leq 1$ , we can show that  $\Delta_1^R(\beta = 0) \leq \Delta_2^R(\beta = 0)$ . Lastly, we show that  $\Delta_1^R(\beta = 1) \geq \Delta_2^R(\beta = 1)$ . Note that  $\Delta_1^R(\beta = 1)$ , and  $\Delta_2^R(\beta = 1)$  can be written as follows:

$$\Delta_1^R(\beta = 1) = \frac{\Delta_1^A(\beta = 1)}{p_{\text{hi}}^1} \quad \text{and} \quad \Delta_2^R(\beta = 1) = \frac{\Delta_2^A(\beta = 1)}{p_{\text{hi}}^2}$$

Using the facts that  $p_{\text{hi}}^1 \leq p_{\text{hi}}^2$  and  $\Delta_1^A(\beta = 1) \geq \Delta_2^A(\beta = 1)$ , we can show that  $\Delta_1^R(\beta = 1) \geq \Delta_2^R(\beta = 1)$ .  $\square$

## B.5 Proof of Proposition 4

We consider each case separately:

1.  $\beta = 0$  case: First, we show that an *alternating* one-period high and one period low pricing strategy generates more profit than a *simultaneous* pricing strategy. Note that products are symmetric (i.e., demand and cost parameters are the same) and cost and demand parameters are stationary (i.e., they do not depend on time). This implies that if simultaneous pricing strategy were adopted, high and low prices would be offered at the same time. This would lead to low demand when high prices are offered, and high demand when low prices are offered. If the capacity at each period, i.e.,  $q$ , is less than the level of the high demand, some capacity will be unused. This reduces the capacity utilization in high period. However, we can avoid this completely by offering high and low prices in an alternating fashion. Then, the demand in each period will be constant, thus completely eliminate completely the unused portion of capacity.

Using the stationary and symmetric setting of the problem, we can decompose problem (4) into independent sub-problems for each product:

$$\max_{1 \geq p_{hi}^i \geq p_{lo}^i \geq 0} \{f(p_{hi}^i, p_{lo}^i) \text{ s. t. } d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q\}$$

where  $d_{hi}^i(p_{hi}^i, p_{lo}^i) = (1 - p_{hi}^i)$ ,  $d_{lo}^i(p_{hi}^i, p_{lo}^i) = (1 - p_{lo}^i) + \alpha(p_{hi}^i - p_{lo}^i)$ , and  $f(p_{hi}^i, p_{lo}^i) = (p_{hi}^i - c)d_{hi}^i(p_{hi}^i, p_{lo}^i) + (p_{lo}^i - c)d_{lo}^i(p_{hi}^i, p_{lo}^i)$ . Let  $\lambda \geq 0$  be a lagrangian multiplier for constraint  $d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q$ . Let  $\mathbf{p}^i = (p_{hi}^i, p_{lo}^i)$ . Since the above model is a concave optimization problem with linear constraints, the optimal solution ( $\mathbf{p}^*$ ) must satisfy the following KKT conditions:  $\{\nabla f(\mathbf{p}^i) = \lambda(\nabla d_{hi}^i(\mathbf{p}^i) + \nabla d_{lo}^i(\mathbf{p}^i)), d_{hi}^i(\mathbf{p}^i) + d_{lo}^i(\mathbf{p}^i) \leq q, \lambda(d_{hi}^i(\mathbf{p}^i) + d_{lo}^i(\mathbf{p}^i) - q) = 0, p_{hi}^i \geq p_{lo}^i\}$ . Depending on whether  $\lambda = 0$  or  $\lambda > 0$ , we have 2 cases:

