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for the maximum k -cut problem**

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Computational study of valid inequalities for the maximum k -cut problem

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Abstract: We consider the maximum k -cut problem that consists in partitioning the vertex set of a graph into k subsets such that the sum of the weights of edges joining vertices in different subsets is maximized. We focus on identifying effective classes of inequalities to tighten the semidefinite programming relaxation. We carry out an experimental study of four classes of inequalities from the literature: **clique**, **general clique**, **wheel** and **bicycle wheel**. We considered 10 combinations of these classes and tested them on both dense and sparse instances for $k \in \{3, 4, 5, 7\}$. Our computational results suggest that the **bicycle wheel** and **wheel** are the strongest inequalities for $k = 3$, and that for $k \in \{4, 5, 7\}$ the **wheel** inequalities are the strongest by far. Furthermore, we observe an improvement in the performance for all choices of k when both **bicycle wheel** and **wheel** are used, at the cost of 72% more CPU time on average when compared with using only one of them.

Key Words: Maximum k -Cut, graph partitioning, semidefinite programming, computational study.

Résumé: Nous considérons le problème de la k -coupe maximale qui consiste à partitionner l'ensemble des sommets d'un graphe en k sous-ensembles tels que la somme des poids des arêtes entre les différents sous-ensembles soit maximale. Nous nous concentrons sur l'identification de classes efficaces d'inégalités pour améliorer la valeur de la relaxation semi-définie du problème. Nous réalisons l'étude expérimentale de quatre classes d'inégalités : **clique**, **general clique**, **wheel**, et **bicycle wheel**. Nous avons considéré 10 combinaisons de ces classes que nous avons testées sur des instances denses et creuses pour $k \in \{3, 4, 5, 7\}$. Les résultats suggèrent que **bicycle wheel** et **wheel** sont les inégalités les plus fortes pour $k = 3$, et que pour $k \in \{4, 5, 7\}$, les inégalités de type **wheel** sont de loin les plus fortes. Par ailleurs, on note une amélioration de la performance pour tout k lorsque les inégalités de type **bicycle wheel** et **wheel** sont utilisées conjointement, au coût de 72% de plus de temps CPU, en moyenne, par rapport à l'utilisation d'un seul de ces types.

Mots clés: k -coupe maximale, partitionnement de graphe, programmation semi-définie, étude computationnelle.

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1 Introduction

Graph partitioning problems (GPPs) are an important class of combinatorial optimization. There are various types of GPP based on the number of partitions allowed, the objective function, the types of edge weights, and the possible presence of additional constraints such as restrictions on the number of vertices allowed in each partition. This work considers the GPP known as the maximum k -cut (max- k -cut) problem: Given a connected graph $G = (V, E)$ with edge weights w_{ij} for all $(i, j) \in E$, partition the vertex set V into at most k subsets so as to maximize the total weight of cut edges, i.e., the edges with end points in two different subsets. This problem is sometimes also called minimum k -partition problem [15]. The special case of max- k -cut with $k = 2$ is known as the max-cut problem and has received most attention in the literature, see e.g. [3, 9, 16, 18, 22, 24].

Graph partitioning and max- k -cut problems have myriad applications as VLSI layout design [3], sports team scheduling [20], statistical physics [19], placement of television commercials [14], network planning [11], and floorplanning [7].

The unweighted version of max- k -cut is known to be $\mathcal{NP}$ -complete [23], therefore researchers have proposed heuristics [6], approximation algorithms [6, 13] and exact methods [1, 15] for max- k -cut. In particular the approximation method proposed in [13] extends the famous semidefinite programming (SDP) results for max-cut in [16] to max- k -cut. The authors of [8] subsequently improved the approximation guarantees for small values of k .

Ghaddar et al. [15] proposed the SBC algorithm, a branch-and-cut algorithm for the minimum k -partition problem based on the SDP relaxation of max- k -cut. They found experimentally that for $k > 2$ the SDP relaxation yields much stronger bounds than the LP relaxation, for both sparse and dense instances. This was improved in [1] which introduced the bundleBC algorithm to solve max- k -cut for large instances by combining the approach of [15] with the design principles of **Biq Mac** [24]. The results in [1] confirm that, computationally speaking, the SDP relaxations often yield stronger bounds than LP relaxations.

The matrix lifting SDP relaxation of the GPP, and the use the **triangle** and **clique** inequalities to strengthen it, were studied in [27, 28]. In particular [27] shows that this relaxation is equivalent to the relaxation in [13] and is as competitive as any other known SDP bound for the GPP. The results also show that the **triangle** inequalities are stronger than the **clique** (independent set) inequalities.

The objective of this work is to carry out a computational study of valid inequalities that strengthen the SDP relaxation for the max- k -cut problem, and to identify the strongest ones in practice. Valid and facet-inducing inequalities for k -partition were studied in [5, 4]. We focus our analysis on the following inequalities: **Triangle**, **Clique**, **General clique**, **Wheel** and **Bicycle wheel**. To the best of our knowledge, no research has specifically focused on comparing the strength of the range of inequalities considered here.

This paper is organized as follows. Sections 1.1 and 1.2 review the SDP formulation and relaxation of the max- k -cut problem. Section 2 presents the classes of inequalities and discusses the separation methods used. Section 3 describes the cutting plane algorithm used, and Section 4 presents and discusses the test results. The conclusions from this study are provided in Section 5.

1.1 Problem formulation

We assume without loss of generality that the graph $G = (V, E)$ is a complete graph because missing edges can be added with a corresponding weight of zero. Let $k \geq 2$ and define the matrix variable $X = (X_{ij})$, $i, j \in V$, as:

$$X_{ij} = \begin{cases} \frac{-1}{k-1} & \text{if vertices } i \text{ and } j \text{ are in different partitions of the } k\text{-cut of } G, \\ 1 & \text{otherwise.} \end{cases}$$

The max- k -cut problem on G can be expressed as the following SDP problem that we refer to as **(MkP)**:

$$\max_X \quad \sum_{i,j \in V, i < j} w_{ij} \frac{(k-1)(1-X_{ij})}{k} \quad (1)$$

$$\text{s.t.} \quad X_{ii} = 1 \quad \forall i \in V, \quad (2)$$

$$X_{ij} \in \left\{ \frac{-1}{k-1}, 1 \right\} \quad \forall i, j \in V, i < j, \quad (3)$$

$$X \succeq 0. \quad (4)$$

1.2 Semidefinite relaxation

Replacing Constraint (3) by $\frac{-1}{k-1} \leq X_{ij} \leq 1$ defines the SDP relaxation. However, the constraint $X_{ij} \leq 1$ can be removed since it is enforced implicitly by the constraints $X_{ii} = 1$ and $X \succeq 0$. Therefore the SDP relaxation, denoted (RMkP), is:

$$\max_X \quad \sum_{i,j \in V, i < j} w_{ij} \frac{(k-1)(1-X_{ij})}{k} \quad (5)$$

$$\text{s.t.} \quad X_{ii} = 1 \quad \forall i \in V, \quad (6)$$

$$X_{ij} \geq \frac{-1}{k-1} \quad \forall i, j \in V, i < j, \quad (7)$$

$$X \succeq 0. \quad (8)$$

2 Formulation and separation of inequalities

The SDP relaxation (RMkP) can be tightened by adding valid inequalities, i.e., cutting planes that are satisfied for all positive semidefinite matrices that are feasible for problem (MkP) but are violated by solution X' of the (current) SDP relaxation. In this section we present the classes of inequalities from [4, 5] that we consider adding to tighten the SDP relaxation. For each class, we describe its properties, its formulation in terms of the SDP variable X , and the separation routine used.

The first two classes of inequalities (Triangle and Clique) were used in the SBC branch-and-cut method [15] and in the matrix lifting approach [27, 28]. We are unaware of the other classes having been computationally tested or used.

2.1 Triangle inequalities

The triangle inequalities are based on the observation that if vertices i and j are in the same partition, and vertices j and h in the same partition, then vertices i and h necessarily have to be in the same partition. The resulting $3 \binom{|V|}{3}$ triangle inequalities are:

$$X_{ih} + X_{hj} - X_{ij} \leq 1 \quad \forall i, j, h \in V. \quad (9)$$

2.1.1 Separation of triangle inequalities

As pointed out in [1], the enumeration of all triangle inequalities is computationally inexpensive, even for large instances. Therefore the complete enumeration is done to find the ones that violate (9). The tolerance used to detect a violation is 10^{-5} .

2.2 Clique inequalities

The $\binom{|V|}{k+1}$ clique inequalities ensure that for every subset of $k+1$ vertices, at least two of the vertices belong to the same partition. We express them as:

$$\sum_{i,j \in Q, i < j} X_{ij} \geq -\frac{k}{2} \quad \forall Q \subseteq V \text{ with } |Q| = k+1. \quad (10)$$

2.2.1 Separation of clique inequalities

The exact separation of clique is $\mathcal{NP}$ -hard in general, and the complete enumeration is intractable even for small values of k [15]. We use the heuristic described in [15] to find a violated clique inequality for a solution X' of the SDP relaxation. This heuristic is described here as Algorithm 1, and returns up to $|V|$ clique inequalities. We use a tolerance $\epsilon = 10^{-5}$.

Algorithm 1: Heuristic to find violated clique inequalities

FOR all $j \in V$ **DO**

1. $C = \{j\}$

2. **WHILE** ($|C| \leq k$) **DO**

(a) Select $u \in V \setminus C$ such that $u \in \left\{ \arg \min_v \sum_{i \in C} X'_{i,v} \mid v \in V \setminus C \right\}$

(b) $C = C \cup \{u\}$

3. **IF** $\left(\sum_{i,z \in C, i < z} X'_{iz} < -\frac{k}{2} - \epsilon \right)$ **THEN**

(a) $Q = C$

(b) Construct one inequality of type (10)

2.3 General clique inequalities

The General clique inequalities in [4] are a generalization of the clique inequality (10).

