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Cross-sectional Dependence in Idiosyncratic Volatility

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Abstract

This paper introduces a framework for analysis of cross-sectional dependence in the idiosyncratic volatilities of assets using high frequency data. We first consider the estimation of standard measures of dependence in the idiosyncratic volatilities such as covariances and correlations. Next, we study an idiosyncratic volatility factor model, in which we decompose the co-movements in idiosyncratic volatilities into two parts: those related to factors such as the market volatility, and the residual co-movements. When using high frequency data, naive estimators of all of the above measures are biased due to the estimation errors in idiosyncratic volatility. We provide bias-corrected estimators and establish their asymptotic properties. We apply our estimators to high-frequency data on 27 individual stocks from nine different sectors, and document strong cross-sectional dependence in their idiosyncratic volatilities. We also find that on average 74% of this dependence can be explained by the market volatility.

Keywords: high frequency data; idiosyncratic volatility; factor structure; cross-sectional returns.

JEL Codes: C22, C14.

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1 Introduction

Volatility of returns of an asset or a portfolio is a key ingredient of traditional empirical asset pricing models. That asset returns have strong cross-sectional correlations is a well documented empirical fact. These correlations arise due to the common risk factors such as the market factor. Due to these return factors, the total stock volatilities are also correlated in the cross-section.

It is then natural to investigate whether the volatilities of the factor-adjusted returns, otherwise known as the Idiosyncratic Volatilities (IVs), also have substantial co-movements. Recent papers by [Herskovic, Kelly, Lustig, and Nieuwerburgh \(2014\)](#) and [Duarte, Kamara, Siegel, and Sun \(2014\)](#) present evidence of considerable cross-sectional co-movements in large low-frequency panels of IVs of daily stock returns. Moreover, they argue that the common co-movements in IVs arise due to a priced risk factor.

We provide a flexible framework for studying cross-sectional dependencies in the IVs using high-frequency data. Our framework incorporates important stylized facts about asset returns and their volatilities. Our estimators provide a solution to the bias problems caused by the pre-estimation of volatilities. We also provide valid inference methods.

First, we study behavior of standard measures of cross-sectional dependence in IVs using high-frequency data. We show that the naive estimators of these measures are biased, and provide bias-corrected estimators. We then obtain the relevant asymptotic distributions, which allow us to perform statistical tests.

Second, we study an idiosyncratic volatility factor model (IV-FM). The IV-FM decomposes the cross-sectional dependence in IVs into two components. The first component is the cross-sectional dependence due to popular factors. The IV factors can include the volatility of the return factors, non-linear transforms of the spot covariance matrices such as correlations, as well as the average variance factor of [Chen and Petkova \(2012\)](#). The second component in the IV-FM is residual dependence in IVs not explained by the IV factors. Again, the standard estimators of this decomposition are biased due to the latency in volatility. We provide bias-corrected estimators, and derive their asymptotic distributions. We build a test for whether the IV-FM can fully account for the dependence between the IVs. Our test could be useful, for example, in assessing the relevance of some classical assumptions made in conditionally heteroscedastic factor models for returns ([Engle, Ng, and Rothschild \(1993\)](#), among others). In particular, a common assumption in this literature restricts the asset's total volatilities to be some linear combination of the volatility of the return factors, which precludes additional factors affecting the IV.¹ We investigate the finite sample properties of the methods in a Monte Carlo experiment, and find they have reasonable size and power.

We apply our estimators to high-frequency data on 27 individual US stocks from nine different sectors. We study idiosyncratic volatilities with respect to two models, CAPM and the three-factor Fama-French model. In both cases, the average correlation between the idiosyncratic volatilities is above 0.55. Moreover, the average correlation between the IVs is on average the same among those pairs of stocks, which have close to zero correlations between their idiosyncratic returns. In other words, the dependencies in IVs cannot be explained by a missing return factor. This is in line with the recent findings of [Herskovic, Kelly, Lustig, and](#)

¹This condition arises when one assumes the latent factors are uncorrelated.

[Nieuwerburgh \(2014\)](#) who use daily and monthly return data. We then consider the idiosyncratic volatility factor model with market volatility as the factor. We find that on average, the systemic component of IV that arises due to IV exposure to market volatility, accounts for 74% of the cross-sectional dependence in the IVs. We find that in 110 out of 351 pairs of stocks analyzed, this idiosyncratic volatility factor model fully accounts for the cross-sectional dependence in IVs, so that their non-systemic components are no longer significantly correlated.

The importance of accounting for estimation errors in volatilities has been demonstrated in other contexts. Recently, [Aït-Sahalia, Fan, and Li \(2013\)](#) show that failure to account for the latency of volatility drives the leverage effect puzzle.² An important aspect of our methods is that we fully account for the latency of IV.

We now describe the theoretical framework and the strategy for constructing our estimators and tests. We define the IV with respect to a continuous-time factor model for returns with observable factors. This framework was originally studied in [Mykland and Zhang \(2006\)](#) in the case of one factor and in the absence of jumps. It was extended to multiple factors and jumps in [Aït-Sahalia, Kalnina, and Xiu \(2014\)](#). We measure the dependence between two IV processes as their quadratic covariation (or quadratic covariation-based correlation). In the case of co-movements in returns, standard estimators of quadratic covariation are unbiased (see ? and [Barndorff-Nielsen and Shephard \(2004\)](#)). However, since we study co-movements in latent IVs and not in observable returns, naive estimators are biased, see, e.g., [Wang and Mykland \(2014\)](#) and [Vetter \(2012\)](#) for related results. We then go further and consider an IV factor model (IV-FM), which decomposes the total IV into a systemic (or common) IV and the non-systemic IV. The IV-FM allows us to quantify what portion of the cross-sectional dependence in IVs is driven by the IV factors. We show that the IV dependence measures (and their ratios, scaled versions, or other functions) can be identified and estimated in a unified way. In particular, all of them can be written as smooth known functions of multiple quantities, each representing the quadratic covariation between possibly non-linear transforms of the spot covariance matrix of the vector of observable processes. Our main statistical contribution (Theorem 1) is derivation of the joint asymptotic distribution for the bias-corrected estimators of these quantities. By the delta method, we obtain the asymptotic distributions of all quantities of interest. The resulting asymptotic distributions allow us to conduct various statistical tests; for example, to test whether the IV factor model can fully explain the cross-sectional dependence in the IVs.

Our paper is related to several strands of the literature. Our inference theory extends the results on estimation of the integrated one-dimensional (total) volatility of volatility ([Vetter \(2012\)](#), [Aït-Sahalia and Jacod \(2014\)](#)). The (total) leverage effect is also a quantity, for which the naive nonparametric estimators are inconsistent due to the measurement errors in volatilities, see [Wang and Mykland \(2014\)](#), [Kalnina and Xiu \(2014\)](#), [Aït-Sahalia, Fan, Laeven, Wang, and Yang \(2013\)](#) and [Aït-Sahalia, Fan, and Li \(2013\)](#) for one-dimensional results. Due to the decomposition of total returns into systemic and idiosyncratic part, our estimators involve aggregation of non-linear functionals of the return volatility matrix, hence our bias-correction terms are related to the general theory developed in [Jacod and Rosenbaum \(2012\)](#) and [Jacod and](#)

²See [Wang and Mykland \(2014\)](#), [Kalnina and Xiu \(2014\)](#), and [Aït-Sahalia, Fan, Laeven, Wang, and Yang \(2013\)](#) for related results on the leverage effect.

Rosenbaum (2013). The decomposition of total volatility into systemic and idiosyncratic volatilities is also considered in Mykland and Zhang (2006) and Aït-Sahalia, Kalnina, and Xiu (2014), but their theoretical results only consider univariate idiosyncratic volatility. Barigozzi, Brownlees, Gallo, and Veredas (2014) and Luciani and Veredas (2012) model large panels of total stock volatilities using high frequency data.

The remainder of the paper is organized as follows. Section 2 introduces the model and describes quantities of interest. Section 3 describes the identification and estimation of these quantities of interest. Section 4 presents the asymptotic properties of our estimators. Section 5 investigates their finite sample properties. Section 6 uses high-frequency stock return data to study the cross-sectional dependence in IVs using our framework. Section 7 concludes. The Appendix contains the proofs.

2 Model and Quantities of Interest

We first describe a general factor model for the returns, in which the idiosyncratic volatility is defined. We then introduce the idiosyncratic volatility factor model (IV-FM).

Suppose we have (log) prices on d_S assets such as stocks and on d_F factors. We stack them into the d -dimensional process $Y_t = (S_{1,t}, \dots, S_{d_S,t}, F_{1,t}, \dots, F_{d_F,t})^\top$ where $d = d_S + d_F$. We assume Y_t follows an Itô semimartingale,

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t,$$

where W is a d' -dimensional Brownian motion ($d' \geq d$), σ_s is a $d \times d'$ stochastic volatility process, and J_t denotes a finite variation jump process. We assume also that the spot variance process $c_t = \sigma_t \sigma_t^\top$ of Y_t is a continuous Itô semimartingale,³

$$c_t = c_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s, \quad (1)$$

see Section 4 for the full list of assumptions.

We assume a standard continuous-time factor model for the (log) prices of the assets:

Definition (Factor Model for Prices). For all $0 \leq t \leq T$ and $j = 1, \dots, d_S$,⁴

$$\begin{aligned} dY_{j,t}^c &= \beta_{j,t}^\top dF_t^c + dZ_{j,t}^c \quad \text{with} \\ [Z_j^c, F^c]_t &= 0, \end{aligned} \quad (2)$$

We do not need the factors F_t to be the same across assets to identify the model, but without loss of generality, we keep this structure because it is standard in empirical finance. In the empirical application,

³Note that assuming that Y and c are driven by the same d' -dimensional Brownian motion W is without loss of generality provided that d' is large enough, see, e.g., equation (8.12) of Aït-Sahalia and Jacod (2014).

⁴If X and Y are two vector-valued Itô semimartingales, their quadratic covariation over the time span $[0, T]$ is defined

$$[X, Y]_T = p\text{-}\lim_{M \rightarrow \infty} \sum_{j=0}^{M-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j})^\top,$$

for any sequence $t_0 < t_1 < \dots < t_M = T$ with $\sup_j \{t_{j+1} - t_j\} \rightarrow 0$ as $M \rightarrow \infty$, where p-lim stands for the probability limit. Barndorff-Nielsen and Shephard (2004) discuss its estimation when both X and Y are observed.

we use two sets of factors: the market portfolio and the three Fama-French factors. The process $Z_{j,t}$ is the idiosyncratic component of the price of the j^{th} stock with respect to the factors. We use the superscript c to emphasize that the above factor model only involves the continuous martingale parts of the two observable processes $Y_{j,t}$ and F_t . The jump parts of these processes are left unrestricted. For $j = 1, \dots, d_S$, $\beta_{j,t}$ is a $\mathbb{R}^{d_F}$ -valued process which represents the continuous beta.⁵ The k -th component of $\beta_{j,t}$ captures the time-varying sensitivity of the continuous part of the return on stock j to the continuous part of the return on the k -th factor. We set $\beta_t = (\beta_{1,t}, \dots, \beta_{d_S,t})^\top$ and $Z_t = (Z_{1,t}, \dots, Z_{d_S,t})^\top$. This framework was originally studied in Mykland and Zhang (2006) in the case of one factor and in the absence of jumps. It was extended to multiple factors and jumps in Ait-Sahalia, Kalnina, and Xiu (2014). See also Li, Todorov, and Tauchen (2013), ?, and Reiß, Todorov, and Tauchen (2015). Our framework can be potentially extended to use principal components instead of observable factors as in ?.

Idiosyncratic Volatility of stock j is the spot volatility of the residual process Z_j , and we denote it by c_{Zj} . It can be written as

$$c_{Zj,t} = c_{Yj,t} - (c_{Fj,t})^\top (c_{FF,t})^{-1} c_{Fj,t}, \quad (3)$$

for $j = 1, \dots, d_S$, where $c_{FF,t}$ denotes the spot covariance of the factors F , which is the lower $d_F \times d_F$ sub-matrix of c_t , and $c_{Fj,t}$ denotes the covariance of the factors and the j^{th} stock, which are the last d_F elements of the j^{th} column of c_t .

We take the following quadratic-covariation based quantity as the natural measure of dependence between the IV shocks of stocks i and j ,

$$\rho_{Zi,Zj} = \frac{[c_{Zi}, c_{Zj}]_T}{\sqrt{[c_{Zi}, c_{Zi}]_T} \sqrt{[c_{Zj}, c_{Zj}]_T}}. \quad (4)$$

Alternatively, one can consider the raw quadratic covariation $[c_{Zi}, c_{Zj}]_T$. We use it later as the basis for testing the presence of dependence in IVs.

To assess the importance of factors in driving the IV dependence, we introduce the Idiosyncratic Volatility Factor model. The IV factors can include the volatility of the return factors, non-linear transforms of the spot covariance matrices such as correlations, as well as the average variance factor of Chen and Petkova (2012).

Definition (Idiosyncratic Volatility Factor Model, IV-FM). For all $0 \leq t \leq T$ and $j = 1, \dots, d_S$, the idiosyncratic volatility c_{Zj} follows,

$$dc_{Zj,t} = b_{Zj}^\top dq_t + dc_{Zj,t}^{NS} \quad \text{with} \quad (5)$$

$$[c_{Zj}^{NS}, q]_t = 0. \quad (6)$$

where q_t is the vector of IV factors, which a sub-vector of $\text{vech}(c_t)$, and where $c_{Zj,t}^{NS}$ is the non-systemic idiosyncratic volatility.

We refer to b_{Zj} as the idiosyncratic volatility beta (IV beta). It is time-invariant. The residual component

⁵ Interestingly it is possible to define a discontinuous beta, see Bollerslev and Todorov (2010) and Li, Todorov, and Tauchen (2014).

$c_{Zj,t}^{NS}$ denotes j^{th} asset's non-systematic idiosyncratic volatility (NS-IV henceforth). In the IV-FM, both the regressand and the regressor are latent, and the components of the IV-FM, $c_{Zj,t}$, q_t and $c_{Zj,t}^{NS}$, are continuous Itô semimartingales.

To measure the residual cross-sectional dependence between two IVs after accounting for the effect of the IV factors, we use a quadratic-covariation based correlation measure between NS-IVs,

$$\rho_{Zi,Zj}^{NS} = \frac{[c_{Zi}^{NS}, c_{Zj}^{NS}]_T}{\sqrt{[c_{Zi}^{NS}, c_{Zi}^{NS}]_T} \sqrt{[c_{Zj}^{NS}, c_{Zj}^{NS}]_T}}. \quad (7)$$

When testing for the presence of residual correlation between NS-IVs, we use the quadratic covariation $[c_{Zi}^{NS}, c_{Zj}^{NS}]_T$ without normalization.

We want to capture how well the IV factors explain the time variation of j^{th} IV. For this purpose, we use the quadratic-covariation based analog of the coefficient of determination. For $j = 1, \dots, d_S$,

$$R_{Zj}^{2,IV-FM} = \frac{b_{Zj}^\top [q, q]_T b_{Zj}}{[c_{Zj}, c_{Zj}]_T}. \quad (8)$$

It is interesting to compare the correlation measure between IVs in equation (4) with the correlation between the non-systemic parts of IVs in (7). We consider their difference,

$$\rho_{Zi,Zj} - \rho_{Zi,Zj}^{NS}, \quad (9)$$

to see how much of the dependence between IVs can be attributed to the IV factors. In practice, if we compare assets that are known to have positive covolatilities (typically, stocks have that property), another useful measure of the systemic part in the overall covariation between IVs is the following quantity,

$$Q_{Zi,Zj}^{IV-FM} = \frac{b_{Zi}^\top [q, q]_T b_{Zj}}{[c_{Zi}, c_{Zj}]_T}. \quad (10)$$

This measure is bounded by 1 if the covariations between NS-IVs are nonnegative and smaller than the covariations between IVs, which is what we find for every pair in our empirical application with high-frequency observations on stock returns.