- **Case 1:** ( $\lambda = 0$ ) Note that in this case, the first KKT condition transforms into  $\nabla f(\mathbf{p}^i) = 0$ . The following prices satisfy this condition:

$$p_t^i = 1 - \frac{1}{4} \frac{4(1-c)(1+\alpha)}{4+4\alpha-\alpha^2} \begin{cases} 2-\alpha & \text{for } t = hi \\ \frac{2+3\alpha-\alpha^2}{1+\alpha} & \text{for } t = lo. \end{cases} \quad (25)$$

In order to meet capacity constraints in each period, i.e.,  $d_{hi}^i(\mathbf{p}^i) + d_{lo}^i(\mathbf{p}^i) \leq q$ ,  $q$  must satisfy  $q \geq \bar{q} = \frac{4(1-c)(1+\alpha)}{4+4\alpha-\alpha^2}$ . Therefore, as long as  $q \geq \bar{q}$ , the prices in equations (25) satisfy all KKT conditions, and hence are optimal solution.

- **Case 2:** ( $\lambda > 0$ ) In this case, the KKT conditions consist of  $\{\nabla f(\mathbf{p}) = \lambda(\nabla d_{hi}^i(\mathbf{p}^i) + \nabla d_{lo}^i(\mathbf{p}^i)), d_{hi}^i(\mathbf{p}^i) + d_{lo}^i(\mathbf{p}^i) = q, p_{hi}^i \geq p_{lo}^i\}$ . The prices that satisfy these equations are as follows:

$$p_t^i = 1 - \frac{1}{4} q \begin{cases} 2-\alpha & \text{for } t = hi \\ \frac{2+3\alpha-\alpha^2}{1+\alpha} & \text{for } t = lo. \end{cases} \quad (26)$$

Note that  $p_{hi}^i \geq p_{lo}^i$ . It is easy to check that as long as  $q < \bar{q}$ ,  $\lambda > 0$ . Therefore, as long as  $q < \bar{q}$ , the prices in equations (26) satisfy all KKT conditions, and hence are optimal solution.

2.  $\beta = 1$  case: Before we start the analysis, we show that when  $\beta = 1$ , the *simultaneous* one period high and one period low pricing policy leads to higher profit than the *alternating* policy. Since we assume that  $c^1 = c^2 = c$ , without loss of generality, we can pick any product and charge the lowest price for it throughout the planning horizon. Let  $i = 1$  be the index for lowest-priced product. Hence, using Lemma 1, we can express product 2's optimal price as follows:

$$p_t^2 = \frac{1 + p_t^1}{2} + \frac{c^2 - c^1}{2} = \frac{1 + p_t^1}{2}$$

where  $t \in \{hi, lo\}$ . Using this equation, we can eliminate price variables for product 2 (i.e.,  $p_t^2$ ) in Model (4), and reduce the problem to one with 2 decision variables, i.e., the high and low prices of product 1:

$$\max_{1 \geq p_{hi}^1 \geq p_{lo}^1 \geq 0} \{f(p_{hi}^1, p_{lo}^1) \text{ s. t. } d_{hi}^1(p_{hi}^1, p_{lo}^1) + d_{lo}^1(p_{hi}^1, p_{lo}^1) \leq 2q, d_{lo}^1(p_{hi}^1, p_{lo}^1) \leq q, p_{hi}^1 \geq p_{lo}^1\}$$

where  $d_{hi}(p_{hi}^1, p_{lo}^1)$  and  $d_{lo}(p_{hi}^1, p_{lo}^1)$  are total demands in the high and low price periods, respectively, i.e.,

$$d_{hi}(p_{hi}^1, p_{lo}^1) = 2 - 2p_{hi}^1 \text{ and } d_{lo}(p_{hi}^1, p_{lo}^1) = 2 + 2(\alpha p_{hi}^1 - (1 + \alpha)p_{lo}^1)$$

and  $f(p_{hi}^1, p_{lo}^1)$  is profit before inventory costs, i.e.,

$$f(p_{hi}^1, p_{lo}^1) = \sum_{t \in \{hi, lo\}} (2(p_t^1 - c)(1 - p_t^1)) + \sum_{t \in \{hi, lo\}} \frac{1}{4} ((1 - p_t^1))^2 + 2\alpha(p_{lo}^1 - c)(p_{hi}^1 - p_{lo}^1).$$