Let $Q = (V(Q), E(Q))$ be a clique of size $p := tk + q$ where $t \geq 1$ and $0 \leq q < k$, i.e., t is the largest integer such that $tk \leq p$. In terms of the SDP variable X , the general clique inequalities have the form:

$$\sum_{i,j \in Q} X_{ij} \geq \frac{k}{k-1} \left[z + w \left(\frac{k-1}{k} - 1 \right) \right] \quad \forall Q \subseteq V, |Q| = p, \quad (11)$$

where $z = \frac{1}{2}t(t-1)(k-q) + \frac{1}{2}t(t+1)q$ and $w = \frac{p(p-1)}{2}$. It is proved in [4] that (11) is valid for the polytope of the max- k -cut problem if $p \geq k$, i.e., $t \geq 1$. Furthermore, it is facet-defining if and only if $1 \leq q \leq k-1$, $t \geq 1$, i.e., p is not an integer multiple of k .

2.3.1 Separation of general clique inequalities

Because the clique inequalities are included in general clique, it is clear that enumeration is also impracticable for general clique. Therefore we propose a heuristic to find the most violated general clique inequalities for X' .

Our heuristic is a small modification of the one used to separate the clique inequalities (Algorithm 1). Specifically:

- The argument in the **WHILE** condition (2) becomes $|C| \leq p - 1$.
- The inequality in the **IF** condition (3) is replaced by

$$\left(w(j) < \frac{k}{k-1} \left[z + w \left(\frac{k-1}{k} - 1 \right) \right] - \epsilon \right).$$

The parameter p can vary with k . Based on preliminary testing, we settled on with the following choices: $p = 5$ for $k = 3$; $p = 6$ for $k = 4$; $p = 7$ for $k = 5$; and $p = 11$ for $k = 7$.

2.4 Wheel inequalities

In [4], the q -wheel is defined as $W_q = (V_q, E_q)$, where

- $V_q = \{v, v_i | i = 1, 2, \dots, q\}$
- $E_q = \bar{E} \cup \hat{E}$, where $\bar{E} = \{(v, v_i), i = 1, 2, \dots, q\}$ and $\hat{E} = \{(v_i, v_{i+1}), i = 1, 2, \dots, q\}$.

where $v \in V$ and all indices are modulo q with $q \pmod q = 0$. For $q = 6$, W_6 is illustrated in Figure 1.

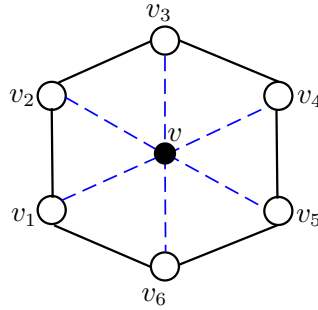


Figure 1: Example of wheel with $q = 6$

Let W_q be a subgraph of G for $q \geq 3$. In terms of the SDP variable X , the wheel inequality is defined as:

$$\sum_{i=1}^q X_{v, v_i} - \sum_{i=1}^q X_{v_i, v_{i+1}} \leq \left\lfloor \frac{1}{2}q \right\rfloor \frac{k}{k-1}. \quad (12)$$

The idea of this inequality is that vertex v can be in the same partition as vertices v_i and v_{i+1} if and only if the edge $e = (v_i, v_{i+1})$ is not cut. It is proved in [4] that (12) is a valid inequality for max- k -cut, and that it defines a facet if q is odd and $k \geq 4$.

2.4.1 Separation of wheel inequalities

By Inequality (12), a q -wheel is violated by X' if:

$$\sum_{i=1}^q X'_{v, v_i} - \sum_{i=1}^q X'_{v_i, v_{i+1}} - \left(\left\lfloor \frac{1}{2}q \right\rfloor \frac{k}{k-1} \right) > 0. \quad (13)$$

For each vertex $v \in V$, we wish to maximize the left-hand side of Inequality (13) to find the most violated wheel inequality.

Finding the minimum value for the second term of (13) is equivalent to finding the minimum Hamiltonian cycle of size q in a complete graph, and is thus $\mathcal{NP}$ -complete. Therefore we tested four heuristics to identify wheel inequalities violated by X' :

- The GRASP meta-heuristic [12].
- A genetic algorithm (GA) from [26].
- the K-step greedy lookahead [25] with K=2.
- Our own greedy heuristic for multiple sizes of wheel (GMSW), described in Algorithm 2.

Algorithm 2: Heuristic to find violated wheel inequalities

```

1. FOR all  $j \in V$  DO
  (a)  $i = 1$ ;
  (b)  $v_i = j$ ;
  (c)  $W = \{v_i\}$ 
  (d) WHILE ( $i \leq NbIte$ ) DO
    i.  $q = |W|$ 
    ii.  $z = 0$ 
    iii. IF  $q \geq 3$  THEN
      Select  $v \in V \setminus W$  s.t.  $z = \max_v \sum_{i=1}^q X'_{v,v_i} - \sum_{i=1}^q X'_{v_i,v_{i+1}} - \left(\lfloor \frac{1}{2}q \rfloor \frac{k}{k-1}\right)$ 
    iv. IF  $z > 0$  THEN
      A. Build Inequality (12) with  $W$  and  $v$ .
      B.  $j = j + 1$ .
      C. Go to 1.
    ELSE
      A. Select  $u \in V \setminus W$  that:
         $u \in \{\arg \min_v X'_{v,q,v} + X'_{v,v_1} \mid v \in V \setminus W\}$ 
      B.  $i = i + 1$ ;
      C.  $v_i = \{u\}$ 
      D.  $W = W \cup \{v_i\}$ 

```

The GMSW heuristic gave the best results (in terms of quality) for 70% of the tests. In GSMW, the parameter $NbIte \leq |V| - 1$ is the maximum size allowed for the wheel. We obtained the best results using $NbIte = 2k$ when $2k \leq |V| - 1$.

2.5 Bicycle wheel inequalities

The fourth class of inequalities corresponds to a q -bicycle wheel $BW_q = (V_q, E_q)$ subgraph of G [4], where:

$$\begin{aligned}
 V_q &= \{u_1, u_2\} \cup \{v_i, i = 1, 2, \dots, q\}, & E_q &= E_1 \cup E_2 \cup \{\bar{e}\}, \\
 E_1 &= \{(u_i, v_j), i = 1, 2, j = 1, \dots, q\}, & E_2 &= \{(v_j, v_{j+1}), j = 1, \dots, q\}, \\
 \bar{e} &= \{(u_1, u_2)\},
 \end{aligned}$$

where vertices $v, u \in V$. For $q = 5$, BW_5 is illustrated in Figure 2.

For BW_q with $q \geq 3$, the corresponding bicycle wheel inequality is defined as:

$$\sum_{i=1}^2 \sum_{j=1}^q X_{u_i, v_j} - \sum_{j=1}^q X_{v_j, v_{j+1}} - X_{u_1, u_2} \leq \frac{k}{k-1} \left[2 \lfloor \frac{1}{2}q \rfloor - q + 1 \right] + q - 1, \quad (14)$$

where all indices are modulo q . It is proved in [4] that the bicycle wheel inequality is a valid inequality to (MSkP), and that for $k \geq 4$, this inequality induces a facet if q is odd and BW_q is an induced subgraph of G .

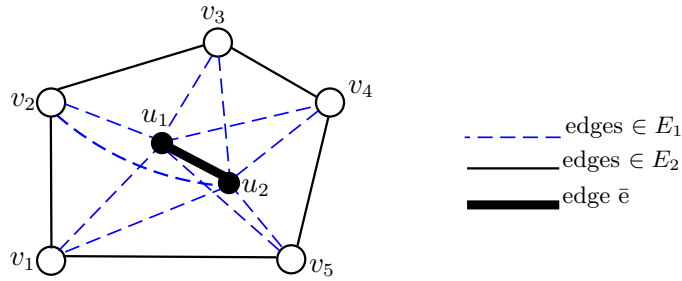


Figure 2: Example of bicycle wheel with $q = 5$.

2.5.1 Separation of bicycle wheel inequalities

The complexity of finding the most violated bicycle wheel inequality is at least as difficult as that for the wheel inequalities. This section presents a heuristic that is used to find violated inequalities of the form (14) for a given X' .

The basic idea of this heuristic is:

- Search for a cycle that minimizes the sum $\left(\sum_{j=1}^{j \leq q} X_{v_j, v_{j+1}} \right)$.
- Find the best points u_1 and u_2 that maximize $\left(\sum_{j=1}^{i=1,2, j \leq q} X_{u_i, v_j} \right)$ and $(-X_{u_1, u_2})$.

For finding good cycles in the first step of this heuristic, we tested the same four heuristics mentioned in Section 2.4.1. Our tests showed that the GA from [26] with $q = 3$ obtained the best results in 95% of the cases. The GA we used to find good cycles to build bicycle wheel inequalities has the following features:

- Size of population is 100.
- Half of the initial population was constructed with the GRASP and other half was randomly selected.
- In crossover process the parents with higher fitness are favored.
- Offspring genes can accept some slight mutations.

We keep the best cycle C_w found in the final population after 100 iterations of the GA. Using this cycle, the main heuristic then checks all the combinations of $u_1, u_2 \in V \setminus C_w$ to find the one that maximizes the right-hand side of (14).

3 Cutting plane algorithm

This section presents the cutting plane algorithm (CPA) used for our computational tests. This algorithm is similar to the one described in [15]. The basic idea is to start by solving the SDP problem (RMkP), then check for violated inequalities, add them to the SDP problem, and resolve to obtain a better upper bound on the global optimal value of the max- k -cut instance.