The next section outlines identification and estimation of the above key quantities. It also presents the asymptotic distributions, which can be used to conduct statistical tests. We conduct three tests. First, we test whether the total cross-correlation in IVs is nonzero for a given pair of assets, which corresponds to the hypothesis $[c_{Zi}, c_{Zj}]_T = 0$. Second, we test whether the IV factors contribute to the cross-correlation in IVs by considering the null hypothesis $[c_{Zj}, q]_T = 0$. Third, we test the hypothesis of whether IV-FM can explain all the cross-sectional IV dependence, i.e., $[c_{Zi}^{NS}, c_{Zj}^{NS}]_T = 0$.

It is interesting to compare our framework with the following null hypothesis studied in [Li, Todorov, and Tauchen \(2013\)](#), $H_0 : c_{Zj,t} = a_{Zj} + b_{Zj}^\top q_t$, $0 \leq t \leq T$. This H_0 implies that the IV is a deterministic function of the factors, which does not allow for a non-systemic error term. In particular, this null hypothesis implies $R_{Zj}^{2,IV-FM} = 1$.

3 Estimation

We now discuss the estimation of our main quantities of interest introduced in Section 2,

$$[c_{Zi}, c_{Zj}]_T, \rho_{Zi, Zj}, [c_{Zj}^{NS}, c_{Zj}^{NS}]_T, \rho_{Zi, Zj}^{NS}, Q_{Zi, Zj}^{IV-FM}, \text{ and } R_{Zi}^{2, IV-FM}, \quad (11)$$

for $i, j = 1, \dots, d_S$. We first show that each of them can be written as $\varphi([H_1(c), G_1(c)]_T, \dots, [H_\kappa(c), G_\kappa(c)]_T)$ where φ as well as H_r and G_r , for $r = 1, \dots, \kappa$, are known real-valued functions. Each element in this expression is of the form $[H(c), G(c)]_T$, i.e., it is a quadratic covariation between functions of c_t . We then show how to estimate $[H(c), G(c)]_T$.

First, consider the quadratic covariation between i^{th} and j^{th} IV, $[c_{Zi}, c_{Zj}]_T$. It can be written as $[H(c), G(c)]_T$ if we choose $H(c_t) = c_{Zi,t}$ and $G(c_t) = c_{Zj,t}$. By (3), both $c_{Zi,t}$ and $c_{Zj,t}$ are smooth functions of c_t . Next, consider the correlation $\rho_{Zi, Zj}$ defined in (4). By the argument above, its numerator and each of the two components in the denominator can be written as $[H(c), G(c)]_T$ for different functions H and G . Therefore, $\rho_{Zi, Zj}$ is itself a known smooth function of three objects of the form $[H(c), G(c)]_T$.

To show that the remaining quantities in (11) can also be expressed in terms of objects of the form $[H(c), G(c)]_T$, note that the IV-FM implies

$$b_{Zj} = ([q, q]_T)^{-1} [q, c_{Zj}]_T \text{ and } [c_{Zi}^{NS}, c_{Zj}^{NS}]_T = [c_{Zi}, c_{Zj}]_T - b_{Zi}^\top [q, q]_T b_{Zj},$$

for $i, j = 1, \dots, d_S$. Since $c_{Zi,t}$, $c_{Zj,t}$ and every element in q_t are real-valued functions of c_t , the above equalities imply that all quantities of interest in (11) can be written as real-valued, known smooth functions of a finite number of quantities of the form $[H(c), G(c)]_T$.

To estimate $[H(c), G(c)]_T$, suppose we have discrete observations on Y_t over an interval $[0, T]$. Denote by Δ_n the distance between observations. Note that we can estimate the spot covariance matrix c_t at time $(i-1)\Delta_n$ with a local truncated realized volatility estimator (Mancini (2001)),

$$\widehat{c}_{i\Delta_n} = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} (\Delta_{i+j}^n Y)(\Delta_{i+j}^n Y)^\top 1_{\{\|\Delta_{i+j}^n Y\| \leq \chi \Delta_n^\varpi\}}, \quad (12)$$

where $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ and where k_n is the number of observations in a local window.⁶

We propose two estimators for the general quantity $[H(c), G(c)]_T$. The first is based on the analog of the definition of quadratic covariation between two Itô processes,

$$\begin{aligned} [\widehat{H(c)}, \widehat{G(c)}]_T^{AN} &= \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left(\left(H(\widehat{c}_{(i+k_n)\Delta_n}) - H(\widehat{c}_{i\Delta_n}) \right) \left(G(\widehat{c}_{(i+k_n)\Delta_n}) - G(\widehat{c}_{i\Delta_n}) \right) \right. \\ &\quad \left. - \frac{2}{k_n} \sum_{g,h,a,b=1}^d (\partial_{gh} H \partial_{ab} G)(\widehat{c}_{i\Delta_n}) \left(\widehat{c}_{ga,i\Delta_n} \widehat{c}_{gb,i\Delta_n} + \widehat{c}_{gb,i\Delta_n} \widehat{c}_{ha,i\Delta_n} \right) \right), \end{aligned} \quad (13)$$

where the factor 3/2 and last term correct for the biases arising due to the estimation of volatility c_t . The increments used in the above expression are computed over overlapping blocks, which results in a smaller asymptotic variance compared to the version using non-overlapping blocks.

⁶It is also possible to define kernel-based estimators as in Kristensen (2010).

Our second estimator is based on the following equality, which follows by the Itô lemma,

$$[H(c), G(c)]_T = \sum_{g,h,a,b=1}^d \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) \bar{c}_t^{gh,ab} dt, \quad (14)$$

where $\bar{c}_t^{gh,ab}$ denotes the covariation between the volatility processes $c_{gh,t}$ and $c_{ab,t}$. The quantity is thus a non-linear functional of the spot covariance and spot volatility of volatility matrices. Our second estimator is based on this “linearized” expression,

$$\begin{aligned} [H(\widehat{c}), G(\widehat{c})]_T^{LIN} &= \frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial_{gh} H \partial_{ab} G)(\widehat{c}_{i\Delta_n}) \left((\widehat{c}_{gh,(i+k_n)\Delta_n} - \widehat{c}_{gh,i\Delta_n})(\widehat{c}_{ab,(i+k_n)\Delta_n} - \widehat{c}_{ab,i\Delta_n}) \right. \\ &\quad \left. - \frac{2}{k_n} (\widehat{c}_{ga,i\Delta_n} \widehat{c}_{gb,i\Delta_n} + \widehat{c}_{gb,i\Delta_n} \widehat{c}_{ha,i\Delta_n}) \right). \end{aligned} \quad (15)$$

Consistency for a similar estimator has been established by [Jacod and Rosenbaum \(2012\)](#).⁷ We go beyond their result by deriving the asymptotic distribution and proposing a consistent estimator of its asymptotic variance.

Note that the same additive bias-correcting term,

$$- \frac{3}{k_n^2} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left(\sum_{g,h,a,b=1}^d (\partial_{gh} H \partial_{ab} G)(\widehat{c}_{i\Delta_n}) (\widehat{c}_{ga,i\Delta_n} \widehat{c}_{gb,i\Delta_n} + \widehat{c}_{gb,i\Delta_n} \widehat{c}_{ha,i\Delta_n}) \right), \quad (16)$$

is used for the two estimators. This term is (up to a scale factor) an estimator of the asymptotic covariance between the sampling errors embedded in estimators of $\int_0^T H(c_t) dt$ and $\int_0^T G(c_t) dt$ defined in [Jacod and Rosenbaum \(2013\)](#).

The two estimators are identical when H and G are linear, for example, when estimating the covariation between two volatility processes. In the univariate case $d = k = 1$, our estimator coincides with the volatility of volatility estimator of [Vetter \(2012\)](#), which was extended to allow for jumps in [Jacod and Rosenbaum \(2012\)](#). Our contribution is the extension of this theory to the multivariate $k > 1$ and/or $d > 1$ case with nonlinear functionals.

4 Asymptotic Properties

We start by outlining the full list of assumptions for our asymptotic results. We then state the asymptotic distribution for the general functionals introduced in the previous section, and develop estimators for the asymptotic variance. Finally, we outline three statistical tests of interest that follow from our theoretical results.

⁷[Jacod and Rosenbaum \(2012\)](#) derive the probability limit of the following estimator:

$$\frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial_{gh,ab}^2 H)(\widehat{c}_{i\Delta_n}) \left((\widehat{c}_{(i+k_n)\Delta_n} - \widehat{c}_{i\Delta_n})(\widehat{c}_{(i+k_n)\Delta_n} - \widehat{c}_{i\Delta_n}) - \frac{2}{k_n} (\widehat{c}_{ga,i\Delta_n} \widehat{c}_{gb,i\Delta_n} + \widehat{c}_{gb,i\Delta_n} \widehat{c}_{ha,i\Delta_n}) \right).$$

4.1 Assumptions

Recall that the d -dimensional process Y_t represents the (log) prices of stocks and factors.

Assumption 1. Suppose Y is an Itô semimartingale on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$,

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_E \delta(s, z) \mu(ds, dz),$$

where W is a d' -dimensional Brownian motion ($d' \geq d$) and μ is a Poisson random measure on $\mathbb{R}_+ \times E$, with E an auxiliary Polish space with intensity measure $\nu(dt, dz) = dt \otimes \lambda(dz)$ for some σ -finite measure λ on E . The process b_t is $\mathbb{R}^d$ -valued optional, σ_t is $\mathbb{R}^d \times \mathbb{R}^{d'}$ -valued, and $\delta = \delta(w, t, z)$ is a predictable $\mathbb{R}^d$ -valued function on $\Omega \times \mathbb{R}_+ \times E$. Moreover, $\|\delta(w, t \wedge \tau_m(w), z)\| \wedge 1 \leq \Gamma_m(z)$, for all (w, t, z) , where (τ_m) is a localizing sequence of stopping times and, for some $r \in [0, 1]$, the function Γ_m on E satisfies $\int_E \Gamma_m(z)^r \lambda(dz) < \infty$. The spot volatility matrix of Y is then defined as $c_t = \sigma_t \sigma_t^\top$. We assume that c_t is a continuous Itô semimartingale,⁸

$$c_t = c_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s. \quad (17)$$

With the above notation, the elements of the spot volatility of volatility matrix and spot covariation of the continuous martingale parts of X and c are defined as follows,

$$\bar{c}_t^{gh,ab} = \sum_{m=1}^{d'} \tilde{\sigma}_t^{gh,m} \tilde{\sigma}_t^{ab,m}, \quad \bar{c}_t'^{g,ab} = \sum_{m=1}^{d'} \sigma_t^{gm} \tilde{\sigma}_t^{ab,m}. \quad (18)$$

The process $\tilde{\sigma}_t$ is restricted as follows:

Assumption 2. $\tilde{\sigma}_t$ is a continuous Itô semimartingale with its characteristics satisfying the same requirements as that of c_t .

Assumption 1 is very general and nests most of the multivariate continuous-time models used in economics and finance. It allows for potential stochastic volatility and jumps in prices. Assumption 2 is required to obtain the asymptotic distribution of estimators of the quadratic covariation between functionals of the spot covariance matrix c_t . It is not needed to prove consistency. This restriction also appears in [Vetter \(2012\)](#), [Kalnina and Xiu \(2014\)](#) and [Wang and Mykland \(2014\)](#).

4.2 Asymptotic Distribution

We have seen in Section 3 that all quantities of interest in (11) are functions of multiple objects of the form $[H(c), G(c)]_T$. Therefore, if we can obtain a multivariate asymptotic distribution for a vector with elements of the form $[H(c), G(c)]_T$, the asymptotic distributions for all our estimators follow by the delta method. Presenting this asymptotic distribution is the purpose of the current section.

We first specify the smoothness restrictions on the functions H and G . For this purpose, we denote by $\mathcal{G}(p)$ the set of real-valued functions H that satisfy the following polynomial growth condition. For $p \geq 3$,

$$\mathcal{G}(p) = \{H : H \text{ is three-times continuously differentiable and for some } K > 0,$$

⁸Note that $\tilde{\sigma}_s = (\tilde{\sigma}_s^{gh,m})$ is $(d \times d \times d')$ -dimensional and $\tilde{\sigma}_s dW_s$ is $(d \times d)$ -dimensional with $(\tilde{\sigma}_s dW_s)^{gh} = \sum_{m=1}^{d'} \tilde{\sigma}_s^{gh,m} dW_s^m$.

$$\|\partial^j H(x)\| \leq K(1 + \|x\|)^{p-j}, j = 0, 1, 2, 3\}.$$

The same assumption is used in other contexts by [Jacod and Rosenbaum \(2012\)](#), [Jacod and Rosenbaum \(2013\)](#), and [Li, Todorov, and Tauchen \(2013\)](#).

Let $H_1, G_1, \dots, H_\kappa, G_\kappa$ be some arbitrary elements of $\mathcal{G}(p)$. We are interested in the asymptotic behavior of vectors $\left([H_1(\widehat{c}), \widehat{G}_1(c)]_T^{AN}, \dots, [H_\kappa(\widehat{c}), \widehat{G}_\kappa(c)]_T^{AN}\right)^\top$ and $\left([H_1(\widehat{c}), \widehat{G}_1(c)]_T^{LIN}, \dots, [H_\kappa(\widehat{c}), \widehat{G}_\kappa(c)]_T^{LIN}\right)^\top$. The following theorem summarizes the joint asymptotic behavior of the estimators.

