

**A Separable Approximation Dynamic  
Programming Algorithm for Economic Dispatch  
with Transmission Losses**

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## Abstract

The standard way to solve the static economic dispatch problem with transmission losses is the penalty factor method. The problem is solved iteratively by a Lagrange multiplier method or by dynamic programming, using values obtained at one iteration to compute penalty factors for the next until stability is attained. A new iterative method is proposed for the case where transmission losses are represented by a quadratic formula (i.e., by the traditional B-coefficients). A separable approximation is made at each iteration, which is much closer to the initial problem than the penalty factor approximation. Consequently, lower cost solutions may be obtained in some cases, and convergence is faster.

**Keywords:** Economic Dispatch, Transmission Losses, B-coefficients, Penalty Factors, Separable Approximation, Dynamic Programming.

## Résumé

La méthode classique de résolution du problème de chargement statique avec pertes de transmission est la méthode des facteurs de pénalité. Le problème est résolu itérativement à l'aide de la méthode des multiplicateurs de Lagrange ou par programmation dynamique, en utilisant les valeurs obtenues à une itération pour calculer les facteurs de pénalité pour la suivante, jusqu'à ce que la stabilité soit atteinte. Une nouvelle méthode itérative est proposée pour le cas où les pertes de transmission peuvent être représentées par une formule quadratique (c'est-à-dire par les coefficients-B traditionnels). Une approximation séparable est faite à chaque itération, beaucoup plus proche du problème initial que l'approximation des facteurs de pénalités. En conséquence, des solutions de moindre coût peuvent parfois être obtenues et la convergence est plus rapide.

**Mots clés:** Problème de chargement, Pertes de transmission, Coefficients-B, Approximation séparable, Programmation dynamique.

## 1 Introduction

Due to the enormous costs involved, optimizing the use of equipment for power generation and transmission is a lasting concern. Given several thermal or hydro units, often with different characteristics, one must decide how to distribute the load considered between them. This problem, called *economic dispatch*, has been much studied, both in the static and dynamic cases. It is discussed at length in the review paper [1] and the book of Wood and Wollenberg [2] *Power generation, operation and control*, an enlarged second edition of which has recently appeared.

The standard way to solve the static case with transmission losses is the *penalty factor* method. The problem is solved iteratively by a Lagrangian multiplier method or by dynamic programming, using values obtained at one iteration to compute penalty factors for the next until stability is attained.

The purpose of this paper is to propose a new iterative method for the case where transmission losses are represented by a quadratic formula (i.e. by the traditional B-coefficients). A separable approximation is made at each iteration, which is much closer to the initial problem than the penalty factor approximation. Moreover, the problem considered at each iteration can be solved by dynamic programming. Convergence is faster than in another scheme, due to Liang and Glover [3], which also uses dynamic programming.

The paper is organized as follows: a mathematical formulation of the problem is given in the next section. Previous solution methods are reviewed in Section III. The particular case of a separable loss function is studied in Section IV, and illustrated by an example of Wood and Wollenberg [2]. The new method is described in full in Section V and applied to examples from [2] and [3]. Conclusions are drawn in Section VI.

## 2 Formulation

Consider  $N$  thermal units committed to serve a load of  $P_R$ , at minimum cost. Assume production of each unit is bounded below and above. Assume further transmission losses in the network are incurred and can be represented by a quadratic formula (with the so-called B-coefficients). This problem can be stated mathematically as

$$\text{Minimize } F_T = \sum_{i=1}^N F_i(P_i) \quad (1)$$

subject to

$$\sum_{i=1}^N P_i = P_R + P_L, \quad (2)$$

and

$$P_{i,min} \leq P_i \leq P_{i,max} \quad i = 1, 2, \dots, N, \quad (3)$$

where

- $P_i$  is the output of unit  $i$  (in MW),
- $F_i(P_i)$  is the input of unit  $i$ , or its cost rate (in \$ /h),
- $P_R$  is the required load (in MW),
- $P_L$  is the transmission losses (in MW).