The pseudo-code of the CPA is given in Algorithm 3. We use the following notation: i is the iterations counter, F^i is the set of inequalities in iteration i , X^i is the optimal solution of the SDP relaxation at iteration i , and $S(X^i)$ is the set of valid inequalities for (MkP) that are violated by X^i .

The set $S(X^i)$ of inequalities is obtained by the separation routines of **triangle**, **clique**, **general clique**, **wheel** and **bicycle wheel** inequalities described in Section 2. We say that a class of inequalities is *active* if its separation routine is applied to find violations in the relaxed solution at each iteration of the CPA.

The size of the set $S(X^i)$ is controlled by the parameter $NbIneq$: $|S(X^i)| \leq NbIneq$ at each iteration of the CPA. After some tests, the CPA with $NbIneq = 300$ was found to give the best results for a variety of graphs and choices of k .

Algorithm 3: Cutting Plane Algorithm

-
1. Initializations: $i = 0$, $F^0 = \emptyset$.
 2. Solve (RMkP) with *inequalities* in F^i added.
 3. Test:
 - **IF** X^i is integer ($rank(X^i) = 1$) **THEN**
 STOP: X^i is also optimal for (MkP).
 4. **IF** $S(X^i) \neq \emptyset$ **THEN**
 - $F^{i+1} = F^i \cup S(X^i)$,
 - $i = i + 1$,
 - go to 2.
 5. STOP: no more violated inequalities for X^i have been found.
-

Since the size of the set $S(X^i)$ is limited, the sequence in which the separation routines are applied will impact the final CPU time. Our preliminary tests showed that the best sequence is: **triangle**, **wheel**, **bicycle wheel**, **clique** and finally **general clique** inequalities. Thus, the CPA will search for **bicycle wheel** only if the total number of **triangle** and **wheel** inequalities found is less than $NbIneq$.

4 Computational tests

The SDP relaxations of the max- k -cut were solved using the **mosek** solver [2] on a linux PC with two Intel(R) Xeon(R) 3.07 GHz processors.

Tests were performed for each value of $k \in \{4, 3, 5, 7\}$ on 147 test instances. Our instances were taken from the **Biq Mac Library** [29] or generated using **rudy** [17].

Section 4.1 introduces the combinations of inequalities tested, and the instances used. Section 4.2 explains the benchmarking methodology used for analyzing the different algorithms, and Section 4.3 reports the computational results. Finally, a summary of our results is given in Section 4.4.

4.1 Combinations of inequalities and test instances

This section defines the ten combinations of active inequalities developed to analyze the performance of the four classes of inequalities, and the instances used in the tests.

4.1.1 Combinations of inequalities

This section presents ten combinations of classes of inequalities. Each combination differ from each other by the classes that are activated or deactivated when running the CPA described in Section 3.

The **triangle** inequalities have been extensively studied in the literature and their efficiency has been confirmed by various researchers [27, 24, 1]. For this reason, we activated them in all combinations.

From the sixteen possible combinations of the remaining four classes of inequalities (**clique**, **wheel**, **bicycle wheel** and **general clique**), we used the ten combinations defined in Table 1 that emphasize the impact of individual classes. The first column of Table 1 gives the name of each combination, and the other five columns indicate with an X which classes are activated in that combination.

The ten combinations can be summarized as: one that activates none of the four classes (**Tri**), another that activates them all (**ALL**), and for each class “Inequality”, two additional combinations (**Tri + Inequality**) and (**ALL – Inequality**).

Moreover, the ten combinations can be split into two groups: The first group, **G1**, is formed by the combinations that activate at most two classes of inequalities, and the second group, **G2**, corresponds to the combinations that deactivate at most one class.

Table 1: The ten combinations

Name	Class of instances				
	Triangle	Clique	General clique	Wheel	Bicycle wheel
Tri	X				
T+C1	X	X			
T+GC	X		X		
T+Wh	X			X	
T+BW	X				X
ALL	X	X	X	X	X
A-C1	X		X	X	X
A-GC	X	X		X	X
A-Wh	X	X	X		X
A-BW	X	X	X	X	

4.1.2 Instances

Of a total of 147 instances, 77 are dense (edge density greater than or equal to 0.5), and the remaining 70 are sparse.

- *Random*: Twelve complete graphs with the edge weights randomly selected and generated by **rudy** [17]. The dimension varies from 50 to 100 and the density between 0.5 and 1.
- *spinglass_2g(3g)*: Eleven instances (3g : four instances) of a toroidal 2D(3D) grid for a spin glass model with gaussian interactions. The grid has size $(n_{row}) \times (n_{columns})$. They are sparse with density between 0.05 and 0.3.
- *spinglass_2pm(3pm)*: Two instances (3pm : six instances) of toroidal 2D(3D)-grid for a spin glass model with +/-J interactions. The grid has size $(n_{row}) \times (n_{columns}) \times (n_{layers})$. The percentage of negative weights is 50 %, dimension n vary from 36 to 64 and density between 0.05 and 0.3.
- *pm1s_n.i*: Ten weighted instances with edge weights drawn uniformly from $\{-1, 0, 1\}$, density 0.1 and dimension 80.
- *pm1d_n.i*: Ten weighted graphs with edge weights drawn uniformly from $\{-1, 0, 1\}$ and density 0.99. For densities $n \in \{80, 100\}$.
- *gkai_a(b or c)*: For each type (a,b or c) five instances with dimensions from 30 to 100, densities from 0.0625 to 0.5 (b: density 1 and for c: densities from 0.1 to 0.8). Diagonal coefficients from $[-100, 100]$ (b: $[-63, 0]$ and c: $[-100, 100]$), off-diagonal coefficients from $[-100, 100]$ (b: $[0, 100]$ and c: $[-50, 50]$).
- *bqp_n-i*: For each dimension n , ten weighted graphs with dimension $n \in \{50, 100\}$ and density 0.1.
- *g05_n.i*: Ten unweighted graphs with edge probability 0.5 and dimension $n \in \{60, 80, 100\}$.
- *w01_100.i*: Ten graphs with integer edge weights chosen from $[-10, 10]$, density 0.1 and dimension 100.

4.2 Comparison methodology

This section explains the tools used to analyze our results. First, we present the performance profiles proposed in [10], then the data profiles as used in [21], followed by the performance versus time graphs, and the relevance histograms.

We define a comparison in terms of a set $\mathcal{P}$ of problems (the 147 instances) and a set $\mathcal{S}$ of optimization algorithms (the ten combinations).

4.2.1 Performance profiles

As in [10, 21], the performance profiles are defined in terms of final objective solution $z_{p,s} > 0$ obtained for each instance $p \in \mathcal{P}$ and combination $s \in \mathcal{S}$. In our case (relaxation of a maximization), larger values of $z_{p,s}$ indicate worse performance. For any pair (p, s) of problem p and combination s , the performance ratio is defined as:

$$r_{p,s} = 100 \left(\frac{z_{p,s} - Best_p}{Best_p} \right) \quad (15)$$

where $Best_p = \min\{z_{p,s} \mid p \in \mathcal{P} : s \in \mathcal{S}\}$.

The performance profiles of a combination $s \in \mathcal{S}$ is defined as the fraction of problems for which the performance ratio is at most α , that is,

$$\rho_s(\alpha) = \frac{1}{|\mathcal{P}|} \text{size}\{p \in \mathcal{P} : r_{p,s} \leq \alpha\}. \quad (16)$$

Thus, for a given α one knows the percentage of instances $p \in \mathcal{P}$ (y axis) that are solved for each combination $s \in \mathcal{S}$ ($\rho_s(\alpha)$ in the x axis).

4.2.2 Data profiles

As pointed out in [21], data profiles are useful to users with a specific computational time limit who need to choose a combination that is likely to reach a given performance. Hence, it shows the temporal evolution of combinations to a specific performance ratio (r_{max}).

The data profiles are defined in terms of CPU time $t_{p,s}$ obtained for each $p \in \mathcal{P}$ and $s \in \mathcal{S}$. So, for a given time β we define the data profiles of a combination $s \in \mathcal{S}$ by:

$$d_s(\beta) = \frac{1}{|\mathcal{P}|} \text{size}\{p \in \mathcal{P} : t_{p,s} \leq \beta \text{ and } r_{p,s} \leq r_{max}\}. \quad (17)$$

Thus, for given r_{max} and time β , one sees the percentage of instances that can be solved for each combination $s \in \mathcal{S}$.

Note that in the computational results we do not analyze the data profiles for dense and sparse graphs separately because the obtained results did not contribute additional insights.

4.2.3 Performance versus time graphs

This benchmark shows the average performance ratio ($\Delta r_{p,s} \mid p \in \mathcal{P} : s \in \mathcal{S}$) in the x -axis, and the average time of execution ($\Delta t_{p,s} \mid p \in \mathcal{P} : s \in \mathcal{S}$) in the y -axis.

A combination ω dominates σ if $\Delta r_{p,\omega} < r_{p,\sigma}$ and $\Delta t_{p,\omega} < \Delta t_{p,\sigma}$. Thus, we can analyze domination between our combinations.

4.2.4 Relevance histograms

We use relevance histograms to quantify the impact of each inequality on the quality of the SDP bound. Let $\mathcal{I} = \{Cl, Wh, BW, GC\}$ be the set of four classes of inequalities, and $\Delta r_{p,s}, \forall p \in \mathcal{P}$ be the average performance ratio of a combination $s \in \mathcal{S}$. We calculate the relevance μ_i of the inequality $i \in \mathcal{I}$ as:

$$\mu_i = \frac{(\Delta r_{p, Tri} - \Delta r_{p, T+i}) + (\Delta r_{p, A-i} - \Delta r_{p, ALL})}{2 \times (\Delta r_{p, Tri} - \Delta r_{p, ALL})}. \quad (18)$$

Equation (18) is normalized by the largest gap, which is the gap between combinations **Tri** and **ALL**. Thus, if an inequality $i \in \mathcal{I}$ has relevance 1 it means that **ALL** $-i$ = **Tri** and **Tri** $+i$ = **ALL**.