Theorem 1. *Let $[H_r(\widehat{c}), \widehat{G}_r(c)]_T$ be either $[H_r(\widehat{c}), \widehat{G}_r(c)]_T^{AN}$ or $[H_r(\widehat{c}), \widehat{G}_r(c)]_T^{LIN}$ defined in (13) and (15), respectively. Suppose Assumption 1 and Assumption 2 hold. Fix $k_n = \theta \Delta_n^{-1/2}$ and set $(8p-1)/4(4p-r) \leq \varpi < \frac{1}{2}$. Then, as $\Delta_n \rightarrow 0$,*

$$\Delta_n^{-1/4} \begin{pmatrix} [H_1(\widehat{c}), \widehat{G}_1(c)]_T - [H_1(c), G_1(c)]_T \\ \vdots \\ [H_\kappa(\widehat{c}), \widehat{G}_\kappa(c)]_T - [H_\kappa(c), G_\kappa(c)]_T \end{pmatrix} \xrightarrow{L-\xi} MN(0, \Sigma_T), \quad (19)$$

where $\Sigma_T = \left(\Sigma_T^{r,s}\right)_{1 \leq r,s \leq q}$ denotes the asymptotic covariance between the estimators $[H_r(\widehat{c}), \widehat{G}_r(c)]_T$ and $[H_s(\widehat{c}), \widehat{G}_s(c)]_T$. The elements of the matrix Σ_T are

$$\begin{aligned} \Sigma_T^{r,s} &= \Sigma_T^{r,s,(1)} + \Sigma_T^{r,s,(2)} + \Sigma_T^{r,s,(3)}, \\ \Sigma_T^{r,s,(1)} &= \frac{6}{\theta^3} \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_s)) \left[C_t(gh, jk) C_t(ab, lm) \right. \\ &\quad \left. + C_t(ab, jk) C_t(gh, lm) \right] dt, \\ \Sigma_T^{r,s,(2)} &= \frac{151\theta}{140} \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_t)) \left[\bar{c}_t^{gh,jk} \bar{c}_t^{ab,lm} + \bar{c}_t^{ab,jk} \bar{c}_t^{gh,lm} \right] dt, \\ \Sigma_T^{r,s,(3)} &= \frac{3}{2\theta} \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_t)) \left[C_t(gh, jk) \bar{c}_t^{ab,lm} + C_t(ab, lm) \bar{c}_t^{gh,jk} \right. \\ &\quad \left. + C_t(gh, lm) \bar{c}_t^{ab,jk} + C_t(ab, jk) \bar{c}_t^{gh,lm} \right] dt, \end{aligned}$$

with

$$C_t(gh, jk) = c_{gj,t} c_{hk,t} + c_{gk,t} c_{hj,t}.$$

The convergence in Theorem 1 is stable in law (denoted L -s, see for example [Aldous and Eagleson \(1978\)](#) and [Jacod and Protter \(2012\)](#)). The limit is mixed gaussian and the precision of the estimators depends on the paths of the spot covariance and the volatility of volatility process. The rate of convergence $\Delta_n^{-1/4}$ has been shown to be the optimal for volatility of volatility estimation (under the assumption of no volatility jumps).

The asymptotic variance of the estimators depends on the tuning parameter θ whose choice may be crucial for the reliability of the inference. We document the sensitivity of the inference theory to the choice of the parameter θ in a Monte Carlo experiment (see Section 5).

4.3 Estimation of the Asymptotic Covariance Matrix

To provide a consistent estimator for the element $\Sigma_T^{r,s}$ of the asymptotic covariance matrix in Theorem 1, we introduce the following quantities:

$$\begin{aligned}\widehat{\Omega}_T^{r,s,(1)} &= \Delta_n \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{c}_i^n)) \left[\widehat{C}_{i\Delta_n}(gh, jk) \widehat{C}_{i\Delta_n}(ab, lm) \right. \\ &\quad \left. + \widehat{C}_{i\Delta_n}(ab, jk) \widehat{C}_{i\Delta_n}(gh, lm) \right], \\ \widehat{\Omega}_T^{r,s,(2)} &= \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{c}_i^n)) \left[\frac{1}{2} \widehat{\gamma}_i^{n,gh} \widehat{\gamma}_i^{n,jk} \widehat{\gamma}_{i+2k_n}^{n,ab} \widehat{\gamma}_{i+2k_n}^{n,lm} + \right. \\ &\quad \left. \frac{1}{2} \widehat{\gamma}_i^{n,ab} \widehat{\gamma}_i^{n,lm} \widehat{\gamma}_{i+2k_n}^{n,gh} \widehat{\gamma}_{i+2k_n}^{n,jk} + \frac{1}{2} \widehat{\gamma}_i^{n,ab} \widehat{\gamma}_i^{n,jk} \widehat{\gamma}_{i+2k_n}^{n,gh} \widehat{\gamma}_{i+2k_n}^{n,lm} + \frac{1}{2} \widehat{\gamma}_i^{n,gh} \widehat{\gamma}_i^{n,lm} \widehat{\gamma}_{i+2k_n}^{n,ab} \widehat{\gamma}_{i+2k_n}^{n,jk} \right], \\ \widehat{\Omega}_T^{r,s,(3)} &= \frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{c}_i^n)) \times \\ &\quad \left[\widehat{C}_{i\Delta_n}(gh, jk) \widehat{\gamma}_i^{n,ab} \widehat{\gamma}_i^{n,lm} + \widehat{C}_{i\Delta_n}(ab, lm) \widehat{\gamma}_i^{n,gh} \widehat{\gamma}_i^{n,jk} + \widehat{C}_{i\Delta_n}(gh, lm) \widehat{\gamma}_i^{n,ab} \widehat{\gamma}_i^{n,jk} + (\widehat{C}_{i\Delta_n}(ab, jk) \widehat{\gamma}_i^{n,gh} \widehat{\gamma}_i^{n,lm}) \right],\end{aligned}$$

with $\widehat{\gamma}_i^{n,jk} = \widehat{c}_{i+k_n}^{n,jk} - \widehat{c}_i^{n,jk}$ and $\widehat{C}_{i\Delta_n}(gh, jk) = (\widehat{c}_{gj,i\Delta_n} \widehat{c}_{hk,i\Delta_n} + \widehat{c}_{gk,i\Delta_n} \widehat{c}_{hj,i\Delta_n})$.

The following result holds,

Theorem 2. Suppose the assumptions of Theorem 1 hold, then, as $\Delta_n \rightarrow 0$

$$\begin{aligned}\frac{6}{\theta^3} \widehat{\Omega}_T^{r,s,(1)} &\xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(1)} \\ \frac{3}{2\theta} [\widehat{\Omega}_T^{r,s,(3)} - \frac{6}{\theta} \widehat{\Omega}_T^{r,s,(1)}] &\xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(3)} \\ \frac{151\theta}{140} \frac{9}{4\theta^2} [\widehat{\Omega}_T^{r,s,(2)} + \frac{4}{\theta^2} \widehat{\Omega}_T^{r,s,(1)} - \frac{4}{3} \widehat{\Omega}_T^{r,s,(3)}] &\xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(2)}.\end{aligned}$$

The estimated matrix $\widehat{\Sigma}_T$ is symmetric but is not guaranteed to be positive semi-definite. By Theorem 1, $\widehat{\Sigma}_T$ is positive semi-definite in large samples. The form of the asymptotic variance is relatively complicated. Estimating it using subsampling or bootstrap techniques is an interesting research question that is beyond the scope of this paper.

Remark 1: Using results in Jacod and Rosenbaum (2012), it can be shown that the first convergence in Theorem 2 holds at a rate of $\Delta_n^{-1/2}$ while the last convergence rate is $\Delta_n^{-1/4}$ by a straightforward extension of Theorem 1. Our estimator of $\Sigma_T^{r,s,(2)}$ can be shown to have a rate of convergence $\Delta_n^{-1/4}$.

Remark 2: In the one-dimensional case ($d = 1$), much simpler estimators of $\Sigma_T^{r,s,(2)}$ can be constructed using the quantities $\widehat{\gamma}_i^{n,jk} \widehat{\gamma}_i^{n,lm} \widehat{\gamma}_{i+k_n}^{n,gh} \widehat{\gamma}_{i+k_n}^{n,xy}$ or $\widehat{\gamma}_i^{n,jk} \widehat{\gamma}_i^{n,lm} \widehat{\gamma}_i^{n,gh} \widehat{\gamma}_i^{n,xy}$ as in Vetter (2012). However, in the multidimensional case, the latter quantities do not identify separately the quantity $\overline{c}_t^{jk,lm} \overline{c}_t^{gh,xy}$ since the combination $\overline{c}_t^{jk,lm} \overline{c}_t^{gh,xy} + \overline{c}_t^{jk,gh} \overline{c}_t^{lm,xy} + \overline{c}_t^{jk,xy} \overline{c}_t^{gh,lm}$ shows up in a non-trivial way in the limit of the estimator.

Corollary 3. For $1 \leq r \leq \kappa$, let $[H_r(\widehat{c}), \widehat{G}_r(c)]_T$ be either $[H_r(\widehat{c}), \widehat{G}_r(c)]_T^{AN}$ or $[H_r(\widehat{c}), \widehat{G}_r(c)]_T^{LIN}$ defined in

(15) and (13), respectively. Suppose the assumptions of theorem 1 hold, then we have:

$$\Delta_n^{-1/4} \widehat{\Sigma}_T^{-1/2} \begin{pmatrix} [\widehat{H_1(c)}, \widehat{G_1(c)}]_T - [H_1(c), G_1(c)]_T \\ \vdots \\ [\widehat{H_\kappa(c)}, \widehat{G_\kappa(c)}]_T - [H_\kappa(c), G_\kappa(c)]_T \end{pmatrix} \xrightarrow{L} N(0, I_\kappa), \quad (20)$$

In the above, we use the notation L to denote the convergence in distribution and I_κ the identity matrix of order κ . Corollary 3 states the standardized asymptotic distribution, which follows directly from the properties of stable-in-law convergence. Similarly, by the delta method, standardized asymptotic distribution can also be derived for the estimators of the quantities in (11). These standardized distributions allow the construction of confidence intervals for all the latent quantities of the form $[H_r(c), G_r(c)]_T$ and, more generally, functions of these quantities.

4.4 Testing procedure

We now describe the three statistical tests that we are interested in. The test of absence of dependence between the IV of the returns on asset i and j can be formulated as:

$$H_0^1 : [c_{Zi}, c_{Zj}]_T = 0 \quad \text{vs} \quad H_1^1 : [c_{Zi}, c_{Zj}]_T \neq 0.$$

The null hypothesis H_0^1 is rejected whenever,

$$\Delta_n^{-1/4} \frac{|\widehat{[c_{Zi}, c_{Zj}]_T}|}{\sqrt{\widehat{AVAR}(c_{Zi}, c_{Zj})}} > Z_\alpha.$$

The test of absence of dependence between the IV of stock j and all IV factors q takes the following form:

$$H_0^2 : [c_{Zj}, q]_T = 0 \quad \text{vs} \quad H_1^2 : [c_{Zj}, q]_T \neq 0.$$

Denoting by d_q the number of IV factors, we reject the above null hypothesis H_0^2 when,

$$\Delta_n^{-1/4} \left(\widehat{[c_{Zj}, q]_T} \right)^\top \left(\widehat{AVAR}(c_{Zj}, q) \right)^{-1} \widehat{[c_{Zj}, q]_T} > \mathcal{X}_{d_q, 1-\alpha}^2.$$

The test of absence of dependence between the NS-IVs can be stated as:

$$H_0^3 : [c_{Zi}^{NS}, c_{Zj}^{NS}]_T = 0 \quad \text{vs} \quad H_1^3 : [c_{Zi}^{NS}, c_{Zj}^{NS}]_T \neq 0,$$

with the null rejected if

$$\Delta_n^{-1/4} \frac{|\widehat{[c_{Zi}^{NS}, c_{Zj}^{NS}]_T}|}{\sqrt{\widehat{AVAR}(c_{Zi}^{NS}, c_{Zj}^{NS})}} > Z_\alpha.$$

Our inference theory also allows to test more general hypotheses, which are joint across any subset of the panel. In the above statements, $[\widehat{H(c)}, \widehat{G(c)}]_T$ can be either $[\widehat{H(c)}, \widehat{G(c)}]_T^{AN}$ or $[\widehat{H(c)}, \widehat{G(c)}]_T^{LIN}$, $\widehat{AVAR}(H(c), G(c))$ is an estimate of the asymptotic variance of $[\widehat{H(c)}, \widehat{G(c)}]_T$, Z_α stands for the $(1 - \alpha)$ quantile of the $N(0, 1)$, and $\mathcal{X}_{d_q, 1-\alpha}^2$ stands for $(1 - \alpha)$ quantile of the $\mathcal{X}_{d_q}^2$ distribution. For the first two

tests, the expression for the true asymptotic variance is obtained using Theorem 1 and its estimation follows from Theorem 2. The asymptotic variance of the third test is obtained by an application of the delta method to the convergence result in Theorem 1. The expression of the AVAR for the third test involves some of the latent quantities defined in (11), which can be estimated using either AN- or LIN-type estimators. Therefore in general, we have two tests for each null hypothesis, corresponding to the two type of estimators for $[H(c), G(c)]_T$. Under (2) and the assumptions of Theorem 1, Corollary 3 implies that the asymptotic size of the two types of tests for the null hypotheses H_0^1 and H_0^2 is α , and their power approaches 1. The same properties apply for the tests of the null hypotheses H_0^3 as long as (2) and our IV-FM representation (5) hold.

Theoretically, it is possible to test for absence of dependence in the IVs at the spot level. In this case the null hypothesis is $H_0^1 : [c_{Zi}, c_{Zj}]_t = 0$ for all $0 \leq t \leq T$, which is, in theory, stronger than our H_0^1 . In particular, Theorem 1 can be used to set up Kolmogorov-Smirnov type of tests for H_0^1 in the same spirit as Vetter (2012). However, we do not pursue this direction in the current paper for two reasons. First, the testing procedure would be more involved. Second, empirical evidence suggests nonnegative dependence between IVs, which means that in practice, it is not too restrictive to assume $[c_{Zi}, c_{Zj}]_t \geq 0 \forall t$, under which H_0^1 and H_0^1 are equivalent.

5 Monte Carlo

This section investigates the finite sample properties of our estimators and tests. The data generating process (DGP) is similar to that of LTT and is constructed as follows. Denote by Y_1 and Y_2 log-prices of two individual stocks, and by X the log-price of the market portfolio. Recall that the superscript c indicates the continuous part of a process. We assume

$$dX_t = dX_t^c + dJ_{3,t}, \quad dX_t^c = \sqrt{c_{X,t}} dW_t,$$

and, for $j = 1, 2$,

$$dY_{j,t} = \beta_t dX_t^c + d\tilde{Y}_{j,t}^c + dJ_{j,t}, \quad d\tilde{Y}_{j,t}^c = \sqrt{c_{Zj,t}} d\tilde{W}_{j,t}.$$

In the above, c_X is the spot volatility of the market portfolio, $\tilde{W}_1$, and $\tilde{W}_2$ are Brownian motions with $\text{Corr}(d\tilde{W}_{1,t}, d\tilde{W}_{2,t}) = 0.4$, and W is an independent Brownian motion; J_1, J_2 , and J_3 are independent compound Poisson processes with intensity equal to 2 jumps per year and jump size distribution $N(0, 0.02^2)$. The beta process is time-varying and is specified as $\beta_t = 0.5 + 0.1 \sin(100t)$.

We next specify the volatility processes. As our building blocks, we first generate four processes $f_1, \dots, f_4$ as mutually independent Cox Ingersoll Ross processes,

$$\begin{aligned} df_{1,t} &= 5(0.09 - f_{1,t})dt + 0.35\sqrt{f_{1,t}} \left(-0.8dW_t + \sqrt{1 - 0.8^2} dB_{1,t} \right), \\ df_{j,t} &= 5(0.09 - f_{j,t})dt + 0.35\sqrt{f_{1,t}} dB_{j,t} \quad , \text{ for } j = 2, 3, 4, \end{aligned}$$

where $B_1, \dots, B_4$ and independent standard Brownian Motions, which are also independent from the Brown-

ian Motions of the return Factor Model.⁹ We use the first process f_1 as the market volatility, i.e., $c_{X,t} = f_{1,t}$. We use the other three processes f_2, f_3 , and f_4 to construct three different specifications for the IV processes $c_{Z1,t}$ and $c_{Z2,t}$, see Table 1 for details. The common Brownian Motion W_t in the market portfolio price process X_t and its volatility process $c_{X,t} = f_{1,t}$ generates a leverage effect for the market portfolio. The value of the leverage effect is -0.8, which is standard in the literature, see Kalnina and Xiu (2014), Aït-Sahalia, Fan, and Li (2013) and Aït-Sahalia, Fan, Laeven, Wang, and Yang (2013).