Moreover,

$$P_L = \sum_{i=1}^N \sum_{j=1}^N P_i B_{ij} P_j + \sum_{i=1}^N B_{oi} P_i + B_{oo}. \quad (4)$$

Derivation of this formula for losses is explained in [2]. Neglecting transmission losses, i.e., setting  $P_L = 0$  in (2), problem (1) - (3) can be solved by a Lagrange multiplier method [2] or by dynamic programming [4], [5], [3].

### 3 Previous Solution Methods

When transmission losses are considered, the standard solution method is the penalty factor approach. Consider the Lagrange function, with a multiplier  $\lambda$  for (2):

$$\text{Minimize } \mathcal{L} = \sum_{i=1}^N F_i(P_i) + \lambda(P_R + P_L - \sum_{i=1}^N P_i). \quad (5)$$

Neglecting bounds on the  $P_i$ , the first order conditions are equation (2) above and

$$\frac{1}{1 - \frac{dP_L}{dP_i}} \cdot \frac{dF_i}{dP_i} = \lambda, \quad i = 1, 2, \dots, N, \quad (6)$$

where

$$\frac{dP_L}{dP_i} = \sum_{j=1}^N 2B_{ij} P_j + B_{oi} \quad (7)$$

are the *incremental losses* and

$$\frac{1}{1 - \frac{dP_L}{dP_i}} = \frac{1}{1 - \sum_{j=1}^N 2B_{ij} P_j - B_{oi}} = PF_i \quad (8)$$

are the *penalty factors*  $PF_i$  for units  $i = 1, \dots, N$ . Equations (2) and (6) are called *coordination equations*.

If bounds on the  $P_i$  are taken into account the first order conditions become

$$PF_i \cdot \frac{dF_i}{dP_i} \begin{cases} \geq \lambda & \text{if } P_i = P_{i,min}, \\ = \lambda & \text{if } P_i \in (P_{i,min}, P_{i,max}), \\ \leq \lambda & \text{if } P_i = P_{i,max}. \end{cases} \quad (9)$$

An iterative method to solve (1)-(3) is the following [1]:

1. Solve the coordination equations with unit penalty factors (i.e. neglecting losses) to get an initial solution  $P_1^o, P_2^o, \dots, P_N^o$ . Set the iteration counter  $k = 1$ .
2. Compute penalty factors with values of the last iteration, i.e.,

$$PF_i^{(k)} = \frac{1}{1 - \sum_{j=1}^N 2B_{ij}P_j^{(k-1)} - B_{oi}},$$

$$i = 1, 2, \dots, N,$$

3. Solve the coordination equations to get a solution  $P_1^{(k)}, P_2^{(k)}, \dots, P_N^{(k)}$ . (This can be done in various ways, e.g. a search technique for the optimal  $\lambda$ , a Newton-Raphson search, etc, see [2]).
4. Compute the current difference between production and load plus losses, i.e.,

$$\begin{aligned} \delta = & \sum_{i=1}^N P_i^{(k)} - P_R - \sum_{i=1}^N \sum_{j=1}^N P_i^{(k)} B_{ij} P_j^{(k)} \\ & - \sum_{i=1}^N B_{oi} P_i^{(k)} - B_{oo}. \end{aligned}$$

If  $\delta \leq \varepsilon$  (a given tolerance), stop. Otherwise, increase  $k$  by 1 and return to step 2.

While no formal proof of convergence for this method seems to have been published, and values of  $\delta$  may oscillate, a local and possibly global optimum is usually reached in a fairly small number of iterations.

Modification to the iterative method just described to allow solution by dynamic programming are proposed in [3]. In Step 1 the initial solution is found by the recursion

$$F_i^*(D_i) = \min_{P_i \in R_i} \{F_{i-1}^*(D_i - P_i) + F_i(P_i)\} \quad (10)$$

where  $R_i$  is the set of integers in  $[P_{i,min}, P_{i,max}]$  the range of production for unit  $i$ , assuming approximation to 1 MW and  $D_i$  is chosen among the range of possible productions for the  $i$  first units, i.e., it is integer and

$$\sum_{k=1}^i P_{k,min} \leq D_i \leq \sum_{k=1}^i P_{k,max}. \quad (11)$$

In Step 3, each cost function  $F_i$  is multiplied by the penalty factor  $P_i^{k-1}$ , the load  $P_R$  is augmented by the losses as estimated and the problem is again solved by the dynamic programming recursion (10).