Note that  $p_t^2 \geq p_t^1$  is automatically satisfied as long as  $0 \leq p_t^1 \leq 1$  since  $p_t^2 = \frac{1}{2} + \frac{1}{2}p_t^1$ . Therefore, we can drop  $p_t^2 \geq p_t^1$  from the formulation. Also, in our analysis, we first assume that  $0 \leq p_t^1 \leq 1$  and afterwards show that this is satisfied by the optimal prices.

We continue by solving the following problem:

$$\max_{p_{hi}^1 \geq p_{lo}^1} \{f(p_{hi}^1, p_{lo}^1) \text{ s. t. } d_{hi}(p_{hi}^1, p_{lo}^1) + d_{lo}(p_{hi}^1, p_{lo}^1) \leq 2q, d_{lo}(p_{hi}^1, p_{lo}^1) \leq q\} \quad (27)$$

Let  $\lambda_1 \geq 0$  and  $\lambda_2 \geq 0$  be the lagrangian multipliers of constraints 1 and 2, respectively. Let  $\mathbf{p} = (p_{hi}^1, p_{lo}^1)$ . Since Model (27) is a concave optimization problem with linear constraints, the optimal solution ( $\mathbf{p}^*$ ) has to satisfy the following KKT conditions:  $\{\nabla f(\mathbf{p}) = \lambda_1(\nabla d_{hi}(\mathbf{p}) + \nabla d_{lo}(\mathbf{p})) + \lambda_2 \nabla d_{lo}(\mathbf{p}), d_{lo}(\mathbf{p}) \leq q, \lambda_2(d_{lo}(\mathbf{p}) - q) = 0, d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) \leq 2q, \lambda_1(d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) - 2q) = 0, p_{hi}^1 \geq p_{lo}^1\}$ . Depending on whether  $\lambda_1 = 0$  or  $\lambda_1 > 0$  and  $\lambda_2 = 0$  or  $\lambda_2 > 0$ , we have  $2 \times 2 = 4$  cases:

– **Case 1:** ( $\lambda_1 = 0, \lambda_2 = 0$ ) Note that in this case, the first KKT condition transform into  $\nabla f(\mathbf{p}) = 0$ .

The following prices satisfy this condition:

$$p_t^1 = \begin{cases} 1 - (\hat{p} - c) \frac{16+10\alpha-8\alpha^2}{49/4+14\alpha-4\alpha^2} & t = hi \\ 1 - (\hat{p} - c) \frac{16+22\alpha-8\alpha^2}{49/4+14\alpha-4\alpha^2} & t = lo. \end{cases} \quad (28)$$

In order to meet the capacity constraint in the second period, i.e.,  $d_{lo}(\mathbf{p}) \leq q$ ,  $q$  must satisfy  $q \geq q_2 = 8(1 - c) \frac{7+11\alpha-2\alpha^2}{49+56\alpha-16\alpha^2}$ . Note that the prices above satisfy  $p_{hi}^1 \geq p_{lo}^1$ . This implies that  $d_{hi}(\mathbf{p}) \leq d_{lo}(\mathbf{p})$ , and hence  $d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) \leq 2q$  automatically holds. Therefore, the prices in equations (28) satisfy all KKT conditions, and thus are the optimal solution to Model (27) when  $q \geq q_2$ .