	$c_{Z1,t}$	$c_{Z2,t}$
Model 1	$0.1 + 1.5f_{2,t}$	$0.1 + 1.5f_{3,t}$
Model 2	$0.1 + 0.6c_{X,t} + 0.4f_{2,t}$	$0.1 + 0.5c_{X,t} + 0.5f_{3,t}$
Model 3	$0.1 + 0.45c_{X,t} + f_{2,t} + 0.4f_{4,t}$	$0.1 + 0.35c_{X,t} + 0.3f_{3,t} + 0.6f_{4,t}$

Table 1: Different specifications for the Idiosyncratic Volatility processes $c_{Z1,t}$ and $c_{Z2,t}$.

We set the time span T equal 1260 or 2520 days, which correspond approximately to 5 and 10 business years. These values are close to those typically used in the nonparametric leverage effect estimation literature (see Aït-Sahalia, Fan, and Li (2013) and Kalnina and Xiu (2014)), which is related to the problem of volatility of volatility estimation. Each day consists of 6.5 trading hours. We consider two different values for the sampling frequency, $\Delta_n = 1$ minute and $\Delta_n = 5$ minutes. We follow LTT and set the truncation threshold u_n in day t at $3\hat{\sigma}_t\Delta_n^{0.49}$, where $\hat{\sigma}_t$ is the squared root of the annualized bipower variation of Barndorff-Nielsen and Shephard (2004). We use 10 000 Monte Carlo replications in all the experiments.

We first investigate the finite sample properties of the estimators under Model 3. The considered estimators include:

- the IV beta of the first stock (b_{Z1}),
- the correlation between the first idiosyncratic volatility and the market volatility (ρ_{Z1}),
- the contribution of the non-systematic idiosyncratic volatility to the variation in the idiosyncratic volatility in the case of the first stock ($1 - R_{Z1}^{2,IV-FM}$),
- the correlation between the idiosyncratic volatilities ($\rho_{Z1,Z2}$),
- the correlation between non-systematic idiosyncratic volatilities ($\rho_{Z1,Z2}^{NS}$),
- the contribution of the NS-IV to the dependence between IVs ($1 - Q_{Z1,Z2}^{IV-FM}$).

The interpretation of simulation results is much simpler when the quantities of interest do not change across simulations. To achieve that, we generate once and keep fixed the paths of the processes $(f_{j,t})_{1 \leq j \leq 4}$ and replicate several times the other parts of the DGP. In Table 2, we report the bias and the interquartile range (IQR) of the two type of estimators for each quantity using 5 minutes data sampled over 10 years. We choose four different values for the width of the subsamples, which corresponds to $\theta = 1.5, 2, 2.5$ and 3 (recall that the number of observations in a window is $k_n = \theta/\sqrt{\Delta_n}$). It seems that larger values of the

⁹The Feller property is satisfied implying the positiveness of the processes $(f_{j,t})_{1 \leq j \leq 4}$.

parameters produce better results. Next, we investigate how these results change when we increase the sampling frequency. In Table 3, we report the results with $\Delta_n = 1$ minute in the same setting. We note a reduction of the bias and IQR at all levels of significance. However, the magnitude of the decrease of the IQR is very small. Finally, we conduct the same experiment using data sampled at one minute over 5 years. Despite using more than twice as many observations than in the first experiment, the precision is not as good. In other words, increasing the time span is more effective for precision gain than increasing the sampling frequency. This result is typical for $\Delta_n^{1/4}$ -convergent estimators, see, e.g., Kalnina and Xiu (2014).

θ	AN				LIN			
	1.5	2	2.5	3	1.5	2	2.5	3
Median Bias								
$\hat{b}_{Z1}$	-0.047	-0.025	-0.011	-0.003	-0.006	0.001	0.009	0.015
$\hat{\rho}_{Z1}$	-0.182	-0.127	-0.096	-0.078	-0.186	-0.136	-0.105	-0.085
$1 - \hat{R}_{Z1}^{2,IV-FM}$	0.176	0.130	0.103	0.085	0.181	0.140	0.112	0.092
$\hat{\rho}_{Z1,Z2}$	-0.288	-0.212	-0.163	-0.133	-0.249	-0.190	-0.146	-0.120
$\hat{\rho}_{Z1,Z2}^{NS}$	-0.189	-0.113	-0.064	-0.034	-0.150	-0.091	-0.047	-0.021
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.139	0.102	0.086	0.077	0.129	0.104	0.092	0.082
IQR								
$\hat{b}_{Z1}$	0.222	0.166	0.138	0.121	0.226	0.168	0.139	0.122
$\hat{\rho}_{Z1}$	0.244	0.200	0.173	0.149	0.218	0.183	0.157	0.139
$1 - \hat{R}_{Z1}^{2,IV-FM}$	0.210	0.188	0.172	0.152	0.181	0.166	0.152	0.140
$\hat{\rho}_{Z1,Z2}$	0.404	0.325	0.263	0.223	0.338	0.283	0.237	0.205
$\hat{\rho}_{Z1,Z2}^{NS}$	0.456	0.384	0.315	0.272	0.388	0.337	0.285	0.250
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.345	0.306	0.257	0.216	0.327	0.276	0.233	0.195

Table 2: Finite sample properties of our estimators using 10 years of data sampled at 5 minutes. The true values are $b_{Z1} = 0.450$, $\rho_{Z1} = 0.585$, $1 - R_{Z1}^{IV-FM} = 0.658$, $\rho_{Z1,Z2} = 0.523$, $\rho_{Z1,Z2}^{NS} = 0.424$, $1 - Q_{Z1,Z2}^{IV-FM} = 0.618$.

Next, we study the size and power of the three statistical tests as outlined in Section 4.4. We use Model 1 to study the size properties of the first two tests: the test of the absence of dependence between the IVs ($H_0^1 : [c_{Z1}, c_{Z2}]_T = 0$), and the absence of dependence between the IV of the first stock and the market volatility ($H_0^2 : [c_{Z1}, c_X]_T = 0$). We use Model 2 to study the size properties of the third test ($H_0^3 : [c_{Z1}^{NS}, c_{Z2}^{NS}]_T = 0$). Finally, we use Model 3 to study power properties of all three tests.

The upper panel Tables 5, 6, and 7 reports the size results while the lower panels shows the results for the power. We present the results for the two sampling frequencies ($\Delta_n = 1$ minute and $\Delta_n = 5$ minutes) and the two type of tests (AN and LIN). We observe that the size of three tests are reasonably close to their nominal levels. The rejection probabilities under the alternatives are rather high, except when the data is sampled at 5 minutes frequency and the nominal level at 1%.¹⁰ We note that the tests based on LIN estimators have better testing power compared to those that build on AN estimators. Increasing the window length induces some size distortions but is very effective for power gain. Consistent with the asymptotic theory, the size of the three tests are closer to the nominal levels and the power is higher at the one minute sampling frequency. Clearly, the test of absence of dependence between IV and the market volatility has the

¹⁰We set the nominal level at 5% in the empirical application.

θ	AN				LIN			
	1.5	2	2.5	3	1.5	2	2.5	3
	Median Bias							
$\hat{b}_{Z1}$	-0.022	-0.012	-0.003	0.004	-0.003	-0.000	0.006	0.012
$\hat{\rho}_{Z1}$	-0.105	-0.085	-0.066	-0.050	-0.110	-0.089	-0.069	-0.052
$1 - \hat{R}_{Z1}^{IV-FM}$	0.107	0.091	0.073	0.056	0.113	0.095	0.075	0.058
$\hat{\rho}_{Z1,Z2}$	-0.147	-0.104	-0.073	-0.048	-0.133	-0.097	-0.067	-0.042
$\hat{\rho}_{Z1,Z2}^{NS}$	-0.135	-0.086	-0.058	-0.039	-0.119	-0.078	-0.052	-0.032
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.071	0.045	0.035	0.029	0.067	0.047	0.038	0.032
	IQR							
$\hat{b}_{Z1}$	0.156	0.112	0.088	0.075	0.157	0.112	0.088	0.075
$\hat{\rho}_{Z1}$	0.200	0.146	0.114	0.094	0.189	0.139	0.110	0.091
$1 - \hat{R}_{Z1}^{IV-FM}$	0.201	0.146	0.118	0.100	0.184	0.138	0.113	0.096
$\hat{\rho}_{Z1,Z2}$	0.340	0.238	0.184	0.150	0.309	0.226	0.177	0.145
$\hat{\rho}_{Z1,Z2}^{NS}$	0.417	0.291	0.228	0.184	0.378	0.274	0.217	0.177
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.355	0.266	0.201	0.158	0.339	0.251	0.191	0.151

Table 3: Finite sample properties of our estimators using 10 years of data sampled at 1 minute. The true values are $b_{Z1} = 0.450$, $\rho_{Z1} = 0.580$, $1 - R_{Z1}^{2,IV-FM} = 0.664$, $\rho_{Z1,Z2} = 0.514$, $\rho_{Z1,Z2}^{NS} = 0.408$, $1 - Q_{Z1,Z2}^{IV-FM} = 0.606$.

θ	AN				LIN			
	1.5	2	2.5	3	1.5	2	2.5	3
	Median Bias							
$\hat{b}_{Z1}$	-0.019	-0.011	-0.007	0.000	-0.001	-0.001	0.002	0.008
$\hat{\rho}_{Z1}$	-0.117	-0.091	-0.074	-0.062	-0.120	-0.094	-0.076	-0.063
$1 - \hat{R}_{Z1}^{2,IV-FM}$	0.115	0.096	0.081	0.069	0.119	0.100	0.084	0.071
$\hat{\rho}_{Z1,Z2}$	-0.168	-0.101	-0.064	-0.038	-0.149	-0.092	-0.057	-0.033
$\hat{\rho}_{Z1,Z2}^{NS}$	-0.141	-0.079	-0.035	-0.007	-0.127	-0.067	-0.029	-0.001
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.121	0.094	0.086	0.087	0.113	0.095	0.088	0.087
	IQR							
$\hat{b}_{Z1}$	0.215	0.159	0.128	0.110	0.216	0.158	0.129	0.110
$\hat{\rho}_{Z1}$	0.263	0.196	0.160	0.135	0.252	0.189	0.155	0.131
$1 - \hat{R}_{Z1}^{2,IV-FM}$	0.282	0.204	0.168	0.144	0.260	0.194	0.161	0.139
$\hat{\rho}_{Z1,Z2}$	0.472	0.337	0.263	0.213	0.436	0.319	0.252	0.206
$\hat{\rho}_{Z1,Z2}^{NS}$	0.541	0.412	0.324	0.266	0.510	0.391	0.311	0.256
$1 - \hat{Q}_{Z1,Z2}^{IV-FM}$	0.357	0.313	0.247	0.198	0.356	0.297	0.238	0.189

Table 4: Finite sample properties of our estimators using 5 years of data sampled at 1 minute. The true values are $b_{Z1} = 0.450$, $\rho_{Z1} = 0.591$, $1 - R_{Z1}^{2,IV-FM} = 0.650$, $\rho_{Z1,Z2} = 0.517$, $\rho_{Z1,Z2}^{NS} = 0.417$, $1 - Q_{Z1,Z2}^{IV-FM} = 0.613$.

best power, followed by the test of absence of dependence between the two IVs. This ranking is compatible with the notion that the finite sample properties of the tests deteriorate with the degree of latency embedded in each null hypothesis.

Type of the test	$\Delta_n = 5$ minutes						$\Delta_n = 1$ minute					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
Panel A : Size Analysis-Model 1												
$\alpha = 20\%$	19.5	21.0	19.4	20.4	19.4	20.7	20.2	19.6	19.7	19.9	19.8	20.1
$\alpha = 10\%$	9.7	10.6	10.6	12.6	9.7	10.3	10.2	9.7	10.0	10.2	9.8	10.2
$\alpha = 5\%$	4.7	5.1	4.5	5.3	4.8	5.6	5.3	5.3	5.2	5.3	4.9	5.1
$\alpha = 1\%$	0.9	1.1	0.9	1.2	0.9	1.1	1.1	1.1	1.2	1.1	1.0	1.0
Panel B : Power Analysis-Model 3												
$\alpha = 20\%$	33.5	45.7	50.4	63.1	67.8	78.1	48.5	56.2	77.7	82.3	94.1	95.8
$\alpha = 10\%$	20.5	31.5	35.7	48.3	53.3	65.8	33.9	41.0	65.6	71.6	88.0	91.2
$\alpha = 5\%$	11.9	21.0	23.9	35.76	40.6	53.4	22.3	29.5	52.8	59.8	79.6	84.4
$\alpha = 1\%$	3.3	6.9	8.7	15.6	18.4	28.6	8.9	12.4	28.6	34.5	57.4	64.1

Table 5: Size and Power of the test of absence of dependence between idiosyncratic volatilities for $T = 10$ years.

Type of test	$\Delta_n = 5$ minutes						$\Delta_n = 1$ minute					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
Panel A : Size Analysis-Model 1												
$\alpha = 20\%$	22.6	20.1	20.0	21.0	19.8	21.5	21.6	20.6	21.6	21.5	21.1	21.5
$\alpha = 10\%$	12.1	10.2	10.0	10.6	9.8	11.0	11.0	10.4	10.3	10.4	10.4	10.4
$\alpha = 5\%$	6.2	5.0	4.5	5.2	4.6	5.4	5.5	5.4	5.2	5.1	5.2	5.3
$\alpha = 1\%$	1.5	1.0	0.8	1.0	0.9	1.2	1.1	1.1	1.0	0.9	0.8	1.0
Panel B : Power Analysis-Model 3												
$\alpha = 20\%$	73.1	80.7	91.4	93.9	97.4	98.3	95.8	97.2	99.7	99.8	100	100
$\alpha = 10\%$	60.0	69.0	84.0	88.3	94.6	96.1	91.1	93.3	99.2	99.4	100	100
$\alpha = 5\%$	47.7	57.2	75.0	81.0	89.6	92.6	84.9	88.2	98.2	98.6	100	100
$\alpha = 1\%$	24.1	32.3	52.2	60.1	73.7	78.9	67.7	72.0	93.0	94.5	99.2	99.4

Table 6: Size and Power of the test of absence of dependence between the idiosyncratic volatility and the market volatility for $T = 10$ years.

Type of test	$\Delta_n = 5 \text{ minutes}$						$\Delta_n = 1 \text{ minute}$					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
Panel A : Size Analysis-Model 2												
$\alpha = 20\%$	19.9	23	20.4	23.7	20.2	23.2	19.7	20.5	20.3	21.7	20.0	22.3
$\alpha = 10\%$	10.0	10.1	12.1	10.8	9.9	12.6	10.1	10.3	10.6	11.3	10.1	11.4
$\alpha = 5\%$	5.0	6.3	5.1	6.3	5.1	6.7	5.5	5.5	5.3	5.9	5.2	6.0
$\alpha = 1\%$	1.1	1.5	0.8	1.6	1.1	1.4	1.1	1.2	1.3	1.3	1.3	1.5
Panel B : Power Analysis-Model 3												
$\alpha = 20\%$	25.1	32.1	29.0	36.5	42.8	51.7	31.0	35	50.0	54.6	68.0	72.3
$\alpha = 10\%$	13.7	19.2	16.8	23.0	28.1	36.9	19.0	22.2	35.0	39.4	53.4	58.3
$\alpha = 5\%$	7.4	11.3	9.3	14.2	18.3	25.2	11.0	13.7	23.9	28.0	40.0	44.9
$\alpha = 1\%$	1.6	3.1	2.3	3.9	6.0	9.5	2.9	4.0	9.3	11.6	18.8	22.2

Table 7: Size and Power of the test of absence of dependence between NS-IVs for $T = 10$ years.