The Lagrange function for this step is

$$\begin{aligned}
& \sum_{i=1}^N \left( \frac{1}{1 - \sum_{j=1}^N 2B_{ij}P_j^{(k-1)} - B_{oi}} \right) F_i(P_i) \\
& + \lambda \left( P_R + \sum_{i=1}^N \sum_{j=1}^N P_i^{(k-1)} B_{ij} P_j^{(k-1)} \right. \\
& \left. + \sum_{i=1}^N B_{oi} P_i^{(k-1)} + B_{oo} - \sum_{i=1}^N P_i \right)
\end{aligned} \tag{12}$$

and the first order condition, as the losses are fixed,

$$\begin{aligned}
& \sum_{i=1}^N \left( \frac{1}{1 - \sum_{j=1}^N 2B_{ij}P_j^{(k-1)} - B_{oi}} \right) \frac{dF_i}{dP_i} \\
& \begin{cases} \geq \lambda & \text{if } P_{i,min} = P_i \\ = \lambda & \text{if } P_{i,min} \leq P_i \leq P_{i,max} \\ \geq \lambda & \text{if } P_i = P_{i,max} \end{cases}
\end{aligned} \tag{13}$$

and are the same as (9) except that the values of  $P_i$  are fixed in the penalty factors.

## 4 Separable Losses

Economic dispatch with transmission losses can be optimized by dynamic programming without using penalty factors. The easiest case is when losses are separable in the  $P_i$ . Assume also that they are quadratic, a particular case considered in [2]. Then the power  $P_i$  produced by unit will be equal to the power  $P'_i$  going to the load and the losses. Thus

$$P'_i = P_i - b_{i0} - b_{i1}P_i - b_{i2}P_i^2 \tag{14}$$

where  $b_{i0}$ ,  $b_{i1}$  and  $b_{i2}$  are the known coefficients of the loss function. Hence

$$P_i = \frac{1 - b_{i1} - \sqrt{(1 - b_{i1})^2 - 4b_{i2}(P'_i + b_{i0})}}{2b_{i2}}. \tag{15}$$

One can associate to  $P'_i$  a cost function  $F'_i(P'_i)$  equal to the cost necessary for unit  $i$  to contribute  $P'_i$  to the load, i.e.,  $F_i(P_i)$  where  $P_i$  is obtained from  $P'_i$  by (14). The correspondence between  $F'_i$  and  $F_i$  is illustrated on Figure 1.

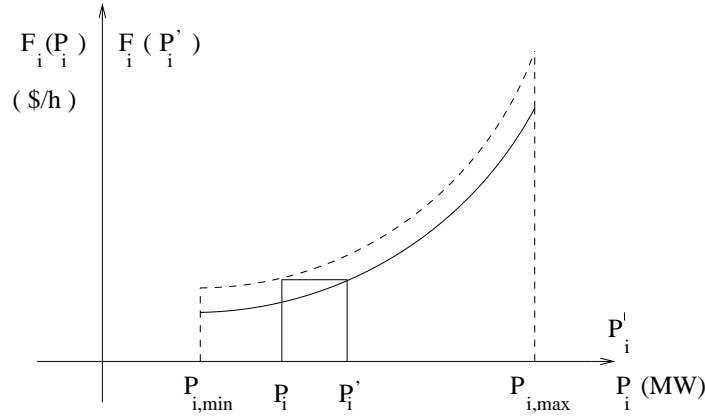


Fig. 1. Input-Output functions  $F_i(P_i)$  and  $F_i(P'_i)$ .