– **Case 2:** ( $\lambda_1 > 0, \lambda_2 = 0$ ) In this case, the KKT conditions consist of  $\{\nabla f(\mathbf{p}) = \lambda_1(\nabla d_{hi}(\mathbf{p}) + \nabla d_{lo}(\mathbf{p})), d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) = 2q, d_{lo}(\mathbf{p}) \leq q, p_{hi}^1 \geq p_{lo}^1\}$ . The prices that satisfy these equations are as follows:

$$p_t^1 = \begin{cases} 1 - q \frac{7/2+11/2\alpha-2\alpha^2}{7+8\alpha-\alpha^2} & t = hi \\ 1 - q \frac{7/2+5/2\alpha-2\alpha^2}{7+8\alpha-\alpha^2} & t = lo. \end{cases} \quad (29)$$

Note that  $p_{hi}^1 \geq p_{lo}^1$ , implying that  $d_{hi}(\mathbf{p}) \leq d_{lo}(\mathbf{p})$ . Therefore,  $d_{lo}(\mathbf{p}) \geq q$ , which contradicts the KKT conditions. Hence, Case 2 does not lead to a feasible solution.

– **Case 3:** ( $\lambda_1 = 0, \lambda_2 > 0$ ) In this case, the KKT conditions consist of  $\{\nabla f(\mathbf{p}) = \lambda_2 \nabla d_{lo}(\mathbf{p}), d_{lo}(\mathbf{p}) = q, d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) \leq 2q, p_{hi}^1 \geq p_{lo}^1\}$ . The prices that satisfy these equations are as follows:

$$p_t^1 = \begin{cases} 1 - (\hat{p} - c) \frac{8+16\alpha+8\alpha^2}{7+14\alpha+6\alpha^2} + q \frac{3/2\alpha+2\alpha^2}{7+14\alpha+6\alpha^2} & t = hi \\ 1 - (\hat{p} - c) \frac{8\alpha+4\alpha^2}{7+14\alpha+6\alpha^2} - q \frac{7/2+7/2\alpha-2\alpha^2}{7+14\alpha+6\alpha^2} & t = lo. \end{cases} \quad (30)$$

Two KKT inequalities (i.e.,  $0 \leq \lambda_2, d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) \leq 2q$ ) give rise to two conditions for  $q$ :  $q \geq q_1 = 8(1 - c) \frac{1+\alpha}{7+10\alpha}$ , and  $q \leq q_2 = \frac{8(1-c)(7+11\alpha-2\alpha^2)}{49+56\alpha-16\alpha^2}$ . It is easy to see that all the KKT conditions are satisfied when these conditions are met. Hence, the prices in equations (30) are the optimal solution to Model (27) when  $q \geq q_1$  and  $q \leq q_2$ .

- **Case 4:** ( $\lambda_1 > 0, \lambda_2 > 0$ ) In this case, the KKT conditions consist of  $\{\nabla f(\mathbf{p}) = \lambda_1(\nabla d_{hi}(\mathbf{p}) + \nabla d_{lo}(\mathbf{p})) + \lambda_2 \nabla d_{lo}(\mathbf{p}), \lambda_2 \leq 0, d_{lo}(\mathbf{p}) = q, d_{hi}(\mathbf{p}) + d_{lo}(\mathbf{p}) = 2q, p_{hi}^1 \geq p_{lo}^1\}$ . The prices that satisfy these equations are as follows:

$$p_t^1 = \begin{cases} 1 - q/2 & t = hi \\ 1 - q/2 & t = lo. \end{cases} \quad (31)$$

Note that  $p_{hi}^1 = p_{lo}^1$ . Two KKT conditions (i.e.,  $\lambda_1 \geq 0, 0 \leq \lambda_2$ ) give rise to two conditions for  $q$ :  $q \geq 0$ , and  $q \leq q_1 = 8(1 - c)\frac{1+\alpha}{7+10\alpha}$ . It is easy to check that all KKT conditions are satisfied when these conditions are met. Hence, the prices in equations (31) are the optimal solution to Model (27) when  $q \geq 0$  and  $q \leq q_1$ .