4.3 Computational results

This section presents and analyzes the results of our tests. Section 4.3.1 demonstrate the benchmarks of all types of graphs. Sections 4.3.2 and 4.3.3 show the benchmarks of sparse and dense instances, respectively.

4.3.1 Results on the full set of instances

This part of the analysis give the results for all 147 instances (sparse and dense), the ten combinations, and $k \in \{3, 4, 5, 7\}$.

Performance profiles:

Figure 3 shows the performance profiles for the ten combinations. To facilitate the analysis, the performance ratio is limited to 0.8.

According to Figure 3a) (for $k = 3$), the combinations with the best performance are ALL, A-GC and A-C1, and the combinations of Group **G2** almost always dominate the ones of Group **G1**. Moreover, results suggest that the bicycle wheel is an important inequality since T+BW dominates all the other combinations of **G1** and A-BW has the worst performance of the **G2**. For the same reasons we can notice that wheel inequality is also very relevant to CPA for $k = 3$.

The results presented in graphs 3b) c) and d) (for $k = 4, 5$ and 7 , respectively) are very similar. It can be noticed that all combinations that activate the wheel inequality dominate the others. Moreover, we observe that all the combinations that have wheel and general clique inequalities can solve more than 70% of the instances with performance ratio less than 0.05. We also observe that the plots of Tri and T+C1 are very similar for those partitions, suggesting that the clique inequalities are ineffective for $k \in \{4, 5, 7\}$.

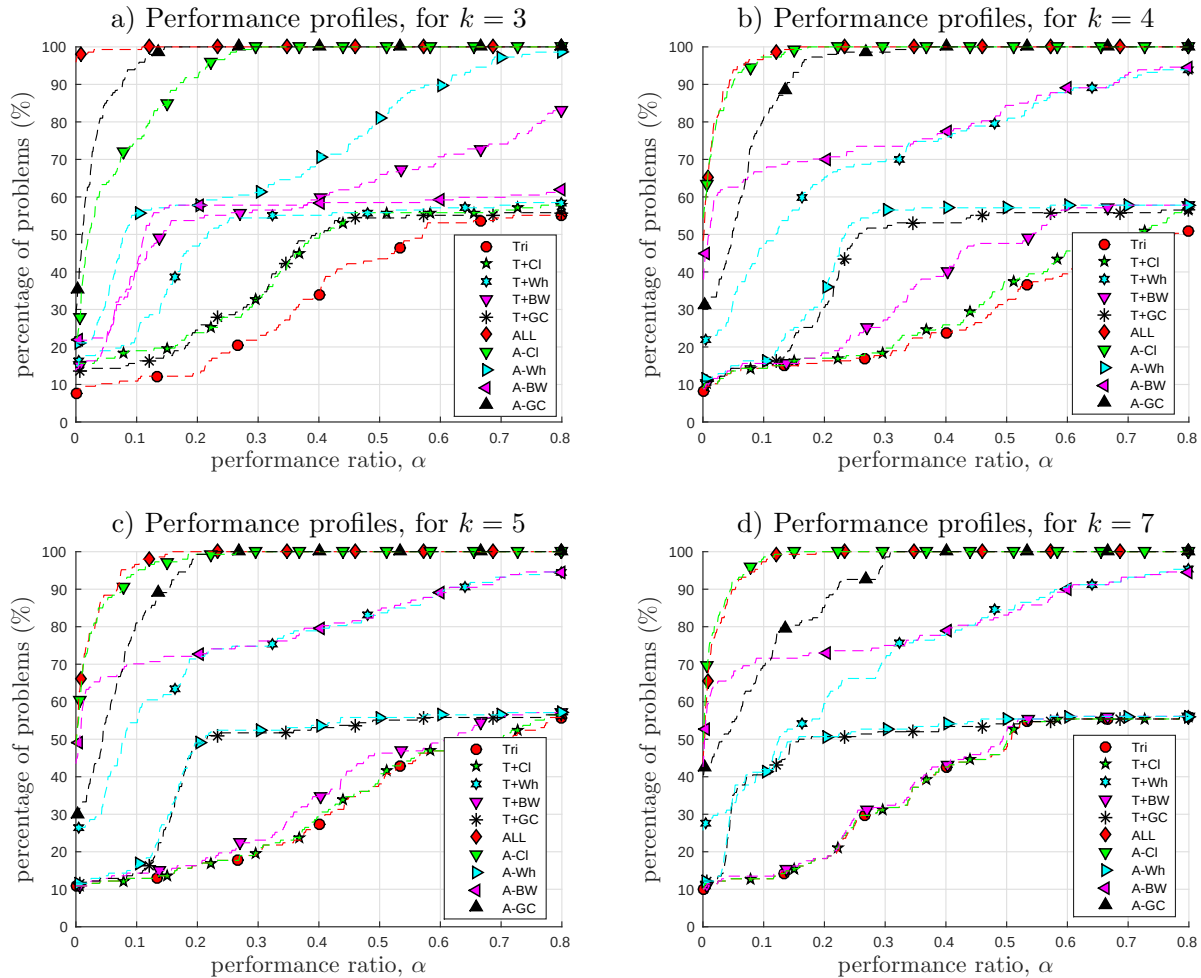


Figure 3: Performance profiles for all instances

Data profiles:

Figure 4 shows the data profiles for a maximum performance ratio (r_{max}) equal to 0.0. Each graph shows the results for one of $k \in \{3, 4, 5, 7\}$. Moreover, the time in this analysis is limited to 5×10^4 seconds (approximately 14 hours).

In Figure 4a), it can be observed that ALL followed by A-GC are the best combinations, and we see that combinations of Group G2 perform better than those of G1.

The graphs in Figure 4b) c) and d) can be separated in combinations with and without wheel inequality, and it can be noticed that all the algorithms that activate wheel inequalities have the better performance, e.g. T+Wh dominates A-Wh for $k \in \{4, 5, 7\}$.

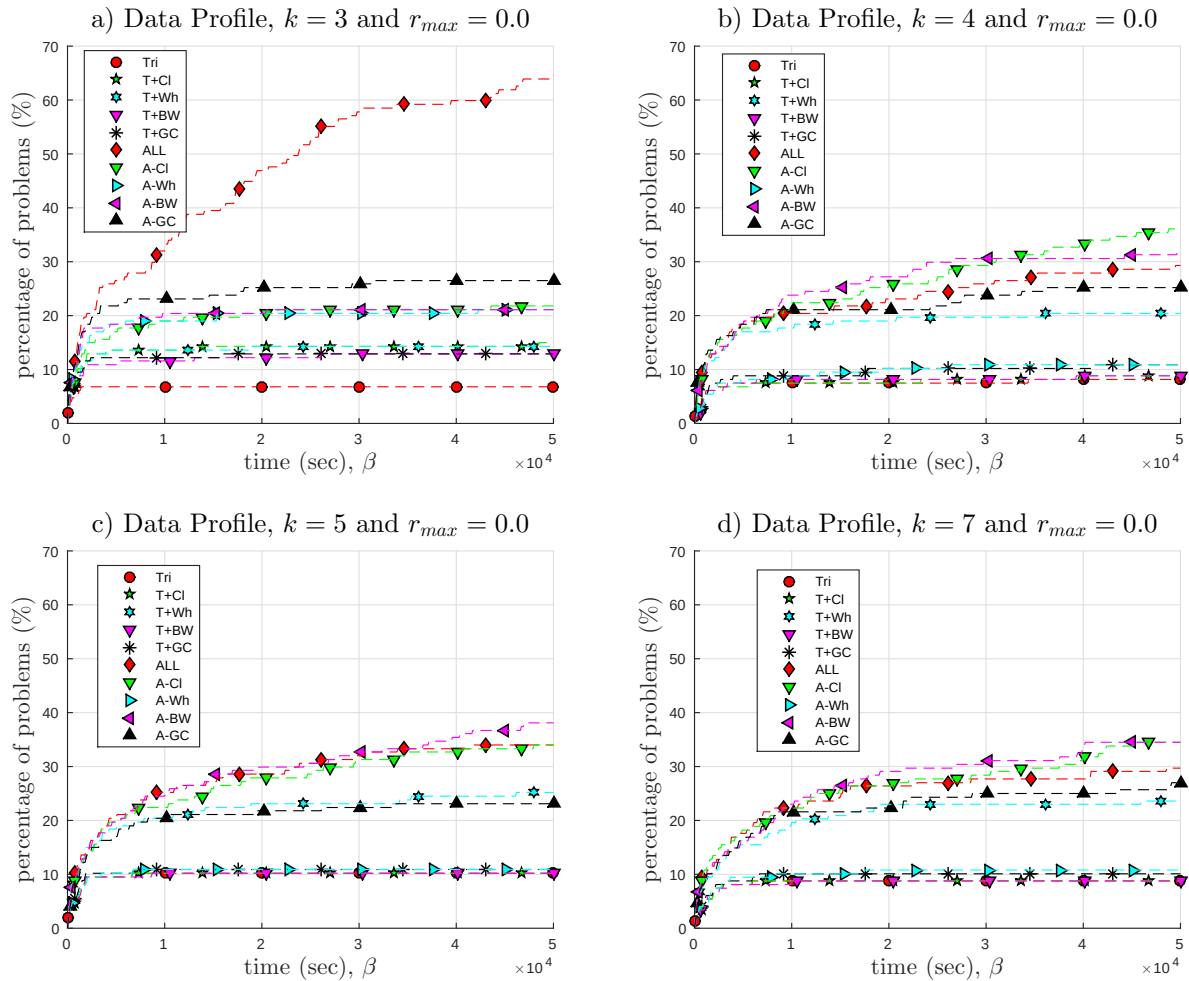


Figure 4: Data profiles for all instances with $r_{max} = 0.0$

Quality versus time:

Figure 5 presents the average performance versus CPU time for the ten combinations of the cutting plane algorithm.