The range of values of  $P'_i$ , deduced from that of  $P_i$  is, assuming integer values, or, in other words an approximation of 1 MW:

$$\begin{aligned} P'_{i,min} &= \lceil P_{i,min} - b_{i0} - b_{i1}P_{i,min} - b_{i2}P_{i,min}^2 \rceil \leq P'_i \\ &\leq \lfloor P_{i,max} - b_{i0} - b_{i1}P_{i,max} - b_{i2}P_{i,max}^2 \rfloor = P'_{i,max} \end{aligned} \quad (16)$$

where  $\lceil a \rceil$  and  $\lfloor a \rfloor$  are respectively the smallest integer not smaller than and the largest integer not larger than  $a$ .

The dynamic programming recursion is

$$F_i^*(D_i) = \min_{P'_i \in R_i} \{F_{i-1}^*(D_i - P'_i) + F_i(P'_i)\} \quad (17)$$

where  $R_i = [P'_{i,min}, P'_{i,max}]$ , the range of effective production of unit  $i$  (i.e. that which is not lost) and

$$D_i = \left[ \sum_{k=1}^i P'_{k,min}, \sum_{k=1}^i P'_{k,max} \right] \quad (18)$$

the range of demand which can be satisfied by the first  $i$  units. This recursion is used to find the optimal policy  $P_1^*, P_2^*, \dots, P_N^*$ , from which the powers  $P_1, P_2, \dots, P_N$  to be produced by each unit are obtained by (15). The cost of this policy is  $F_N^*(P_R) = \sum_{i=1}^N F_i(P_i^*)$ .

Observe that the method proposed gives a globally optimal solution (up to the approximations in the values of the  $P_i$ ) even if the cost functions  $F_i(P_i)$  and/or  $P_i$  are non-convex. In contrast, the classical penalty function method may stop at a local optimum in this case.

**Example 1.** (Wood and Wollenberg [2], p 36). This problem involves three units with input/output functions and production ranges:

$$\begin{aligned}
 F_1(P_1) &= 561.0 + 7.920P_1 + 0.001562P_1^2, \\
 P_{1,min} &= 150MW, \quad P_{1,max} = 600MW, \\
 F_2(P_2) &= 310.0 + 7.850P_2 + 0.001940P_2^2, \\
 P_{2,min} &= 100MW, \quad P_{2,max} = 400MW, \\
 F_3(P_3) &= 78.0 + 7.970P_3 + 0.00482P_3^2, \\
 P_{3,min} &= 50MW, \quad P_{3,max} = 200MW,
 \end{aligned}$$

$P_R = 850$  MW and losses are given by

$$P_L = 0.00003P_1^2 + 0.00009P_2^2 + 0.00012P_3^2.$$

*Solution.* The following solution is obtained by applying dynamic programming to the transformed problem:  $P_1 = 434.668$ ,  $P_2 = 300.106$ ,  $P_3 = 131.061$  with losses equal to 15.8350. Note that the same losses are obtained by the penalty factor approach in 4 iterations ([2]).

## 5 Separable Quadratic Approximation

An iterative method using a separable quadratic approximation of the loss function is easily obtained by fixing at each iteration one value  $P_j$  in each quadratic term involving two such values  $P_i$  and  $P_j$ . Such an approximation is much more precise than that one obtained by fixing both values in all such terms, discussed above. Then the algorithm of the previous section is as follows:

- (a) Obtain an initial solution  $P_1^{(o)}, P_2^{(o)}, \dots, P_N^{(o)}$  by solving the problem after deleting all quadratic terms  $P_i B_{ij} P_j$  with  $i \neq j$  in the loss function (or, which is better, by some heuristic, see below). Set the iteration counter  $k = 1$ .
- (b) Compute the following separable quadratic approximation of the loss function

$$\sum_{i=1}^N B_{ii} P_i^2 + \sum_{i=1}^N \sum_{j=1, i \neq j}^N P_i B_{ij} P_j^{(k-1)} + \sum_{i=1}^N B_{oi} P_i + B_{oo} \quad (19)$$

- (c) Solve the problem with the approximate loss function (19) to obtain a solution  $P_1^{(k)}, P_2^{(k)}, \dots, P_N^{(k)}$ .
- (d) Compute the approximate and exact losses using  $P_1^{(k)}, P_2^{(k)}, \dots, P_N^{(k)}$  in (19) and (4). If these values differ in absolute value by less than  $\varepsilon$ , stop. Otherwise increase  $k$  by 1 and return to (b).