Finally, combining Equations (28), (30), and (31), and using  $p_t^2 = \frac{1}{2} + \frac{1}{2}p_t^1$ , we can obtain the complete characterization for the optimal high and low prices ( $p_{hi}^i$  and  $p_{lo}^i$ ) for products 1 and 2 as summarized in Proposition 4.

□

## B.6 Proof of Proposition 5

We consider each case separately:

- $\beta = 0$ : From Proposition 4, we note that  $p_{hi}^i, i \in \{1, 2\}$  and  $p_{lo}^i, i \in \{1, 2\}$  decrease in  $q$ . Note also that rate of decrease of  $p_{hi}^i$  with respect to  $q$  is  $(2 - \alpha)$ , whereas the rate of decrease of  $p_{lo}^i$  is  $\frac{2+3\alpha-\alpha^2}{1+\alpha}$ . Since  $2 - \alpha < \frac{2+3\alpha-\alpha^2}{1+\alpha}$  for  $\alpha \in \{0, 1\}$ , this implies that the difference between  $p_{hi}^i - p_{lo}^i$  increases in  $q$  for  $i \in \{1, 2\}$ .
- $\beta = 1$ : Similarly, from Proposition 4, we note that  $p_{lo}^i, i \in \{1, 2\}$  decreases in  $q$ , whereas  $p_{hi}^i$  first decreases in  $q$  when  $q < q_1$ , then increases in  $q$  when  $q_1 \leq q < q_2$  and finally stays constant in  $q$  afterwards. Also, when  $q < q_1$ ,  $p_{lo}^i = p_{hi}^i$ , and hence, the depth of the promotion is zero. On the other hand, when  $q_1 \leq q < q_2$ , the depth of the promotion increases in  $q$  since  $p_{hi}^i$  increases and  $p_{lo}^i$  decreases in  $q$ . Finally, when  $q_2 \leq q$ , the depth of promotion is constant in  $q$ .

□

## C Tables for empirical survey

Table 5: Summary statistics.

Retailer	$A_j (m^2)$	Category	$A_{jk}$	$M_{jk}$	$DS_{jk}$	$R_{ijkt}$		$\Delta_{ijk}$	
						N	Mean (Std. Dev.)	N	Mean (Std. Dev.)
IGA	612	Detergent	3.35	52	52	73	29.51% (6.83%)	84	0.562 (0.166)
		Shampoo	8.53	85	68	42	27.76% (6.47%)	52	0.573 (0.27)
		Both		137	120	115	28.87% (6.72%)	136	0.566 (0.211)
Loblaws	709	Detergent	1.62	40	40	43	16.43% (9.22%)	108	0.292 (0.221)
		Shampoo	9.24	67	58	317	17.66% (12.22%)	157	0.815 (0.318)
		Both		107	98	360	17.51% (11.9%)	265	0.602 (0.382)
Metro	1055	Detergent	5.10	87	80	14	16.58% (1.92%)	67	0.192 (0.265)
		Shampoo	8.48	55	44	50	10.85% (6.31%)	40	0.699 (0.178)
		Both		142	124	64	12.1% (6.12%)	107	0.381 (0.341)
Provigo	1589	Detergent	22.88	157	123	45	17.05% (8.25%)	54	0.584 (0.34)
		Shampoo	33.81	109	74	78	14.63% (10.06%)	87	0.602 (0.258)
		Both		266	197	123	15.52% (9.47%)	141	0.595 (0.291)

Table 6: Correlation coefficients for the model demonstrating Result 1 and Result 2.

	Mean	Std Dev	1.	2.	3.	4.	5.	6.
1. $R_{ijkt}$	0.1859	0.1139	1					
2. $A_{jk}$	15.8005	9.7008	-0.0667	1				
3. $A_j$	1235.0	410.6	-0.1901***	0.8948***	1			
4. $U_{ijkt}$	0.5634	0.4963	-0.0354	0.1207**	0.0648	1		
5. $DS_{jk}$	91.1677	32.3945	-0.0570	0.7467***	0.8551***	0.0242	1	
6. $C_k$	0.7357	0.4413	-0.1826***	0.0628	0.3817***	-0.0373	0.5308	1

Table 7: Correlation coefficients for the model showing Result 3.