In Figure 5a) it may be seen that A-Cl and T+GC are dominated by ALL and T+Cl, respectively. Moreover, for $k = 3$, T+Cl, T+GC and Tri are the fastest combinations but they have the worst performance. It is also noticeable that combinations with bicycle wheel show better performance but require more CPU time.

It can be observed that the results for $k \in \{4, 5, 7\}$ (Figure 5b) c) and d)) are very similar. In those graphs **A-Wh** is always dominated by **T+Wh**. We also notice that all combination that activate **wheel** have on average a performance ratio less than 0.2.

Finally, results in Figure 5 suggest that the CPU time of all the combinations is increased with the number of partitions.

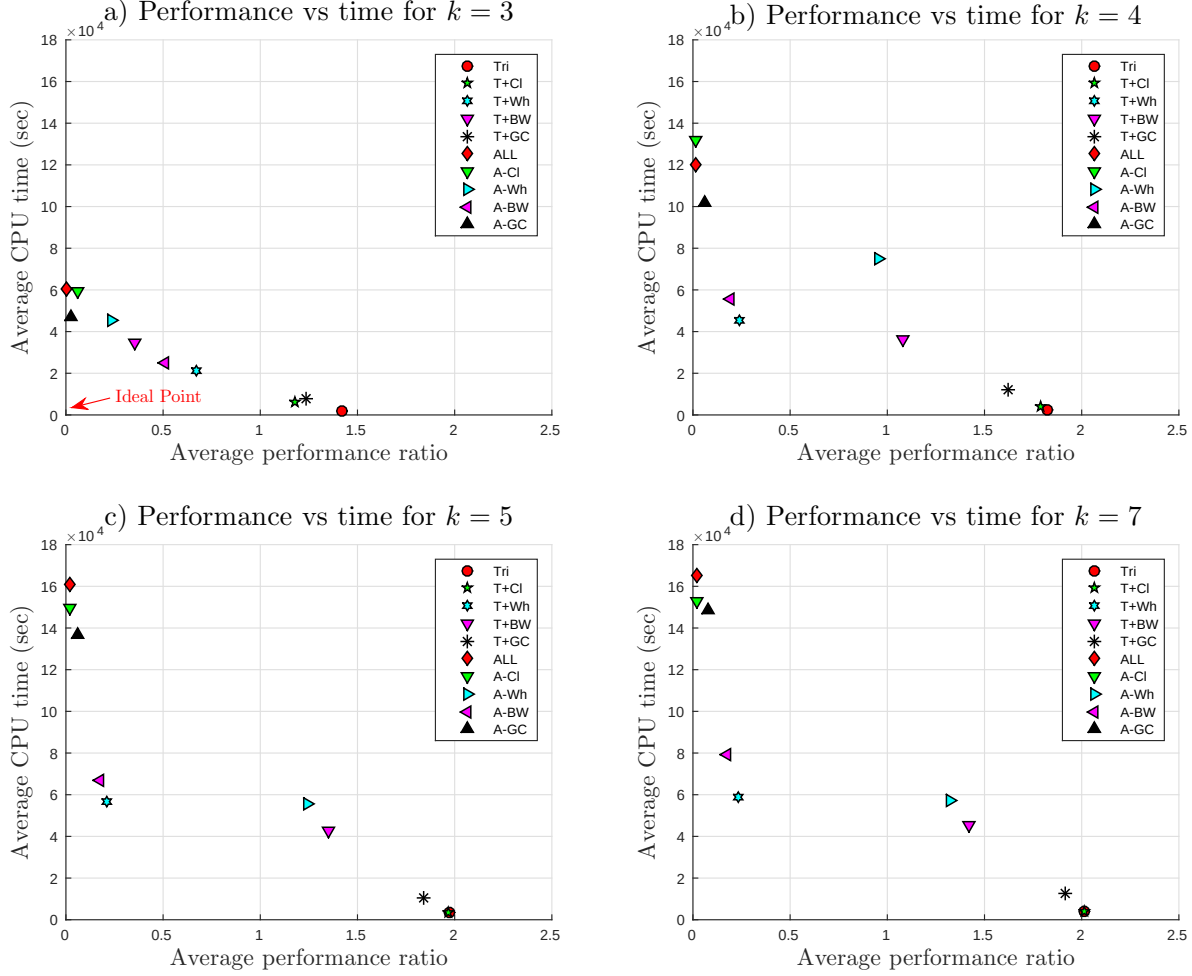


Figure 5: Performance versus CPU time for all instances

Relevance of inequalities:

Figure 6 presents the relevance histograms for partitions $k \in \{3, 4, 5, 7\}$ of four inequalities: Clique, Wheel, Bicycle wheel (B. Wheel) and General clique (Gen. Clique). In Figure 6a), we see that the bicycle wheel is the most relevant inequality followed by wheel. Although for $k \in \{4, 5, 7\}$ the wheel inequality is clearly the most significant. Figure 6 suggests that the relevance of the wheel inequality increases with k .

4.3.2 Results for sparse graphs

This section analyzes the results of the ten combinations on the sparse instances (density below 50 %). We present the performance profiles, performance versus time and the relevance histograms for the ten combinations on these 70 instances.

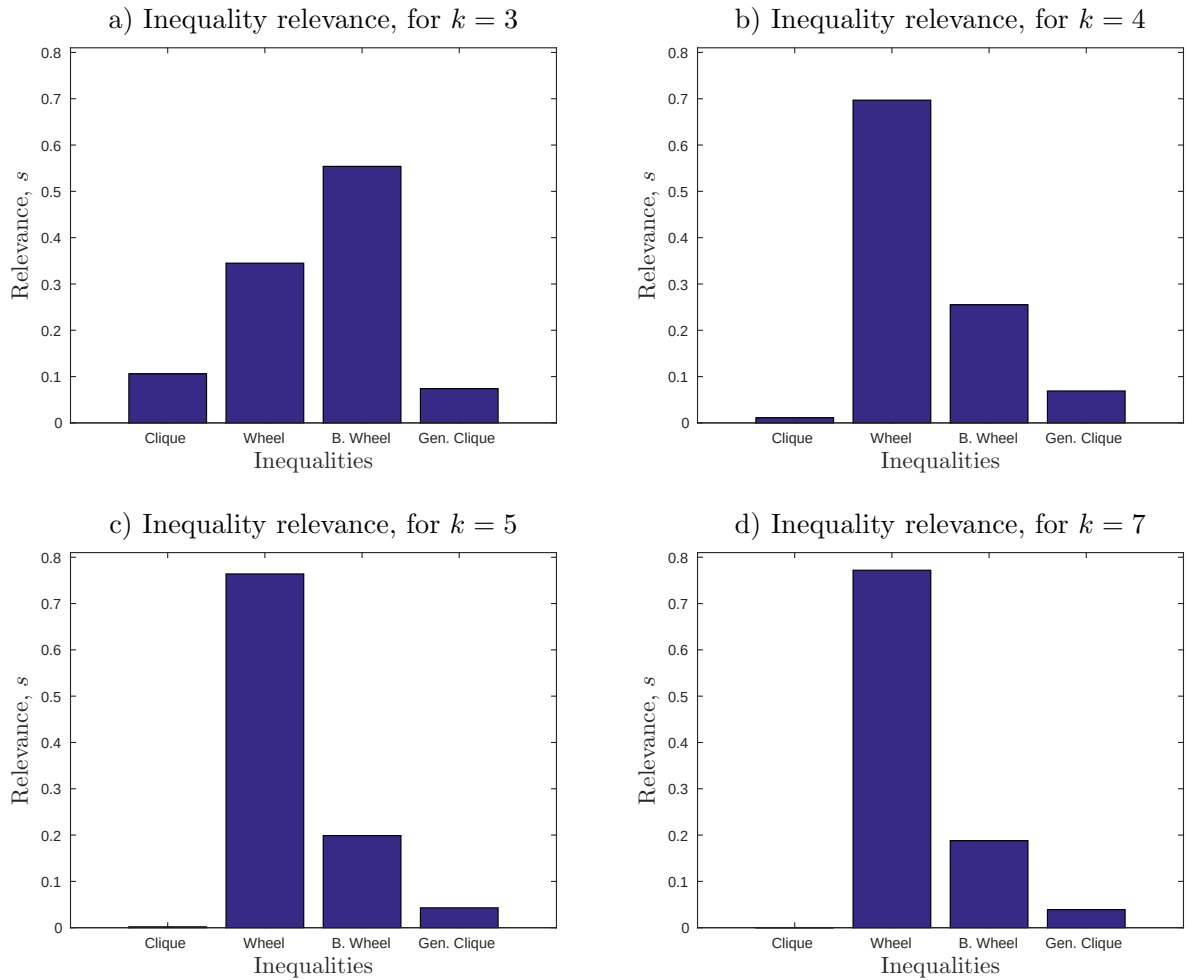


Figure 6: Relevance of the inequalities for all instances

Performance profiles:

Figure 7 shows the performance profiles of the combinations for sparse instances.

In Figure 7a) ($k = 3$), it can be observed that the combinations with the best results are ALL, A-GC and A-C1, followed by A-Wh. For $k = 3$, it is also noticeable that bicycle wheel is the most important inequality, since all combinations that deactivate it have the worst performance. For the same reasons we can include wheel inequalities in second place of importance for $k = 3$.

Graphs 7b) c) and d) (for $k = 4, 5$ and 7 , respectively) are very similar. It can be seen that all algorithms that the wheel inequality produce the best results. A closer look on those results reveals a strong relation between T+Wh (the best result for Group G1) and A-BW, suggesting that wheel is the most significant inequalities to the max- k -cut followed by bicycle wheel.