6 Empirical Analysis

We apply our methods to study the cross-sectional dependence in IV using high frequency data. We find that stocks' idiosyncratic volatilities co-move strongly with the market volatility. Also, we find that market volatility is often the main source of the dependence observed in the IVs. This is a quite surprising finding. It is of course well known that the total volatility of stocks moves with the market volatility. However, we stress that we find that the strong effect is still present when considering the idiosyncratic volatilities.

We use full record transaction prices from NYSE TAQ database for 27 stocks over the time period 2003-2012. After removing the non-trading days, our sample contains 2517 days. The selected stocks have been part of S&P 500 index throughout our sample. Our 27 stocks contain three liquid stocks in each of the nine sectors of the index (Consumer Discretionary, Consumer Staples, Energy, Financial, Health Care, Industrial, Materials, Technology, and Utilities). For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. until 4 p.m. We clean the data following the procedure suggested by ?, remove the overnight returns and then sample at 5 minutes. This sparse sampling has been widely used in the literature because the effect of the microstructure noise and potential asynchronicity of the data is less important at this frequency, see also ?.

The parameter choices for the estimators are as follows. Guided by our Monte Carlo results, we set the length of window to be approximately one week for the estimators in Section 3 (this corresponds to $\theta = 2.5$ where $k_n = \theta \Delta_n^{-1/2}$ is the number of observations in a window). The truncation threshold for all estimators is set as in the Monte Carlo study ($3\hat{\sigma}_t \Delta_n^{0.49}$ where $\hat{\sigma}_t^2$ is the bipower variation).

We consider two sets of factors in the factor model for returns: the S&P500 market index and the three Fama-French factors (FF3 henceforth). All factors are sampled at 5 minutes over 2003-2012.¹¹

Figures 1 and 2 contain plots of the time series of the estimated $1 - R_{Y_j}^2$ of the return regressions, i.e., the estimated monthly contribution of the idiosyncratic volatility to the total volatility, for each stock in the

¹¹The high-frequency data on the Fama-French factors were obtained from Aït-Sahalia, Kalnina, and Xiu (2014).

two models (CAPM and FF3).¹² In Table 8, we report the average of these monthly statistics over the full sample. As we can see in Table 8, the idiosyncratic returns have a relatively high contribution to the total variation of the returns in the two models. The minimum value (across all the stocks) for $1 - R_{Yj}^2$ is 61.5% for the one-factor model and 53.5% in the FF3 model. Figures 1 and 2 show that the time series of all stocks follow approximately the same trend with a considerable drop in the contribution around the crisis year 2008, which shows that the systemic risk became relatively more important during this period. Overall our results suggest that IV contributes to more than a half of the total variation for each stock. Therefore, studying the source of variation in IVs is potentially useful. We proceed to investigate the dynamic properties of the panel of idiosyncratic volatilities.

We first investigate the (total) dependence in the idiosyncratic volatilities. We have 351 pairs of stocks available in the panel. For each pair of stocks, we compute the correlation between the IVs, $\rho_{Zi, Zj}$. The upper and lower panels of Table 13 display the correlations estimated using the LIN-type estimators in the CAPM and FF3 as the factor model for prices.¹³ The values in parenthesis correspond to the p-values of the test of dependence in the IVs (see Section 4.4 for an expression of the test statistic). The reader should be careful when interpreting these p-values because they are not adjusted for multiple testing. Clearly, there is evidence for strong dependence between the IVs. Indeed, the absolute values of the t-statistics are bigger than 1.96 for 350 pairs over 351. Only the dependence between the IV of the Goldman Sachs (GS) and IBM gives rise to the absolute value of the t-statistic smaller than 1.96. Using the Bonferroni correction, the p-value of the test of absence of dependence in all pairs is less than 0.0001. The estimated correlation is positive for each pair of stocks. We also observe substantial heterogeneity in the correlation with a maximum value of 94.4% (Exxon Mobil (XOM) and Chevron Corp (CVX)) and a minimum value of 24.8% (Duke Energy (DUK) and Avery Dennison Corporation (AVY)). Table 9 is a summary of the results of Table 13; it shows the number of pairs with the estimated correlation greater than a set of thresholds. For example, it shows that the correlation is greater than 50% for more than two thirds of the pairs (265). Interestingly, the results of the test are unchanged for the FF3 model, and the estimated correlations are very close to those obtained in the CAPM. This result is not surprising given the relatively small difference between the values of R_{Yj}^2 in the two models.

We next ask the question of whether potential missing factors in the factor model for returns might be responsible for the strong dependence in IVs. Omitted factors in the factor model for returns induce correlation between the estimated idiosyncratic returns, $\text{Corr}(Z_i, Z_j)$.¹⁴ We report in Table 12 the estimated

¹²For the j^{th} stock, our analog of the coefficient of determination in the return factor model for this stock is $R_{Yj}^2 = 1 - \frac{\int_0^T c_{Zj,t} dt}{\int_0^T c_{Yj,t} dt}$. We estimate R_{Yj}^2 using the general method of Jacod and Rosenbaum (2013). The resulting estimator of R_{Yj}^2 requires a choice of a block size for the spot volatility estimation; we choose two hours in practice (the number of observations in a block, say l_n , has to satisfy $l_n^2 \Delta_n \rightarrow 0$ and $l_n^3 \Delta_n \rightarrow \infty$, so it is of smaller order than the number of observations k_n in our estimators of Section 3).

¹³This choice is motivated by our simulation results where LIN type of estimators and tests appear to have better finite sample properties than AN type estimators.

¹⁴Our measure of correlation between the idiosyncratic returns Z_i and Z_j is

$$\text{Corr}(Z_i, Z_j) = \frac{\int_0^T c_{Zij,t} dt}{\sqrt{\int_0^T c_{Zi,t} dt} \sqrt{\int_0^T c_{Zj,t} dt}}, \quad i, j = 1, \dots, d_S, \quad (21)$$

where $c_{Zij,t}$ is the spot covariation between the idiosyncratic returns Z_i and Z_j . Similarly to R_{Yj}^2 , we estimate $\text{Corr}(Z_i, Z_j)$

correlations $\text{Corr}(Z_i, Z_j)$. Table 10 presents a summary of how estimates of $\text{Corr}(Z_i, Z_j)$ in Table 12 are related to the estimates of correlation in IVs, ρ_{Z_i, Z_j} , in Table 13. In particular, different rows in Table 10 display average values of ρ_{Z_i, Z_j} among those pairs, for which $\text{Corr}(Z_i, Z_j)$ is below some threshold. For example, the last-but-one row in Table 10 indicates that there are 56 pairs of stocks with estimated $\text{Corr}(Z_i, Z_j) < 0.01$, and among those stocks, the average correlation between IVs, ρ_{Z_i, Z_j} , is estimated to be 0.579. This estimate ρ_{Z_i, Z_j} is virtually the same among pairs of stocks with high $\text{Corr}(Z_i, Z_j)$. Therefore, we know that among 56 pairs of stocks, a missing return factor cannot explain dependence in IVs. Moreover, these results suggest that missing return factor cannot explain dependence in IVs for all considered stocks. These results are in line with recent findings of [Herskovic, Kelly, Lustig, and Nieuwerburgh \(2014\)](#) with daily and monthly returns.

To understand the source of the strong dependence in the IVs, we consider the Idiosyncratic Volatility Factor Model (IV-FM) of Section 2. We use the market volatility as the single IV factor. We start by considering individual stocks separately. In Table 11, we report the estimates of the idiosyncratic volatility beta ($\hat{b}_{Z_i}$), the correlation between the idiosyncratic volatility and the market volatility ($\hat{\rho}_{Z_i}$), and the contribution of each non-systemic IV (NS-IV) to the aggregate variation in IV ($1 - R_{Z_i}^{2, IV-FM}$). The absolute values of the t-statistics based on the covariation between IV and the market volatility are bigger than 1.96 for each stock. For every stock, the estimated IV beta and the correlation $\hat{\rho}_{Z_i}$ are positive, suggesting that the idiosyncratic volatility co-moves with the market volatility. For 16 stocks out of 27, the NS-IV contributes to more than 50% of the variation in their IV, with the average being 56%.

Next, we turn to the implications of the IV-FM for the cross-section. In Table 14 we report, for each pair of stocks, the correlation between the NS-IVs, ρ_{Z_i, Z_j}^{NS} . The values are much smaller than the correlations between the total IVs (ρ_{Z_i, Z_j}) in Table 13. Next, Table 15 reports, for each pair of stocks, the contribution of the dependence in the NS-IVs to the total dependence in the IVs ($1 - Q_{Z_i, Z_j}^{IV-FM}$). Overall, the fraction of dependence explained by the market volatility is very large. As it is apparent in the tables, there are only two pairs, for which the contribution of the NS-IVs is greater than 50%. Therefore, market volatility seems to be the main source of the dependence. Its average contribution of the systemic IVs to the total dependence in the idiosyncratic volatilities is 73.8% in the CAPM and 73.5% in the FF3 model.

Given the large fraction of the cross-sectional dependence in IVs that is explained by the market volatility, it is interesting to investigate if our IV-FM can fully explain the IV dependence across stocks. For this purpose, we conduct inference on dependence in the NS-IVs. Table 14 displays the estimated correlation between the NS-IVs. The residual correlations are smaller than the total IV correlations. There are only 26 pairs of stocks with this correlation higher than 50% in the CAPM model and 27 pairs in the FF3 model. Interestingly, they all remain positive. The t-statistics based on covariation between NS-IVs are larger than 1.96 for 241 pairs in the CAPM and 244 pairs in the FF3 model (see the values in parenthesis of both Tables 14 and 13). From Table 11, each stock has at least eight other stocks with whom it produces a t-statistic bigger than 1.96. Using the Bonferroni correction, the p-value of the test of absence of dependence in all pairs is less than 0.0001. We conclude that despite the market volatility explaining most of the cross-sectional dependence using the method of [Jacod and Rosenbaum \(2013\)](#).

dependence in IVs, it does not explain all of it. Additional IV factors may help to explain all the dependence in the idiosyncratic volatilities.

Sector	Stock	Ticker	CAPM	FF3 Model
Financial	American Express Co.	AXP	65.2	56.5
	Goldman Sachs Group	GS	65.5	56.9
	JPMorgan Chase & Co.	JPM	63.0	54.8
Energy	Chevron Corp.	CVX	64.2	55.8
	Schlumberger Ltd.	SLB	74.0	64.3
	Exxon Mobil Corp.	XOM	61.5	53.5
Consumer Staples	Coca Cola Company	KO	75.2	65.4
	Procter & Gamble	PG	74.8	65.1
	Wal-Mart Stores	WMT	73.7	64.0
Industrials	Caterpillar Inc.	CAT	63.2	54.8
	3M Company	MMM	63.4	55.0
	United Technologies	UTX	63.8	55.5
Technology	Cisco Systems	CSCO	65.5	56.9
	International Bus. Machines	IBM	63.3	55.2
	Intel Corp.	INTC	63.0	54.8
Health Care	Johnson & Johnson	JNJ	75.6	65.7
	Merck & Co.	MRK	78.6	68.5
	Pfizer Inc.	PFE	75.9	66.1
Consumer Discretionary	Home Depot	HD	69.1	60.1
	McDonald's Corp.	MCD	75.8	65.9
	Nike	NKE	74.5	64.7
Utilities	Air Products & Chemicals Inc	APD	66.8	58.0
	Allegheny Technologies Inc	ATI	71.3	61.8
	Avery Dennison Corp	AVY	70.7	61.5
Material	Duke Energy	DUK	80.9	70.3
	CenterPoint Energy	CNP	82.8	72.0
	Exelon Corp.	EXC	80.8	70.2

Table 8: Average of the monthly contribution of the IV of stocks to their total volatility ($1-R_{Y_j}^2$) over the period 2003:2012 in percentages. The first column provides information on the sectors, the second the names of the companies and the third their tickers. The fourth and fifth columns show $1-R_{Y_j}^2$ for the CAPM and FF3 return model.

$\widehat{\rho}_{Z_i, Z_j}$	CAPM	FF3 Model
> 0.9	2	2
> 0.8	13	13
> 0.7	60	58
> 0.6	158	163
> 0.5	265	265
> 0.4	323	323
> 0.3	350	350
> 0.2	350	350
> 0.1	350	350
$\neq 0$	350	350

Table 9: Number of pairs of stocks with significant dependence between their IVs and the estimated correlation greater than the threshold given in the first column. The second column shows the results for the CAPM model. The results for the FF3 model are reported in the third column.

$ \widehat{\text{Corr}}(Z_i, Z_j) $	CAPM			FF3 Model		
	Pairs	Avg $ \widehat{\text{Corr}}(Z_i, Z_j) $	Avg $\widehat{\rho}_{Z_i, Z_j}$	Pairs	Avg $ \widehat{\text{Corr}}(Z_i, Z_j) $	Avg $\widehat{\rho}_{Z_i, Z_j}$
< 0.6	351	0.043	0.585	351	0.043	0.586
< 0.4	350	0.042	0.584	350	0.042	0.585
< 0.3	348	0.040	0.583	348	0.041	0.584
< 0.2	343	0.037	0.583	343	0.038	0.584
< 0.1	323	0.031	0.580	323	0.031	0.581
< 0.075	303	0.028	0.579	304	0.028	0.581
< 0.05	265	0.023	0.570	266	0.023	0.571
< 0.025	152	0.013	0.568	152	0.013	0.566
< 0.01	56	0.005	0.579	56	0.005	0.574
< 0.005	29	0.003	0.580	27	0.003	0.580

Table 10: We report the number of pairs of stocks with the absolute value of the correlation between their idiosyncratic returns smaller than the threshold given in the first column, the average of the absolute value of the idiosyncratic returns correlation for those pairs as well as the average of the IVs correlation for the same pairs. The results are presented both for the CAPM and the FF3 models.