	Mean	Std Dev	1.	2.	3.
1. $\Delta_{ijk}$	0.5565	0.3350	1		
2. $DS_{jk}$	77.8752	28.0687	0.3520***	1	
3. $C_k$	0.2096	0.4073	0.471***	0.4847***	1
4. $IGA$	0.4083	0.4919	0.0153	-0.2936***	-0.1395**
5. $Loblaws$	0.1649	0.3714	0.1120**	0.7451***	0.1243**
6. $Metro$	0.2173	0.4127	-0.2329***	-0.4214***	-0.1279**
7. $Provigo$	0.5177	0.5001	0.0609	-0.2191***	0.1047**

## References

- Ahn, H., Gümüş, M., and Kaminsky, P. Pricing and manufacturing decisions when demand is a function of prices in multiple periods. *Operations Research*, 55(6):1039–1057, 2007.
- Ahn, H., Gümüş, M., and Kaminsky, P. Inventory, discounts, and the timing effect. *Manufacturing and Service Operations Management*, 11:613–629, 2009.
- Aviv, Y. and Pazgal, A. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing and Service Operations Management*, 10 (3):339–359, 2008. *Manufacturing and Service Operations Management*.
- Aydin, G. and Hausman, W.H. The role of slotting fees in the coordination of assortment decisions. Working paper, Department of Industrial and Operations Engineering, University of Michigan, 2008.
- Aydin, G. and Porteus, E. Joint inventory and pricing decisions for an assortment. *Operations Research*, 56(5): 1247–1255, 2008. Working paper, Department of Industrial and Operations Engineering, University of Michigan.
- Blattberg, R.C., Briesch, R., and Fox, E.J. How promotions work. *Marketing Science*, 14:G122–G132, 1995.
- Cachon, G.P. and Kok, A.G. Category management and coordination in retail assortment planning in the presence of basket shopping consumers. *Management Science*, 53(6):934–951, 2007. Working paper, Operations and Information Management, The Wharton School, University of Pennsylvania.
- Chan, L.M.A., Shen, Z.J.M., Simchi-Levi, D., and Swann, J. Coordination of Pricing and Inventory Decisions: A Survey and Classification. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-business Era*. Kluwer Academic Publishers, 2004.
- Chen, H., Wu, O., and Yao, D. On the Benefit of Inventory-Based Dynamic Pricing Strategies. *Production and Operations Management*, 19(3):249–260, 2010.
- Chen, X. and Simchi-Levi, D. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research*, 52:887–896, 2004.
- Chen, Y., Ray, S., and Song, Y. Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. *Naval Research Logistics (NRL)*, 53(2):117–136, 2006.
- Chen, Y., Narasimhan, C., and Zhang, Z.J. Consumer heterogeneity and competitive price-matching guarantees. *Marketing Science*, 20(3):300–314, 2001.
- Corsten, D. and Gruen, T.W. Desperately seeking shelf availability: An examination of the extent, causes and efforts to reduce retail out-of-stocks. *International Journal of Retail and Distribution Management*, 31:605–617, 2003.
- Dong, L., Kouvelis, P., and Tian, Z. Dynamic pricing and inventory control for substitute products. *Manufacturing and Service Operations Management*, 11(2):317–339, 2009.
- Draganska, M. and Jain, D. Consumer preferences and product-line pricing strategies: An empirical analysis. *Marketing Science*, 25:164–174, 2006.