A better initial approximation to the loss function may be obtained by computing a feasible initial solution by a heuristic such as following:

- (h1) Set  $P_i = P_{i,min}$  for  $i = 1, \dots, N$ ;
- (h2) Compute  $\delta P = \frac{P_R - \sum_{i=1}^N P_i}{N}$ ;
- (h3) If  $P_i + \delta P > P_{i,max}$  for some  $i$ , set  $P_i = P_{i,max}$  delete that index value, and reduce  $N$  by 1. Return to (h2);
- (h4) Set  $P_i + \delta P$  for all remaining  $i$ .

The aim of this heuristic is to obtain quickly a feasible solution in which the productions above the minimum ones of the various units are as close as possible. Therefore, it first attempts to give an equal load to each unit. If some upper bound is not satisfied the largest possible load is assigned to the corresponding unit. The procedure is iterated with the total unassigned load. More sophisticated heuristics could take cost functions into account, but this does not appear to be necessary.

**Example 2.** (Wood and Wollenberg [2], p 117-120)). The fuel cost curves for the three units in the six-bus network considered are given by

$$\begin{aligned} F_1(P_1) &= 213.1 + 11.669P_1 + 0.00533P_1^2 \\ F_2(P_2) &= 200.0 + 10.333P_2 + 0.00889P_2^2 \\ F_3(P_3) &= 240.0 + 10.833P_3 + 0.00741P_3^2 \end{aligned}$$

with a total load to be supplied  $P_R = 210$  and unit dispatch limits

$$\begin{aligned} 50.0 &\leq P_1 \leq 200 \\ 37.5 &\leq P_2 \leq 150 \\ 45.0 &\leq P_3 \leq 180. \end{aligned}$$

The coefficients  $B_{oo} = 4.04$ ,  $B_{oi} = (-0.07660, -0.00342, 0.01890)$ , ( $i=1,2,3$ ) and the  $B$  matrix is

$$B = \begin{bmatrix} 0.0006760 & 0.0000953 & -0.0000507 \\ 0.0000953 & 0.0005210 & 0.0000901 \\ -0.0000507 & 0.0000901 & 0.0002940 \end{bmatrix}.$$

*Solution.* The optimal solution is obtained in 3 iterations:

It.	$F$	$P_1$	$P_2$	$P_3$	$P_R + P_L$	Error
1	3146.28	70.3011	76.1023	71.1801	217.583	1.48846
2	3165.09	71.5868	73.9075	73.5692	219.063	0.01805
3	3164.86	71.5625	73.9348	73.5485	219.046	0.00021

**Example 3.** (Liang and Glover [3]). The operating costs of the generators are represented by the following polynomials of third order:

$$a_o + a_1P_i + a_2P_i^2 + a_3P_i^3, \quad i = 1, 2, 3$$

where coefficients as well as  $P_{min}$  and  $P_{max}$  are listed below:

U	$a_o$	$a_1$	$a_2$	$a_3$	$P_{min}$	$P_{max}$
1	11.200	5.10238	-0.00264290	3.33333E-6	100	500
2	-632.000	13.01000	-0.03057140	3.33333E-5	100	500
3	147.144	4.28997	0.00030845	-1.17677E-7	200	1000

U = Unit

The load is  $P_R = 1400$  and the losses are given by

$$P_L = 0.000075P_1^2 + 0.000015P_2^2 + 0.000045P_3^2.$$

The  $B$ -matrix is

$$B = \begin{bmatrix} 0.0000750 & 0.000005 & 0.0000075 \\ 0.0000050 & 0.000015 & 0.0000100 \\ 0.0000075 & 0.000010 & 0.0000450 \end{bmatrix}.$$