Stock	CAPM					FF3 Model				
	$\hat{b}_z$	$\hat{\rho}_z$	$1 - \hat{R}_Z^{2,IV-FM}(\%)$	p-val	#	$\hat{b}_z$	$\hat{\rho}_z$	$1 - \hat{R}_Z^{2,IV-FM}(\%)$	p-val	#
AXP	1.600	0.685	53.1	0.010	13	1.584	0.684	53.2	0.011	13
GS	2.313	0.498	75.2	0.024	13	2.341	0.505	74.5	0.018	13
JPM	1.899	0.540	70.9	0.004	8	1.894	0.538	71.0	0.004	9
CVX	0.611	0.714	49.0	0.008	20	0.603	0.715	48.9	0.008	20
SLB	1.064	0.723	47.8	0.005	16	1.043	0.718	48.4	0.005	15
XOM	0.576	0.762	42.0	0.004	21	0.575	0.760	42.2	0.004	19
KO	0.327	0.753	43.2	0.013	13	0.328	0.753	43.3	0.012	14
PG	0.427	0.784	38.5	0.002	18	0.424	0.784	38.5	0.002	19
WMT	0.445	0.752	43.5	0.007	22	0.444	0.753	43.3	0.007	22
CAT	0.589	0.649	57.9	0.002	13	0.590	0.648	58.0	0.003	14
MMM	0.389	0.733	46.3	0.000	16	0.386	0.729	46.9	0.000	15
UTX	0.501	0.717	48.6	0.004	17	0.501	0.716	48.7	0.004	17
CSCO	0.571	0.667	55.5	0.002	20	0.562	0.664	55.9	0.002	21
IBM	0.339	0.650	57.7	0.015	9	0.343	0.657	56.8	0.011	9
INTC	0.454	0.669	55.3	0.003	23	0.451	0.663	56.0	0.003	22
JNJ	0.404	0.822	32.4	0.007	22	0.401	0.819	32.9	0.007	21
MRK	0.535	0.558	68.9	0.001	24	0.534	0.557	68.9	0.001	24
PFE	0.434	0.584	65.9	0.002	22	0.425	0.583	66.0	0.001	22
HD	0.500	0.652	57.5	0.005	19	0.499	0.652	57.5	0.004	16
MCD	0.516	0.628	60.5	0.002	15	0.518	0.630	60.3	0.002	15
NKE	0.528	0.664	55.9	0.000	25	0.526	0.664	55.9	0.000	24
APD	0.535	0.724	47.6	0.001	24	0.527	0.716	48.8	0.001	24
ATI	1.698	0.576	66.8	0.001	20	1.726	0.584	65.9	0.001	19
AVY	0.315	0.429	81.6	0.001	15	0.312	0.427	81.7	0.001	14
DUK	0.415	0.461	78.7	0.002	19	0.415	0.466	78.3	0.002	20
CNP	0.740	0.540	70.8	0.001	20	0.737	0.541	70.7	0.001	20
EXC	0.927	0.776	39.8	0.001	21	0.926	0.774	40.1	0.001	21

Table 11: Estimates of the IV beta ($\hat{b}_z$), the correlation between the IV and the market volatility ($\hat{\rho}_z$) and the contribution of the NS-IV to the variation in the IV ($1 - \hat{R}_Z^{2,IV-FM}$). We use the market volatility as the IV factor. P-val is the p-value of the test of the absence of dependence between the IV and the market volatility for a given individual stock. In the column with the heading #, we report the number of stocks with their NS-IV having a relatively large covariation with the NS-IV of the stock listed in the first column (in particular, when the t-statistic based on the covariation between the NS-IVs is larger than 1.96 in the absolute value).

7 Conclusion

This paper provides tools for the analysis of cross-sectional dependencies in idiosyncratic volatilities using high frequency data. First, using a factor model in prices, we develop inference theory for covariances and correlations between the idiosyncratic volatilities. Next, we study an idiosyncratic volatility factor model, in which we decompose the co-movements in idiosyncratic volatilities into two parts: those related to factors such as the market volatility, and the residual co-movements. We provide the asymptotic theory for the estimators in the decomposition.

Empirically, we find that our IV Factor Model with market volatility as the only factor explains a large part of the cross-sectional dependence in IVs. However, it is not able to explain all of it. It therefore opens the room for the construction of additional IV factors based on economic theory, for example, along the lines of the heterogeneous agents model of [Herskovic, Kelly, Lustig, and Nieuwerburgh \(2014\)](#).

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Appendix

A Figures and Tables

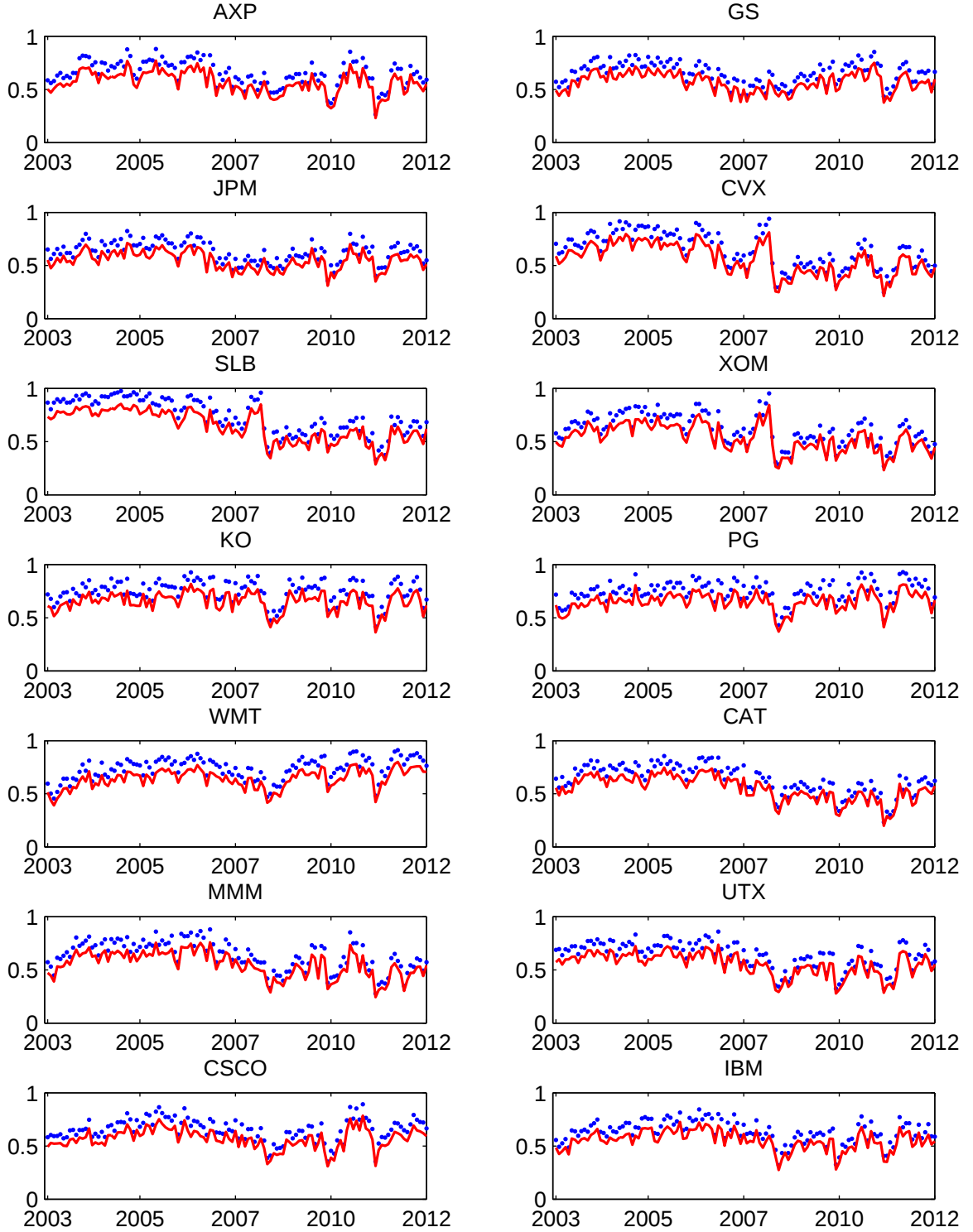


Figure 1: Monthly contribution of the idiosyncratic volatility to the total volatility ($1 - \widehat{R}_{Yj}^2$) over the period 2003:2012. The dotted blue line plots this measure calculated in CAPM model. The solid red line plots the same measure obtained in the FF3 model. We use the ticker of the stocks to label the graphs.

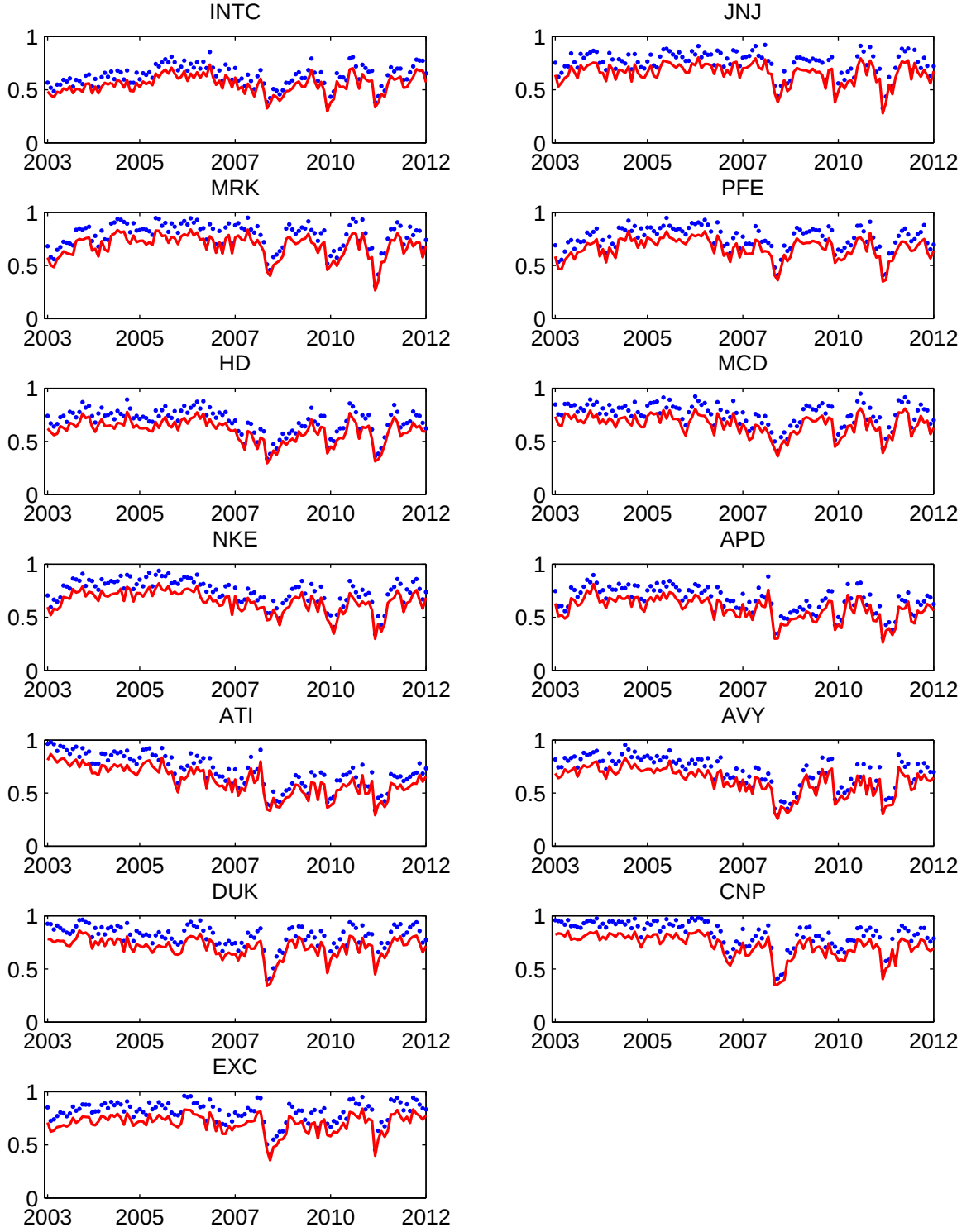


Figure 2: Monthly contribution of the idiosyncratic volatility to the total volatility ($1 - \hat{R}_{Yj}^2$) over the period 2003:2012. The dotted blue line plots this measure calculated in CAPM model. The solid red line plots the same measure obtained in the FF3 model. We use the ticker of the stocks to label the graphs.

	AXP	GS	JPM	CVX	SLB	XOM	KO	PG	WMT	CAT	MMM	UTX	CSCO	IBM	INTC	JNJ	MRK	PFE	HD	MCD	NKE	APD	ATI	AVY	DUK	CNP	EXC
AXP		0.146	0.189	-0.032	-0.018	-0.038	-0.003	-0.009	0.028	0.039	0.016	0.023	0.011	0.016	0.013	-0.009	-0.003	0.002	0.052	0.021	0.029	0.007	0.017	0.022	-0.023	-0.003	-0.013
GS	0.148		0.230	-0.022	-0.001	-0.025	-0.016	-0.009	0.013	0.040	0.025	0.007	0.019	0.016	0.017	-0.025	-0.011	-0.006	0.040	0.013	0.020	0.014	0.043	0.018	-0.018	-0.003	-0.027
JPM	0.190	0.230		-0.049	-0.031	-0.044	-0.012	-0.013	0.015	0.006	-0.003	-0.009	-0.002	0.007	0.007	-0.012	-0.008	0.003	0.040	0.020	0.008	-0.007	0.014	0.002	-0.011	-0.015	-0.032
CVX	-0.032	-0.022	-0.052		0.324	0.515	0.008	-0.003	-0.031	0.025	0.016	0.014	-0.007	-0.002	-0.014	0.004	-0.001	-0.004	-0.047	-0.005	-0.001	0.066	0.088	0.011	0.045	0.048	0.053
SLB	-0.017	-0.002	-0.031	0.327		0.293	-0.030	-0.041	-0.044	0.055	0.011	0.011	-0.012	-0.019	-0.011	-0.025	-0.023	-0.027	-0.037	-0.014	0.000	0.055	0.108	0.013	0.008	0.021	0.021
XOM	-0.038	-0.024	-0.045	0.514	0.295		0.017	0.018	-0.025	0.019	0.013	0.010	-0.003	-0.002	-0.017	0.012	-0.003	0.004	-0.042	0.001	-0.013	0.051	0.074	-0.002	0.040	0.036	0.048
KO	-0.004	-0.015	-0.010	0.007	-0.030	0.016		0.167	0.075	0.001	0.058	0.060	0.025	0.059	0.019	0.119	0.084	0.081	0.023	0.066	0.042	0.023	-0.023	0.036	0.070	0.048	0.076
PG	-0.012	-0.010	-0.014	-0.004	-0.042	0.016	0.170		0.080	-0.005	0.056	0.063	0.021	0.053	0.007	0.151	0.095	0.080	0.040	0.077	0.044	0.027	-0.038	0.031	0.076	0.044	0.068
WMT	0.028	0.012	0.015	-0.031	-0.045	-0.027	0.078	0.079		0.017	0.037	0.035	0.037	0.050	0.034	0.057	0.032	0.045	0.191	0.081	0.086	0.028	0.000	0.035	0.032	0.011	0.019
CAT	0.040	0.040	0.007	0.024	0.054	0.018	0.002	-0.006	0.018		0.102	0.114	0.041	0.030	0.039	-0.012	-0.013	-0.006	0.041	0.038	0.053	0.099	0.107	0.087	0.005	0.022	0.014
MMM	0.016	0.026	-0.004	0.016	0.011	0.012	0.057	0.056	0.035	0.100		0.139	0.028	0.068	0.038	0.054	0.037	0.037	0.044	0.059	0.047	0.115	0.055	0.097	0.043	0.047	0.053
UTX	0.024	0.006	-0.009	0.015	0.010	0.011	0.061	0.061	0.034	0.113	0.139		0.042	0.062	0.027	0.048	0.037	0.027	0.036	0.061	0.067	0.089	0.043	0.091	0.040	0.034	0.044
CSCO	0.011	0.020	-0.003	-0.010	-0.011	-0.004	0.026	0.019	0.038	0.041	0.028	0.043		0.103	0.182	0.020	0.018	0.021	0.036	0.037	0.029	0.024	0.021	0.032	0.010	0.003	0.006
IBM	0.017	0.017	0.007	-0.004	-0.021	-0.003	0.061	0.050	0.048	0.032	0.069	0.060	0.102		0.101	0.046	0.030	0.038	0.041	0.058	0.047	0.033	0.008	0.036	0.016	0.014	0.022
INTC	0.015	0.017	0.007	-0.014	-0.012	-0.017	0.018	0.009	0.034	0.039	0.037	0.028	0.182	0.102		0.009	0.019	0.022	0.046	0.030	0.038	0.025	0.028	0.030	-0.007	-0.008	-0.005
JNJ	-0.011	-0.027	-0.014	0.003	-0.027	0.010	0.119	0.149	0.055	-0.012	0.054	0.048	0.020	0.046	0.008		0.170	0.166	0.017	0.056	0.029	0.019	-0.030	0.014	0.066	0.047	0.065
MRK	-0.004	-0.011	-0.009	-0.002	-0.023	-0.004	0.084	0.094	0.032	-0.012	0.036	0.037	0.019	0.030	0.019	0.169		0.203	0.018	0.046	0.026	0.022	-0.014	0.021	0.055	0.041	0.060
PFE	-0.000	-0.008	0.002	-0.003	-0.028	0.003	0.082	0.078	0.045	-0.007	0.035	0.028	0.021	0.039	0.023	0.166	0.203		0.025	0.038	0.022	0.014	-0.020	0.016	0.056	0.046	0.050
HD	0.051	0.040	0.041	-0.045	-0.038	-0.041	0.025	0.041	0.192	0.041	0.045	0.035	0.035	0.041	0.046	0.017	0.016	0.025		0.094	0.107	0.030	0.017	0.049	0.021	0.008	0.007
MCD	0.019	0.014	0.020	-0.006	-0.015	0.000	0.066	0.074	0.081	0.038	0.058	0.059	0.037	0.055	0.031	0.055	0.046	0.036	0.093		0.082	0.044	0.010	0.043	0.037	0.024	0.029
NKE	0.032	0.022	0.008	-0.000	-0.000	-0.011	0.043	0.045	0.087	0.052	0.047	0.067	0.030	0.045	0.039	0.029	0.026	0.022	0.107	0.083		0.064	0.035	0.071	0.023	0.030	0.031
APD	0.008	0.017	-0.008	0.066	0.054	0.051	0.023	0.026	0.028	0.098	0.116	0.089	0.024	0.033	0.027	0.018	0.020	0.014	0.030	0.044	0.065		0.144	0.162	0.041	0.049	0.051
ATI	0.019	0.043	0.014	0.090	0.110	0.075	-0.023	-0.036	-0.000	0.107	0.055	0.043	0.022	0.009	0.027	-0.030	-0.015	-0.020	0.016	0.011	0.034	0.144		0.086	0.008	0.028	0.013
AVY	0.022	0.020	0.003	0.009	0.012	-0.003	0.037	0.031	0.034	0.088	0.097	0.091	0.032	0.036	0.030	0.014	0.021	0.016	0.048	0.042	0.071	0.163	0.086		0.043	0.056	0.046
DUK	-0.024	-0.020	-0.013	0.044	0.008	0.038	0.070	0.077	0.033	0.007	0.042	0.040	0.010	0.016	-0.007	0.066	0.054	0.055	0.020	0.036	0.023	0.042	0.008	0.042		0.231	0.306
CNP	-0.002	-0.005	-0.016	0.049	0.021	0.036	0.048	0.044	0.011	0.022	0.047	0.035	0.003	0.015	-0.007	0.048	0.042	0.047	0.009	0.024	0.029	0.050	0.028	0.055	0.230		0.243
EXC	-0.013	-0.030	-0.037	0.054	0.021	0.048	0.075	0.070	0.020	0.014	0.054	0.044	0.006	0.022	-0.004	0.067	0.059	0.050	0.007	0.031	0.032	0.053	0.012	0.046	0.306	0.242	