- Elmaghraby, W. and Keskinocak, P. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science*, 49(10):1287–1309, 2003.
- Elmaghraby, W., Gülcü, A., and Keskinocak, P. Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *Manufacturing and Service Operations Management*, 10 (1):126–148, 2008. To appear in *Manufacturing & Service Operations Management*.
- Federgruen, A. and Heching, A. Combined pricing and inventory control under uncertainty. *Operations Research*, 47 (3):454–475, 1999.
- Freimer, M. and Horsky, D. Try it, you will like it—does consumer learning lead to competitive price promotions? *Marketing Science*, 27(5):796–810, 2008.
- Gelb, B., Andrews, D., and Lam, S. Strategic perspective on sales promotions. *MIT Sloan Management Review*, July: W1–W7, 2007.
- Goyal, M. and Netessine, S. Strategic technology choice and capacity investment under demand uncertainty. *Management Science*, 53(2):192, 2007.
- Harris, F.H. and Pinder, J. A revenue management approach to demand management and order booking in assemble-to-order manufacturing. *Journal of Operations Management*, 13(4):299–309, 1995.
- Huh, W.T. and Janakiraman, G. (s, s) optimality in joint inventory-pricing control: An alternate approach. *Operations Research*, 56 (3):783–790, 2008.
- Kimes, S. Yield management: a tool for capacity-considered service firms. *Journal of Operations Management*, 8(4): 348–363, 1989.
- Lal, R. Price promotions: Limiting competitive encroachment. *Marketing Science*, 9(3):247–262, 1990a.
- Lal, R. and Villas-Boas, J.M. Price promotions and trade deals with multiproduct retailers. *Management Science*, 44 (7):935–949, 1998.
- Lus, B. and Muriel, A. Measuring the impact of increased product substitution on pricing and capacity decisions under linear demand models. *Production and Operations Management*, 18(1):95–113, 2009.
- Mathur, S. and Sinitsyn, M. Price promotions in emerging markets. McGill University Working Paper, 2011.
- McGill, J.I. and van Ryzin, G.J. Revenue management: Research overview and prospects. *Transportation Science*, 33(2):233–256, 1999.
- Narasimhan, C. Competitive promotional strategies. *Journal of Business*, 61(4):427–449, 1988.
- Pesendorfer, M. Retail sale: A study of pricing behavior in supermarkets. *Journal of Business*, 75:33–66, 2002.
- Rao, R.C., Arjunji, R.V., and Murthi, B.P.S. Game theory and empirical generalizations concerning competitive promotions. *Marketing Science*, 14:G89–G100, 1995.
- Röller, L. and Tombak, M. Strategic choice of flexible production technologies and welfare implications. *The Journal of Industrial Economics*, pages 417–431, 1990.
- Röller, L. and Tombak, M. Competition and investment in flexible technologies. *Management Science*, pages 107–114, 1993.
- Shen, M. and Su, X. Customer behavior modeling in revenue management and auctions: A review and new research opportunities. *Production and Operations Management*, 16(6):713–728, 2007.
- Silva-Risso, J.M., Bucklin, R.E., and Morrison, D.G. A decision support system for planning manufacturers' sales promotion calendars. *Marketing Science*, 18(3):274–300, 1999.
- Singh, N. and Vives, X. Price and quantity competition in a differentiated duopoly. *The RAND Journal of Economics*, pages 546–554, 1984.
- Song, Y., Ray, S., and Boyaci, T. Technical Note—Optimal Dynamic Joint Inventory-Pricing Control for Multiplicative Demand with Fixed Order Costs and Lost Sales. *Operations research*, 57(1):245–250, 2009.
- Taylor, J. and Fawcett, S. Retail on-shelf performance of advertised items: an assessment of supply chain effectiveness at the point of purchase. *Journal of Business Logistics*, 22:73–89, 2001.
- Varian, H.R. A model of sales. *American Economic Review*, 70(4):651–659, 1980.
- Villas-Boas, J.M. Models of competitive price promotions: Some empirical evidence from the coffee and saltine crackers markets. *Journal of Economics & Management Strategy*, 4(1):85–107, 1995.