Quality versus time:

Figure 8 presents the average performance versus CPU time of the combinations for sparse instances.

In Figure 8a) the combinations T+C1, T+GC and Tri are the fastest but they have the worst performance. The best performance can be credited to A-BW, since its results are the closest to the ideal point. It is apparent that combinations with the bicycle wheel have the best performance but the highest CPU time.

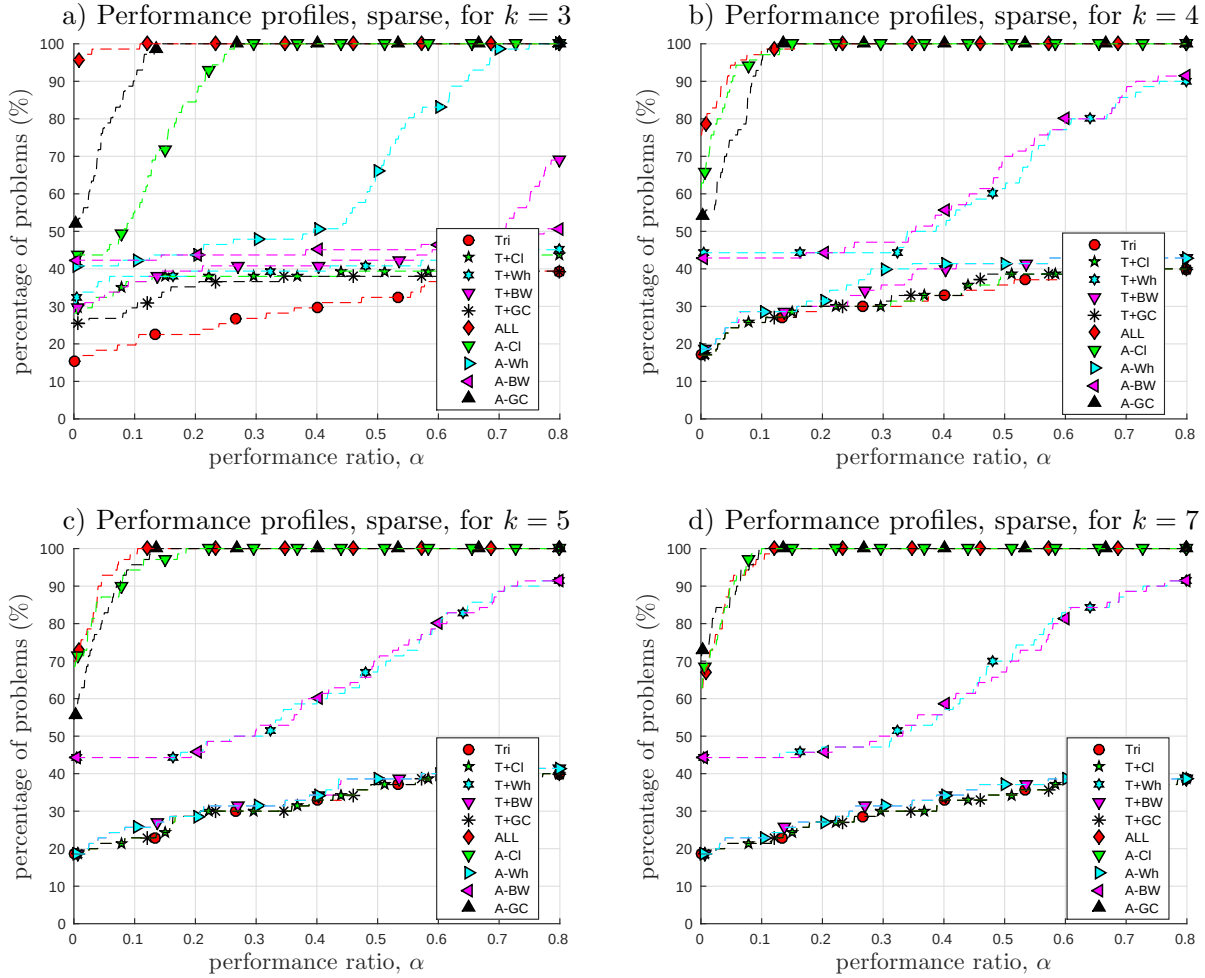


Figure 7: Performance profiles for sparse instances

It can be noticed that the results for $k \in \{4, 5, 7\}$ (Figure 8b) c) and d)) are very similar. Those graphs suggest that **wheel** is the most relevant class of inequalities, because all the combinations where it is activated have the average performance lower than 0.3. Moreover, the combinations that activate **wheel** and **bicycle wheel** inequalities at the same time (A-GC, A-CI and ALL) have the best average performance, but the worst CPU time. They present an average time larger than 6×10^4 seconds.

Finally, we observe that the CPU time increases with the number of partitions. For $k = 7$, for example, the combinations spend 75% more CPU time solving the cutting plane algorithm than for the other types of partition.

Relevance of inequalities:

Figure 9 demonstrates the relevance of the four inequalities for sparse instances. For $k = 3$, results shown in Figure 9a), it can be seen that the **bicycle wheel** is the most relevant inequality followed by **wheel** and **clique**.

For $k \in \{4, 5, 7\}$ it can be noticed that **clique** and **general clique** inequalities are ineffective, and that the **wheel** inequality is the strongest class by far.

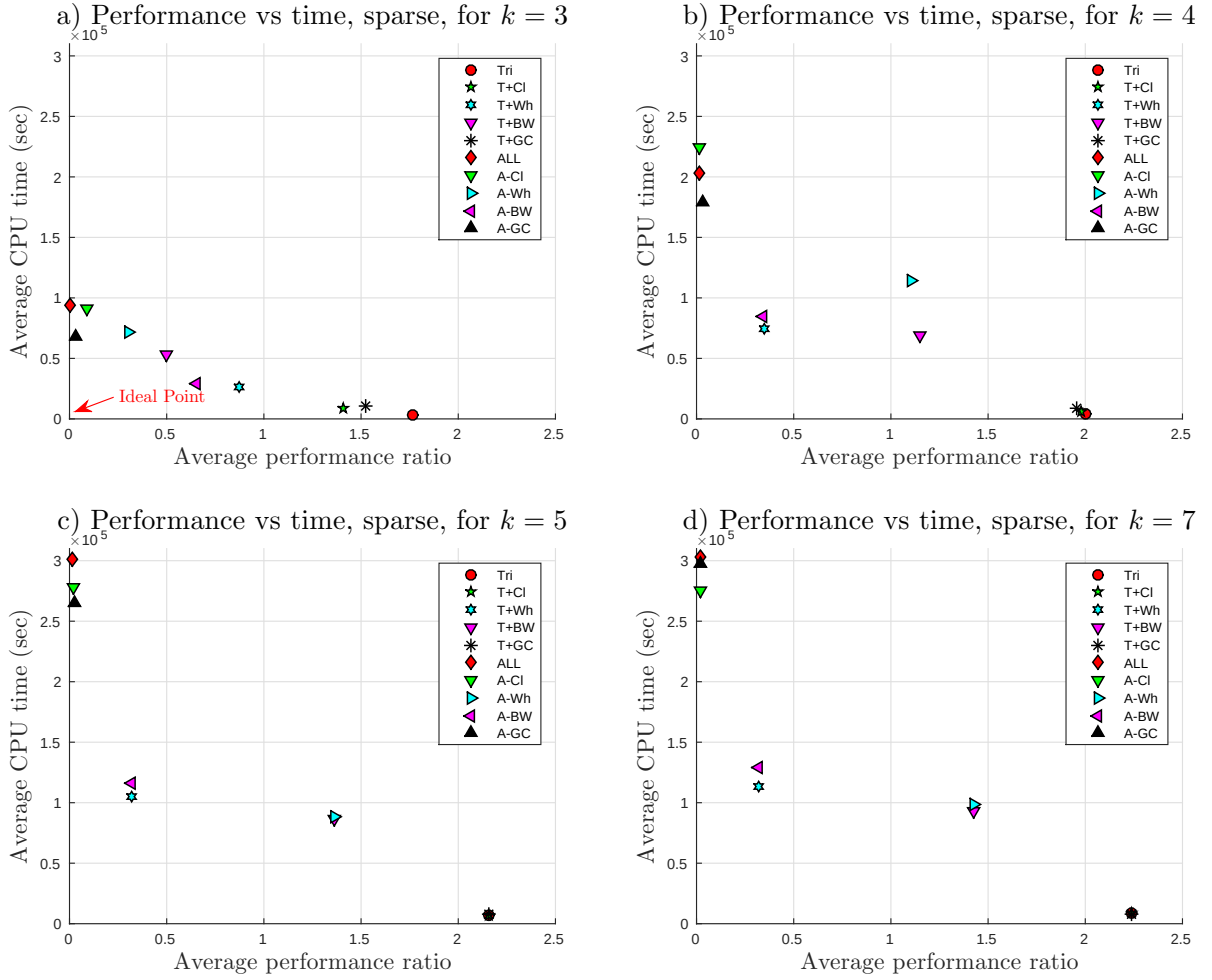


Figure 8: Performance versus CPU time for sparse instances

4.3.3 Results for dense graphs

This section analyzes the results of the ten combinations on the dense instances (density greater than or equal to 50 %).

Performance profiles:

Similarly to the previous performances profiles, it can be observed in Figure 10a) ($k = 3$) that the combinations with the best results are ALL, A-GC and A-Cl, followed by A-Wh.

The results in Figure 10a) also suggest that bicycle wheel is the most significant inequality since combinations that activate it have the best performance of G1 and G2. A similar analysis reveals that wheel is also relevant for this partition.

Figures 10b) c) and d) (for $k = 4, 5, 7$) suggest that wheel and general clique are, respectively, the most relevant, since the activation/deactivation of those inequalities have a huge impact in the results.