Table 12: The correlation between stocks idiosyncratic returns over 2003-2012, $\text{Corr}(Z_i, Z_j)$.
The top panel reports the results for the CAPM, the bottom panel presents the same results for the FF3 model.

B Proofs

Throughout, we denote by K a generic constant which may change from line to line. When it depends on a parameter p we use the notation K_p instead. We assume by convention $\sum_{i=a}^{a'} = 0$ when $a > a'$.

B.1 Proof of Theorem 1

This theorem is proved in three steps. For simplicity, in the first two steps, we focus on the estimation of $[H(c), G(c)]_T$ with $H, G \in \mathcal{G}(p)$. The joint estimation is discussed in Step 3.

By a localization argument (See Lemma 4.4.9 of [Jacod and Protter \(2012\)](#)), there exists a λ -integrable function J on E and a constant such that the stochastic processes in (17) and (18) satisfy

$$\|b\|, \|\tilde{b}\|, \|c\|, \|\tilde{c}\|, J \leq A, \|\delta(w, t, z)\|^r \leq J(z) \quad (22)$$

Setting $b'_t = b_t - \int \delta(t, z) 1_{\{\|\delta(t, z)\| \leq 1\}} \lambda(dz)$ and $Y'_t = \int_0^t b'_s ds + \int_0^t \sigma_s dW_s$, we have

$$Y_t = Y_0 + Y'_t + \sum_{s \leq t} \Delta Y_s.$$

The local estimator of the spot variance of the unobservable process Y' is given by,

$$\hat{c}_i^n = \frac{1}{k_n \Delta_n} \sum_{u=0}^{k_n-1} (\Delta_{i+u}^n Y') (\Delta_{i+u}^n Y')^\top = (\hat{c}_i^{n, gh})_{1 \leq g, h \leq d}. \quad (23)$$

Note that no jumps truncation is needed in the definition of $\hat{c}_i^n$ since the process Y' is continuous. Therefore, it is more convenient to work with $\hat{c}_i^n$ rather than $\hat{c}_i^n$ (defined in (12)). Let $[H(\widehat{c}), \widehat{G}(c)]_T^{LIN'}$ and $[H(\widehat{c}), \widehat{G}(c)]_T^{AN'}$ be the unfeasible estimators obtained by replacing $\hat{c}_i^n$ by $\hat{c}_i^{n, gh}$ in the definition of $[H(\widehat{c}), \widehat{G}(c)]_T^{LIN}$ and $[H(\widehat{c}), \widehat{G}(c)]_T^{AN}$.

Step1: Dealing with price jumps

We prove that, as long as $(8p-1)/4(4p-r) \leq \varpi < \frac{1}{2}$, we have

$$\Delta_n^{-1/4} \left([H(\widehat{c}), \widehat{G}(c)]_T^{LIN} - [H(\widehat{c}), \widehat{G}(c)]_T^{LIN'} \right) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \Delta_n^{-1/4} \left([H(\widehat{c}), \widehat{G}(c)]_T^{AN} - [H(\widehat{c}), \widehat{G}(c)]_T^{AN'} \right) \xrightarrow{\mathbb{P}} 0. \quad (24)$$

To show this result, let define the functions

$$R(x, y) = \sum_{g, h, a, b=1}^d (\partial_{gh} H \partial_{ab} G)(x) (y^{gh} - x^{gh}) (y^{ab} - x^{ab}), \quad S(x, y) = (H(y) - H(x)) (G(y) - G(x))$$

$$U(x) = \sum_{g, h, a, b=1}^d (\partial_{gh} H \partial_{ab} G)(x) (x^{ga} x^{hb} + x^{gb} x^{ha}),$$

for any $\mathbb{R}^d \times \mathbb{R}^d$ matrices x and y . The following decompositions hold,

$$[H(\widehat{c}), \widehat{G}(c)]_T^{AN} - [H(\widehat{c}), \widehat{G}(c)]_T^{AN'} = \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left[(S(\hat{c}_i^n, \hat{c}_{i+k_n}^n) - S(\hat{c}_i^{n, gh}, \hat{c}_{i+k_n}^{n, gh})) - \frac{2}{k_n} (U(\hat{c}_i^n) - U(\hat{c}_i^{n, gh})) \right],$$

$$[H(\widehat{c}), \widehat{G}(c)]_T^{LIN} - [H(\widehat{c}), \widehat{G}(c)]_T^{LIN'} = \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left[(R(\hat{c}_i^n, \hat{c}_{i+k_n}^n) - R(\hat{c}_i^{n, gh}, \hat{c}_{i+k_n}^{n, gh})) - \frac{2}{k_n} (U(\hat{c}_i^n) - U(\hat{c}_i^{n, gh})) \right].$$

By (3.11) in [Jacod and Rosenbaum \(2012\)](#), there exists a sequence of real numbers a_n converging to zero such that

$$\mathbb{E}(\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|^q) \leq K_q a_n \Delta_n^{(2q-r)\varpi+1-q}, \text{ for any } q > 0, \quad (25)$$

Since H and $G \in \mathcal{G}(p)$, it is easy to see that the functions R and S are continuously differentiable and satisfy

$$\|\partial J(x, y)\| \leq K(1 + \|x\| + \|y\|)^{2p-1} \text{ for } 1 \leq g, h, a, b \leq d \text{ and } J \in \{S, R\}, \quad (26)$$

$$\|\partial U(x)\| \leq K(1 + \|x\|)^{2p-1}. \quad (27)$$

where ∂J (resp ∂U) is a vector that collects the first order partial derivatives of the function J (resp U) with respect to all the elements of (x, y) (resp x). By Taylor expansion, Jensen inequality, (26) and (27), it can be shown that, for $J \in \{S, R\}$

$$\begin{aligned} |J(\hat{c}_i^n, \hat{c}_{i+k_n}^n) - J(\hat{c}_i^{\prime n}, \hat{c}_{i+k_n}^{\prime n})| &\leq K(1 + \|\hat{c}_i^{\prime n}\|^{2p-1} + \|\hat{c}_{i+k_n}^{\prime n}\|^{2p-1})(\|\hat{c}_i^n - \hat{c}_i^{\prime n}\| + \|\hat{c}_{i+k_n}^n - \hat{c}_{i+k_n}^{\prime n}\|) + K\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|^{2p} \\ &\quad + K\|\hat{c}_{i+k_n}^n - \hat{c}_{i+k_n}^{\prime n}\|^{2p}, \text{ and} \\ |U(\hat{c}_i^n) - U(\hat{c}_i^{\prime n})| &\leq K(1 + \|\hat{c}_i^{\prime n}\|^{2p-1})(\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|) + K\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|^{2p}. \end{aligned}$$

By (3.20) in [Jacod and Rosenbaum \(2012\)](#), we have $\mathbb{E}(\|\hat{c}_i^n\|^v) \leq K_v$, for any $v \geq 0$. Hence by Hölder inequality, for $\epsilon > 0$ fixed,

$$\begin{aligned} \mathbb{E}(\|\hat{c}_i^n\|^{2p-2}\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|) &\leq \left(\mathbb{E}(\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|^{(1+\epsilon)})\right)^{1/1+\epsilon} \left(\mathbb{E}(\|\hat{c}_i^n\|^{(2p-2)(1+\epsilon)/\epsilon})\right)^{\epsilon/1+\epsilon} \\ &\leq K_p \left(\mathbb{E}(\|\hat{c}_i^n - \hat{c}_i^{\prime n}\|^{(1+\epsilon)})\right)^{1/1+\epsilon} \\ &\leq K_p a_n \Delta_n^{(2-\frac{1}{1+\epsilon})\varpi + \frac{1}{1+\epsilon} - 1} \end{aligned}$$

Using the above result and (25), it is easy to see that, for (24) to hold, the following conditions are sufficient:

$$(2 - \frac{r}{1+\epsilon})\varpi + \frac{1}{1+\epsilon} - 1 - \frac{3}{4} \geq 0, \quad (4p - r)\varpi + 1 - 2p - \frac{3}{4} \geq 0, \quad \text{and } (2 - r)\varpi + -\frac{3}{4} \geq 0.$$

Using the fact that $0 < \varpi < \frac{1}{2}$, and taking ϵ sufficiently close to zero, we can see that the required condition for (24) to hold is, $(8p - 1)/4(4p - r) \leq \varpi < \frac{1}{2}$, which completes the proof.

Step 2 : First approximation for the estimators

Taking advantage of Step 1, it is enough to derive the asymptotic distributions of $[\widehat{H(c)}, \widehat{G(c)}]_T^{LIN'}$ and $[\widehat{H(c)}, \widehat{G(c)}]_T^{AN'}$. We show that the two estimators $[\widehat{H(c)}, \widehat{G(c)}]_T^{LIN'}$ and $[\widehat{H(c)}, \widehat{G(c)}]_T^{AN'}$ can be approximated by some quantity with an error of approximation of order smaller than $\Delta_n^{-1/4}$. To see this, we set

$$\begin{aligned} [\widehat{H(c)}, \widehat{G(c)}]_T^A &= \frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left(\left(\partial_{gh} H \partial_{ab} G \right)(c_i^n) \left[(\hat{c}_{i+k_n}^{\prime n, gh} - \hat{c}_i^{\prime n, gh})(\hat{c}_{i+k_n}^{\prime n, ab} - \hat{c}_i^{\prime n, ab}) \right. \right. \\ &\quad \left. \left. - \frac{2}{k_n} (\hat{c}_i^{\prime n, ga} \hat{c}_i^{\prime n, hb} + \hat{c}_i^{\prime n, gb} \hat{c}_i^{\prime n, ha}) \right] \right), \end{aligned}$$

with $c_i^n = c_{(i-1)\Delta_n}$ and the superscript A being a short for the word "approximate". For notational simplicity, we do not index the above quantity by a prime although it depends on $\hat{c}_i^n$ instead $\hat{c}_i^{\prime n}$. We aim to prove that

$$\Delta_n^{-1/4} \left([\widehat{H(c)}, \widehat{G(c)}]_T^{LIN'} - [\widehat{H(c)}, \widehat{G(c)}]_T^A \right) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \Delta_n^{-1/4} \left([\widehat{H(c)}, \widehat{G(c)}]_T^{AN'} - [\widehat{H(c)}, \widehat{G(c)}]_T^A \right) \xrightarrow{\mathbb{P}} 0. \quad (28)$$

To prove (28), we introduce some new notations. Following Jacod and Rosenbaum (2012), we define

$$\alpha_i^n = (\Delta_i^n Y')(\Delta_i^n Y')^\top - c_i^n \Delta_n, \quad \beta_i^n = \hat{c}_i'^n - c_i^n, \quad \text{and} \quad \gamma_i^n = \hat{c}_{i+k_n}'^n - \hat{c}_i'^n, \quad (29)$$

which satisfy

$$\beta_i^n = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} (\alpha_{i+j}^n + (c_{i+j}^n - c_i^n) \Delta_n) \quad \text{and} \quad \gamma_i^n = \beta_{i+k_n}^n - \beta_i^n + \Delta_n (c_{i+k_n}^n - c_i^n). \quad (30)$$

The following holds

$$\begin{aligned} [H(\widehat{c}), G(c)]_T^{LIN'} - [H(\widehat{c}), G(c)]_T^A &= \frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \psi_i^n(g, h, a, b), \\ [H(\widehat{c}), G(c)]_T^{AN'} - [H(\widehat{c}), G(c)]_T^A &= \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \left(\chi_i^n - \sum_{g,h,a,b=1}^d (\partial_{gh} H \partial_{ab} G)(c_i^n) \gamma_i^{n,gh} \gamma_i^{n,ab} \right), \end{aligned}$$

with

$$\begin{aligned} \psi_i^n(g, h, a, b) &= \left((\partial_{gh} H \partial_{ab} G)(\hat{c}_i'^n) - (\partial_{gh} H \partial_{ab} G)(c_i^n) \right) \gamma_i^{n,gh} \gamma_i^{n,ab}, \\ \chi_i^n &= \left(H(\hat{c}_{i+k_n}'^n) - H(\hat{c}_i'^n) \right) \left(G(\hat{c}_{i+k_n}'^n) - G(\hat{c}_i'^n) \right). \end{aligned}$$