*Solution.* If an initial solution is obtained by neglecting the off-diagonal elements of matrix  $B$ , the optimal solution is obtained in four iterations:

It.	$F$	$P_1$	$P_2$	$P_3$	$P_R + P_L$	$Error$
1	6592.20	359.704	406.478	665.957	1432.14	10.46985
2	6642.32	360.266	406.927	676.145	1443.34	0.075683
3	6642.68	361.361	406.969	675.079	1443.41	0.001587
4	6642.69	360.293	406.967	676.158	1443.42	0.000366

Another way to derive an initial solution is also developed above, in Section V. In this example, the initial solution obtained is  $P_1 = P_2 = 433.333$ ;  $P_3 = 533.333$ . The optimal solution is then reached in only two iterations:

It.	$F$	$P_1$	$P_2$	$P_3$	$P_R + P_L$	$Error$
1	6641.60	363.129	407.547	672.448	1443.12	0.229736
2	6642.68	359.222	406.959	677.244	1443.42	0.000977

**Example 4.** In the previous examples convex programming problems are solved, since both the objective function  $F_T$  and the transmission losses  $P_L$  are convex. Note that convexity of the  $P_L$  function follows from the positive definiteness of the  $B$  matrix (i.e., the main minors of  $B$  are positive in Examples 1, 2 and 3 above). In this example, we change elements  $b_{12}$  and  $b_{21}$  from Example 3, i.e., we now have  $b_{12} = b_{21} = 0.00005$ . Then

the minor of the second order is negative ( $det = 0.000075 \cdot 0.000015 - 0.00005^2 < 0$ ) and thus, there are more than one local minima. Since the penalty factor method is derived from the first order conditions, it is a local optimization method, and may not reach the global optimum.

*Solution.* With our separable approximation dynamic programming method, the following locally optimal solution is obtained in six iterations:

It.	$F$	$P_1$	$P_2$	$P_3$	$P_R + P_L$	$Error$
1	6699.85	375.187	100.875	999.538	1475.60	8.50537
2	6657.20	419.456	98.993	946.974	1465.42	0.259155
3	6684.72	348.296	401.648	703.157	1453.10	3.97778
4	6703.30	328.416	403.299	725.495	1457.21	0.324585
5	6701.74	327.425	404.014	725.391	1456.83	0.009888
6	6701.69	327.437	403.992	725.391	1456.82	0.000122

We then ran the *penalty factor* method to see what solution it gives for this non-convex example. For that purpose we used software attached to the book [2]. The solution obtained is as follows:  $F(P_1, P_2, P_3) = 6940.35$ ;  $P_1 = 428.6$ ,  $P_2 = 100.0$ ,  $P_3 = 1,000.0$ ,  $\lambda = 4.6765$ . Therefore, the objective function value and total losses are 6940.35 and 128.6 respectively (compare those values with 6701.69 and 56.82 obtained by our approach).

We also tried a modification of the penalty factor method: instead of adjusting  $\lambda$  in the inner loop (as suggested in [2] for Step 3 in the algorithm given in Section III), we simply find the closest solution that satisfies the range constraints (3) as well. In other words, within Step 3, we iteratively project the current solution  $(P_1, \dots, P_n)$  onto the hyperplane defined by (2) and the hypercube (3) until a feasible solution is reached. Note that both methods are equivalent if the point obtained after solving the coordination equations is feasible. The result, obtained in 5 iterations, is:  $F(219.21, 243.79, 1000.00) = 6812.31$ . Again, this solution is worse than the dynamic programming one.

## 6 Conclusion

A new method for static unit commitment with a quadratic loss function is proposed. It uses a separable quadratic approximation to the loss function, and refines it iteratively. Convergence is quicker than with previous methods. Moreover, it may lead to better locally optimal solutions than those methods in case of non-convexity of the loss function.

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