It is also apparent that for $k \in \{4, 5, 7\}$ the results of ALL, A-BW and A-Cl are similar and the same occurs to Tri, T+Cl and T+BW. It means that the activation or deactivation of clique and bicycle wheel does not change the performance of the CPA on these instances.

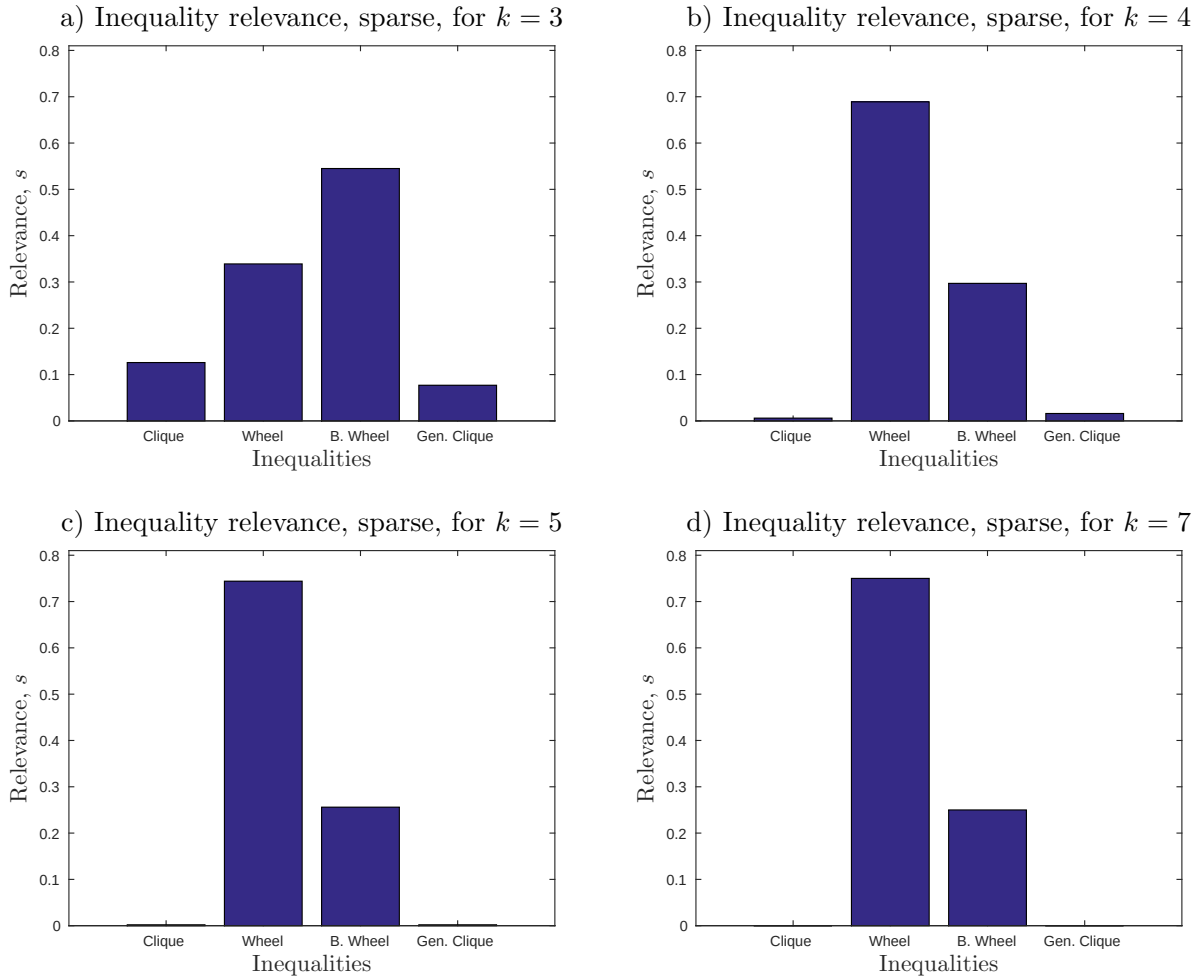


Figure 9: Relevance of the inequalities for sparse instances

Quality versus time:

In Figure 11a) the combinations T+Cl, T+GC and Tri are the fastest. However, they have the worst average performance (higher than 0.8). It can be seen that A-Cl, A-BW, and T+GC are dominated by other combinations and it is also noticeable that algorithms with bicycle wheel have the best average performance.

The results for $k \in \{4, 5, 7\}$ (Figure 11b) c) and d)) are very similar. In those graphs we notice the relevance of the wheel inequality, since all the algorithms using this inequality have average performance lower than 0.2. Furthermore, it can be concluded that T+Wh achieves the best results, because it is the closest to the ideal point.

Finally, we observe that all the combinations spend in average 25% more CPU time for $k = 4$ than for the other partitions.

Relevance of inequalities:

Figure 12 presents the four relevance histograms for dense instances. In Figure 12a) it can be noticed that the bicycle wheel is the most relevant class of inequality followed by wheel and clique.

For $k \in \{4, 5, 7\}$ the results suggests that wheel inequalities are the most relevant. It is also notable that the clique and general clique inequalities are ineffective when the triangle inequalities are activated.

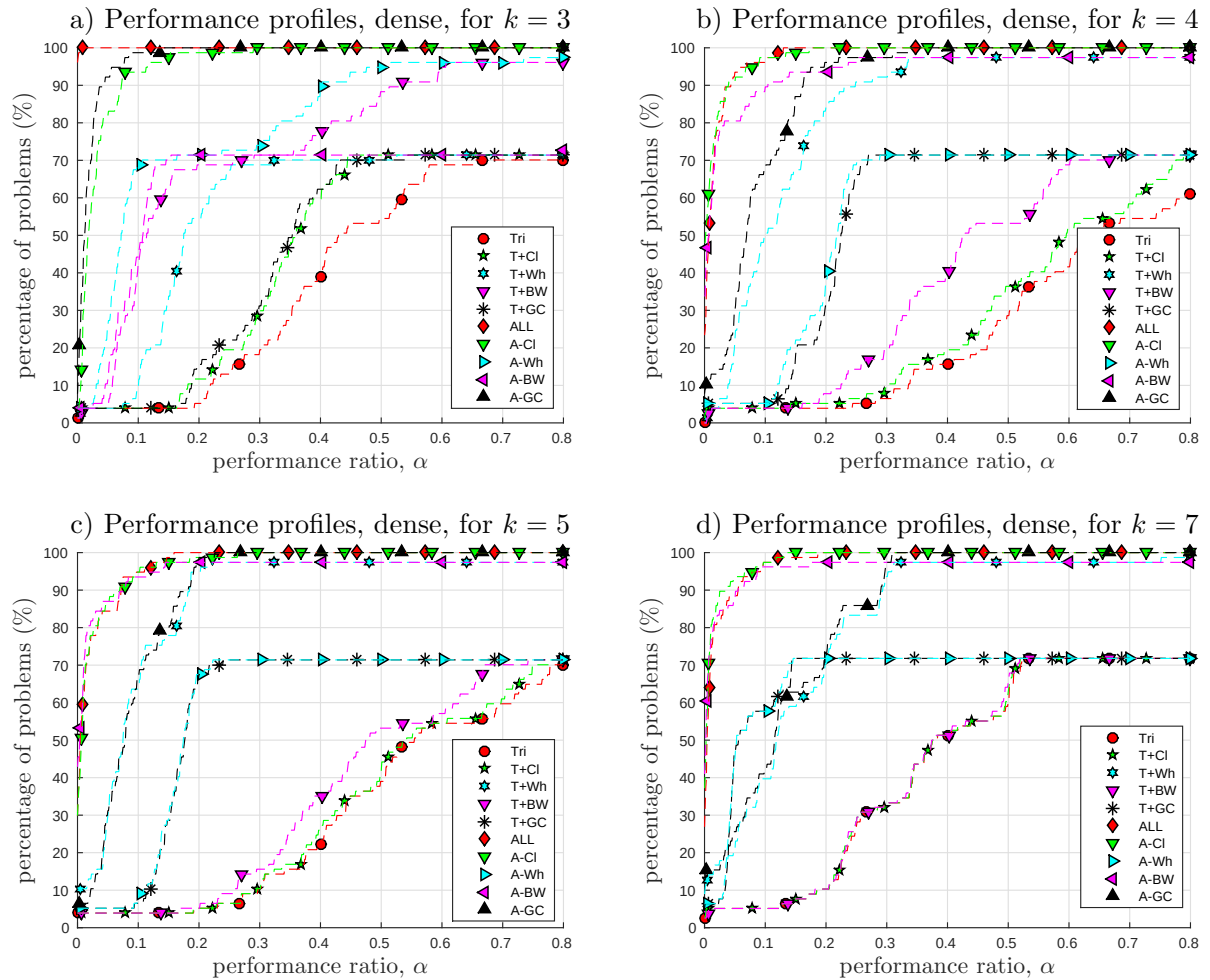


Figure 10: Performance profiles for dense instances

4.4 Summary of the computational tests

While it is normal that the combination **ALL** (activation of all the inequalities) has the best performance, it is also the combination that requires the most CPU time. On the other hand, **Tri** is the fastest combination but delivers the worst performance. The combination **T+Wh** achieves, on average, the best ratio between CPU time and performance for all values of k considered in this study.

Table 2 summarizes the results and shows the best inequalities for each benchmark (performance profile, performance versus time (QvsT) and inequality relevance) for each value of $k \in \{3, 4, 5, 7\}$, and for sparse/dense/all instances. We observe that the strongest inequalities for $k = 3$ are bicycle wheel and wheel, and for $k \in \{4, 5, 7\}$ the wheel inequalities are the strongest.

It is also noticeable that the activation of wheel combined with bicycle wheel (or general clique for dense graphs) improves the performance for all types of partitions. However, it spends in average 72% (32%) more CPU time for the CPA to terminate. Generally, we see that an increase in k causes an increase in the CPU time, especially for the sparse instances.

From a practical point of view, wheel is the most attractive inequality because it shows good performances for all instances and for all types of partitions.

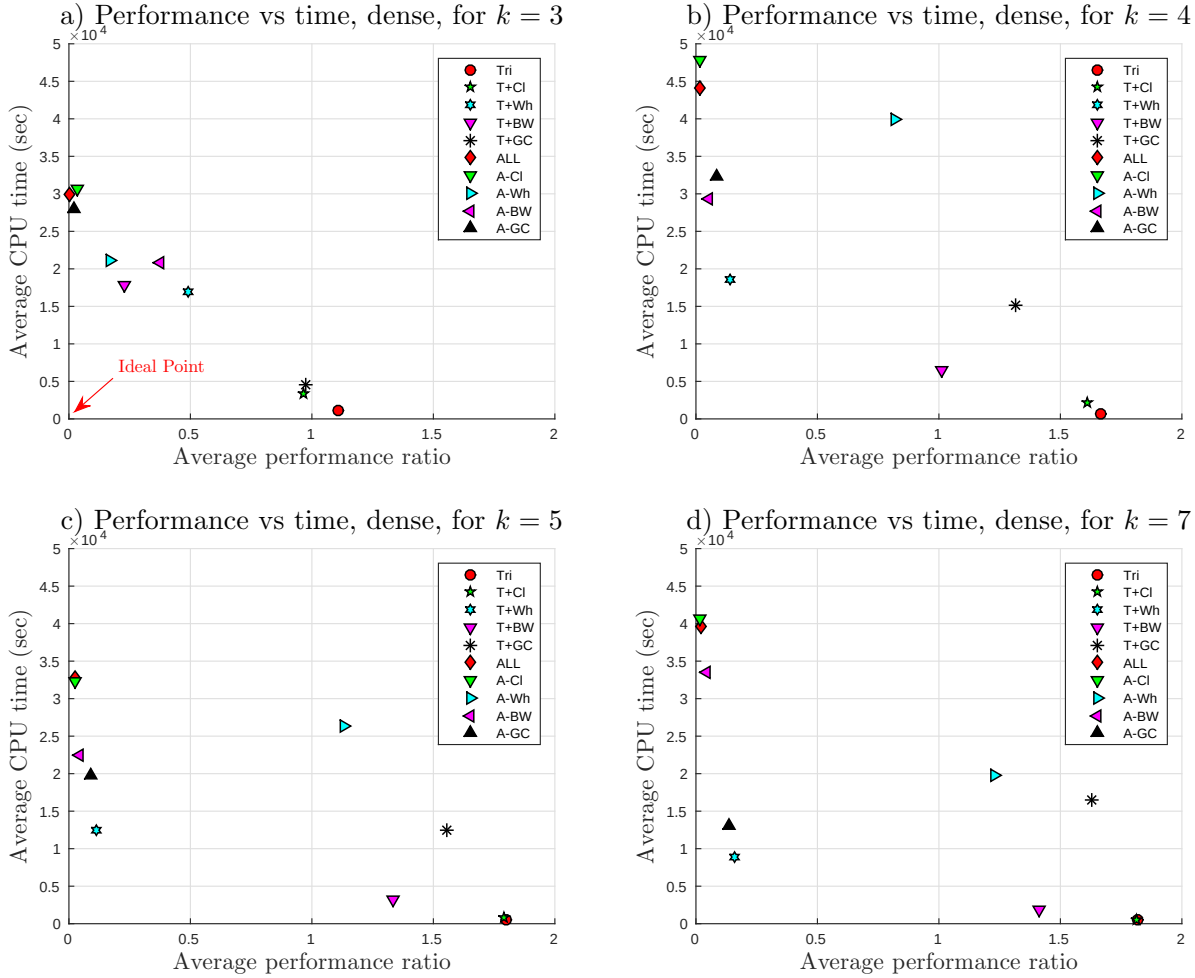


Figure 11: Performance versus CPU time for dense instances

Table 2: Summary of the best inequalities for each benchmark

k	Density	Performance profiles	QvsT	Relevance
3	All	Bicycle Wheel and Wheel	Bicycle Wheel	Bicycle Wheel and Wheel
	Sparse	Bicycle Wheel and Wheel	Bicycle Wheel	Bicycle Wheel and Wheel
	Dense	Bicycle Wheel and Wheel	Bicycle Wheel	Bicycle Wheel and Wheel
4,5,7	All	Wheel and General Clique	Wheel	Wheel
	Sparse	Wheel and Bicycle Wheel	Wheel	Wheel
	Dense	Wheel and General Clique	Wheel	Wheel

5 Discussion

In this paper we carried out an experimental study of several classes of valid inequalities to strengthen the SDP relaxation of the max- k -cut problem. Specifically we considered the following classes of cutting planes: Triangle, Clique, General clique, Wheel and Bicycle wheel. We developed a different separation algorithm for each class. For triangle inequalities we used complete enumeration, for clique, general clique, and wheel inequalities a greedy heuristics was developed, and for bicycle wheel a genetic algorithm was applied. The inequalities were then tested within a standard cutting plane algorithm for $k \in \{3, 4, 5, 7\}$, using nearly 150 instances with varying properties.

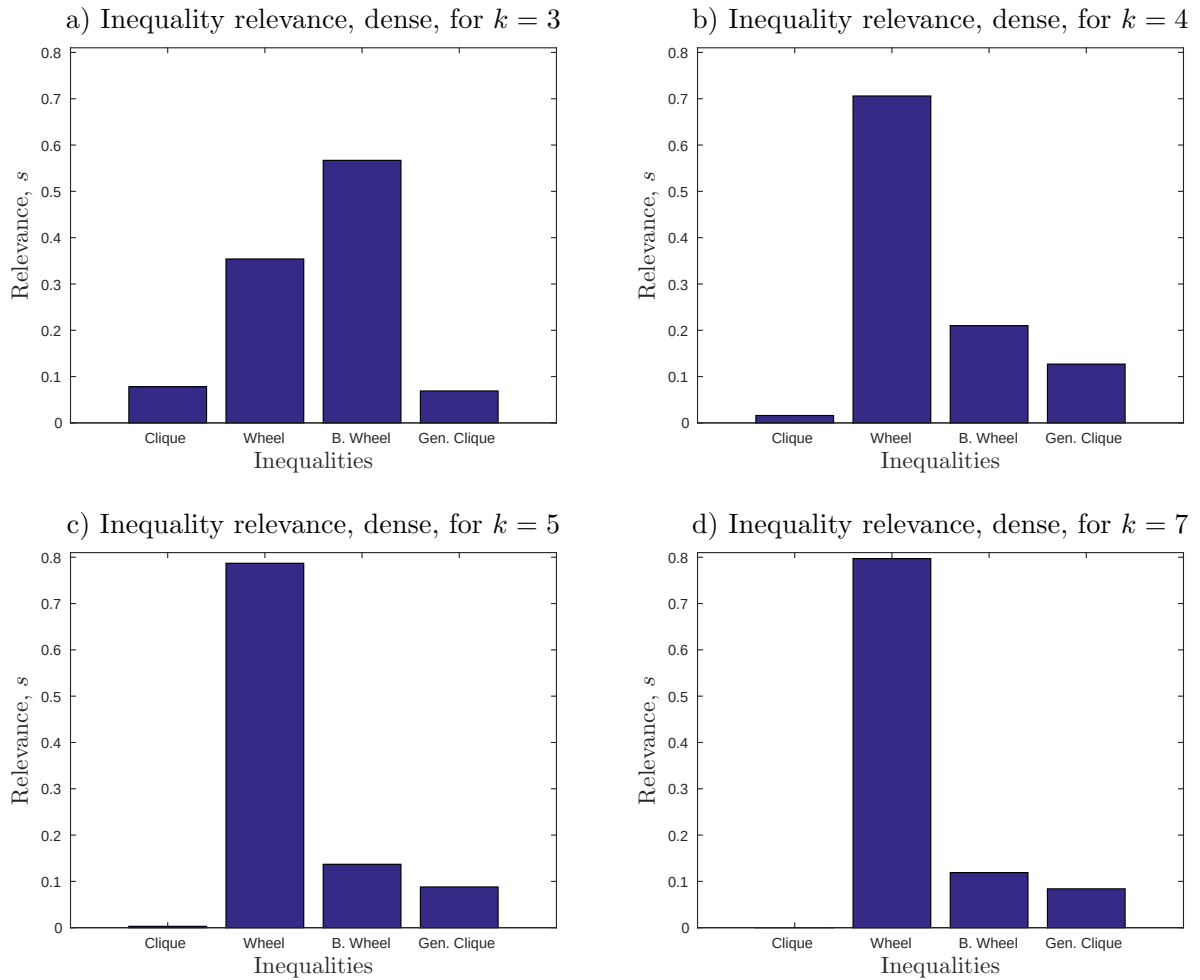


Figure 12: Relevance of the inequalities for dense instances

To investigate the strength of each class of inequalities we created ten combinations by alternating the activation and deactivation of the inequalities. In order to guarantee a fair comparison of the combinations, we analyzed four different benchmarks: performance profiles, data profiles, quality versus time and relevance histograms.

The results are summarized in Table 2. We conclude that *bicycle wheel* is the strongest inequality for $k = 3$ and *wheel* for $k \in \{4, 5, 7\}$. Moreover, the inclusion of *wheel* and *bicycle wheel* inequalities at the same time strengthens even more the SDP relaxation of the max- k -cut at the cost of 72% more CPU time, on average.

In future research we plan to study more efficient algorithms to solve the SDP relaxation with *wheel* and *bicycle wheel* inequalities, and to build a branch-and-cut algorithm to find the integer solution of the max- k -cut problem for both sparse and dense graphs for large instances.

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