By Taylor expansion, we have

$$\begin{aligned} (\partial_{gh} S \partial_{ab} G)(\hat{c}_i'^n) - (\partial_{gh} S \partial_{ab} G)(c_i^n) &= \sum_{x,y=1}^d \left(\partial_{xy,gh}^2 S \partial_{ab} G + \partial_{xy,ab}^2 G \partial_{gh} S \right) (c_i^n) \beta_i^{n,xy} \\ &+ \frac{1}{2} \sum_{j,k,x,y=1}^d \left(\partial_{jk,xy,gh}^3 S \partial_{ab} G + \partial_{xy,gh}^2 S \partial_{jk,ab}^2 G + \partial_{jk,xy,ab}^3 G \partial_{gh} S + \partial_{xy,ab}^2 G \partial_{jk,gh}^2 S \right) (\tilde{c}_i^n) \beta_i^{n,xy} \beta_i^{n,jk} \end{aligned}$$

and

$$\begin{aligned} S(\hat{c}_{i+k_n}'^n) - S(\hat{c}_i'^n) &= \sum_{gh} \partial_{gh} S(c_i^n) \gamma_i^{n,gh} + \sum_{j,k,g,h} \partial_{jk,gh}^2 S(c_i^n) \gamma_i^{n,gh} \beta_i^{n,jk} + \frac{1}{2} \sum_{x,y,g,h} \partial_{xy,gh}^2 S(c_i^n) \gamma_i^{n,gh} \gamma_i^{n,xy} \\ &+ \frac{1}{2} \sum_{x,y,j,k,g,h} \partial_{xy,jk,gh}^3 S(CC_i^{n,S}) \gamma_i^{n,gh} \beta_i^{n,xy} \beta_i^{n,jk} + \frac{1}{6} \sum_{j,k,x,y,g,h} \partial_{jk,xy,gh}^3 S(C_i^{n,S}) \gamma_i^{n,jk} \gamma_i^{n,gh} \gamma_i^{n,xy}, \end{aligned}$$

for $S \in \{H, G\}$, $\tilde{c}_i^n = \lambda c_i^n + (1 - \lambda) \hat{c}_i'^n$, $C_i^{n,S} = \lambda_S \hat{c}_i'^n + (1 - \lambda_S) \hat{c}_{i+k_n}'^n$, $CC_i^{n,S} = \mu_S c_i^n + (1 - \mu_S) \hat{c}_i'^n$ for $\lambda, \lambda_H, \mu_H, \lambda_G, \mu_G \in [0, 1]$. Rigourously $\tilde{c}_i^n$ and λ depend on g, h, a , and b . To avoid too cumbersome notation we do not emphasize this additional dependence while we do take it into account in our derivations.

We recall some well-known results. For any continuous Itô process Z_t , we have

$$\mathbb{E} \left(\sup_{w \in [0, s]} \|Z_{t+w} - Z_t\|^q \middle| \mathcal{F}_t \right) \leq K_q s^{q/2}, \quad \text{and} \quad \left\| \mathbb{E} (Z_{t+s} - Z_t) \middle| \mathcal{F}_t \right\| \leq K s. \quad (31)$$

Set $\mathcal{F}_i^n = \mathcal{F}_{(i-1)\Delta_n}$. By (4.10) in Jacod and Rosenbaum (2013) we have,

$$\mathbb{E} \left(\left\| \alpha_i^n \right\|^q \middle| \mathcal{F}_i^n \right) \leq K_q \Delta_n^q \text{ for all } q \geq 0 \text{ and } \mathbb{E} \left(\left\| \sum_{j=0}^{k_n-1} \alpha_{i+j}^n \right\|^q \middle| \mathcal{F}_i^n \right) \leq K_q \Delta_n^q k_n^{q/2} \text{ whenever } q \geq 2. \quad (32)$$

Combining (40), (38), (39) with $Z = c$ and the Hölder inequality yields for $q \geq 2$,

$$\mathbb{E} \left(\left\| \beta_i^n \right\|^q \middle| \mathcal{F}_i^n \right) \leq K_q \Delta^{q/4}, \quad \text{and} \quad \mathbb{E} \left(\left\| \gamma_i^n \right\|^q \middle| \mathcal{F}_i^n \right) \leq K_q \Delta^{q/4}. \quad (33)$$

The bound in the first equation of (41) is more tighter than that in (4.11) of Jacod and Rosenbaum (2012) due to the absence of volatility jumps. This tighter bound will be useful later on for deriving the CLT for the approximate estimator (Step 3). By the boundedness of c_t and the polynomial growth assumption, we have

$$\left| (\partial_{jk,xy,ab}^3 G \partial_{gh} H + \partial_{xy,gh}^2 H \partial_{jk,ab}^2 G)(\tilde{c}_i^n) \beta_i^{n,xy} \beta_i^{n,jk} \gamma_i^{n,gh} \gamma_i^{n,ab} \right| \leq K(1 + \|\tilde{c}_i^n\|)^{2(p-2)} \|\beta_i^n\|^2 \|\gamma_i^n\|^2.$$

Recalling $\tilde{c}_i^n = \lambda c_i^n + (1 - \lambda) \tilde{c}_i'^n$ and using the convexity of the function $x^{2(p-2)}$, we can refine the last inequality as follows:

$$\left| (\partial_{jk,xy,ab}^3 G \partial_{gh} H + \partial_{xy,gh}^2 H \partial_{jk,ab}^2 G)(\tilde{c}_i^n) \beta_i^{n,xy} \beta_i^{n,jk} \gamma_i^{n,gh} \gamma_i^{n,ab} \right| \leq K(1 + \|\beta_i^n\|^{2(p-2)}) \|\beta_i^n\|^2 \|\gamma_i^n\|^2. \quad (34)$$

By Taylor expansion, the polynomial growth assumption and using similar idea as for (34), we have

$$\chi_i^n - \sum_{g,h,a,b} (\partial_{gh} H \partial_{ab} G)(c_i^n) \gamma_i^{n,gh} \gamma_i^{n,ab} = \sum_{g,h,a,b,j,k} (\partial_{gh} H \partial_{jk,xy}^2 G + \partial_{gh} G \partial_{jk,xy}^2 H)(c_i^n) (\gamma_i^{n,gh} + \frac{1}{2} \beta_i^{n,gh}) \gamma_i^{n,ab} \gamma_i^{n,jk} + \varphi_i^n$$

$$\sum_{g,h,a,b} (\partial_{gh} H \partial_{ab} G)(\tilde{c}_i'^n) - (\partial_{gh} H \partial_{ab} G)(c_i^n) = \sum_{g,h,a,b,xy} (\partial_{gh} H \partial_{ab,xy}^2 G + \partial_{ab} G \partial_{gh,xy}^2 G)(c_i^n) (\beta_i^{n,xy}) \gamma_i^{n,gh} \gamma_i^{n,ab} + \delta_i^n$$

with $\mathbb{E}(|\varphi_i^n| | \mathcal{F}_i^n) \leq K \Delta_n$ and $\mathbb{E}(|\delta_i^n| | \mathcal{F}_i^n) \leq K \Delta_n$ which follow from applying Cauchy-Schwartz inequality together with (41). Given that $k_n = \theta(\Delta_n)^{-1/2}$, a direct implication of the previous inequalities is

$$\frac{3\Delta_n^{-1/4}}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \varphi_i^n \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \frac{3\Delta_n^{-1/4}}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \delta_i^n \xrightarrow{\mathbb{P}} 0.$$

Therefore, in order to prove the two claims in (28), it suffices to show

$$\frac{3\Delta_n^{-1/4}}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{g,h,a,b,j,k} (\partial_{gh} H \partial_{jk,ab}^2 G + \partial_{gh} H \partial_{jk,ab}^2 G)(c_i^n) \gamma_i^{n,gh} \gamma_i^{n,ab} \gamma_i^{n,jk} \xrightarrow{\mathbb{P}} 0, \quad (35)$$

$$\frac{3\Delta_n^{-1/4}}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{g,h,a,b,j,k} (\partial_{gh} H \partial_{jk,ab}^2 G + \partial_{gh} H \partial_{jk,ab}^2 G)(c_i^n) \beta_i^{n,gh} \gamma_i^{n,ab} \gamma_i^{n,jk} \xrightarrow{\mathbb{P}} 0. \quad (36)$$

For any càdlàg bounded process Z , we set

$$\eta_{t,s}(Z) = \sqrt{\mathbb{E} \left(\sup_{0 < u \leq s} \|Z_{t+u} - Z_t\|^2 | \mathcal{F}_t^n \right)},$$

$$\eta_{i,j}^n(Z) = \sqrt{\mathbb{E} \left(\sup_{0 \leq u \leq j\Delta_n} \|Z_{(i-1)\Delta_n+u} - Z_{(i-1)\Delta_n}\|^2 | \mathcal{F}_i^n \right)}.$$

In order to prove (35) and (36), we introduce the following lemmas.

Lemma 1. *For any càdlàg bounded process Z , for all $t, s > 0$, $j, k \geq 0$, set $\eta_{t,s} = \eta_{t,s}(Z)$, then we have:*

$$\Delta_n \mathbb{E} \left(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,k_n} \right) \longrightarrow 0, \quad \Delta_n \mathbb{E} \left(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,2k_n} \right) \longrightarrow 0,$$

$$\mathbb{E}(\eta_{i+j,k} | \mathcal{F}_i^n) \leq \eta_{i,j+k} \quad \text{and} \quad \Delta_n \mathbb{E} \left(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,4k_n} \right) \longrightarrow 0.$$

The first three claims of Lemma 6 are proved in Jacod and Rosenbaum (2012). The last result can be proved

similarly to the first two.

Lemma 2. *Let Z be a continuous Itô process with drift term b_t^Z and spot variance process c_t^Z , set $\eta_{t,s} = \eta_{t,s}(b^Z, c^Z)$, then the following bounds can be established:*

$$\begin{aligned}
|\mathbb{E}(Z_t | \mathcal{F}_0) - tb_0^Z| &\leq Kt\eta_{0,t} \\
|\mathbb{E}(Z_t^j Z_t^k - tc_0^{Z,jk} | \mathcal{F}_0)| &\leq Kt^{3/2}(\sqrt{\Delta_n} + \eta_{0,t}) \\
|\mathbb{E}((Z_t^j Z_t^k - tc_0^{Z,jk})(c_t^{Z,lm} - c_0^{Z,lm}) | \mathcal{F}_0)| &\leq Kt^2 \\
|\mathbb{E}(Z_t^j Z_t^k Z_t^l Z_t^m | \mathcal{F}_0) - \Delta_n^2(c_0^{Z,jk} c_0^{Z,lm} + c_0^{Z,jl} c_0^{Z,km} + c_0^{Z,jm} c_0^{Z,kl})| &\leq Kt^{5/2} \\
|\mathbb{E}(Z_t^j Z_t^k Z_t^l | \mathcal{F}_0)| &\leq Kt^2 \\
|\mathbb{E}(\prod_{l=1}^6 Z_t^{j_l} | \mathcal{F}_0) - \frac{\Delta_n^3}{6} \sum_{l < l'} \sum_{k < k'} \sum_{m < m'} c_0^{Z,j_l j_{l'}} c_0^{Z,j_k j_{k'}} c_0^{Z,j_m j_{m'}}| &\leq Kt^{7/2}
\end{aligned}$$

The first four claims of Lemma 7 are parts of Lemma 4.1 in [Jacod and Rosenbaum \(2012\)](#). The two remaining statements can be shown similarly.

Lemma 3. *Let ζ_i^n be a r -dimensional $\mathcal{F}_i^n$ measurable process satisfying $\|\mathbb{E}(\zeta_i^n | \mathcal{F}_{i-1}^n)\| \leq L'$ and $\mathbb{E}(\|\zeta_i^n\|^q | \mathcal{F}_{i-1}^n) \leq L_q$, let also φ_i^n be a real-valued $\mathcal{F}_i^n$ measurable process that fulfills $\mathbb{E}(\|\varphi_{i+j-1}^n\|^q | \mathcal{F}_{i-1}^n) \leq L^q$ for $q \geq 2$ and $1 \leq j \leq 2k_n - 1$, then we have*

$$\mathbb{E}\left(\left\|\sum_{j=1}^{2k_n-1} \varphi_{i+j-1}^n \zeta_{i+j}^n\right\|^q \middle| \mathcal{F}_{i-1}^n\right) \leq K_q L^q (L_q k_n^{q/2} + L'^q k_n^q).$$

Proof of Lemma 5

Set

$$\xi_i^n = \varphi_{i-1}^n \zeta_i^n, \quad \xi_i'^n = \mathbb{E}(\xi_i^n | \mathcal{F}_{i-1}^n) = \mathbb{E}(\varphi_{i-1}^n \zeta_i^n | \mathcal{F}_{i-1}^n) = \varphi_{i-1}^n \mathbb{E}(\zeta_i^n | \mathcal{F}_{i-1}^n), \quad \text{and} \quad \xi_i''^n = \xi_i^n - \xi_i'^n.$$

Given that $\|\mathbb{E}(\zeta_i^n | \mathcal{F}_{i-1}^n)\| \leq L'$, we have $\|\xi_i'^n\| \leq L' |\varphi_{i-1}^n|$. By the convexity of the function x^q which holds for $q \geq 2$, we have,

$$\left\|\sum_{j=1}^{2k_n-1} \xi_{i+j}^n\right\|^q \leq K \left(\left\|\sum_{j=1}^{2k_n-1} \xi_{i+j}'^n\right\|^q + \left\|\sum_{j=1}^{2k_n-1} \xi_{i+j}''^n\right\|^q\right).$$

Therefore, on one hand we have

$$\left\|\sum_{j=1}^{2k_n-1} \xi_{i+j}'^n\right\|^q \leq K k_n^{q-1} \sum_{j=1}^{2k_n-1} \|\xi_{i+j}'^n\|^q \leq K k_n^{q-1} L'^q \sum_{j=1}^{2k_n-1} |\varphi_{i+j-1}^n|^q,$$

which given that, $\mathbb{E}(\|\varphi_{i+j-1}^n\|^q | \mathcal{F}_{i-1}^n) \leq L^q$, satisfies:

$$\mathbb{E}\left(\left\|\sum_{j=1}^{2k_n-1} \xi_{i+j}'^n\right\|^q \middle| \mathcal{F}_{i-1}^n\right) \leq K L'^q k_n^{q-1} \sum_{j=1}^{2k_n-1} \mathbb{E}(|\varphi_{i+j-1}^n|^q | \mathcal{F}_{i-1}^n) \leq K L'^q k_n^q L^q.$$

On the other hand, we have $\mathbb{E}(\|\xi_{i+j}''^n\|^q | \mathcal{F}_{i-1}^n) \leq \mathbb{E}(\|\xi_{i+j}^n\|^q | \mathcal{F}_{i-1}^n) \leq L_q L^q$ and $\mathbb{E}(\xi_{i+j}''^n | \mathcal{F}_{i-1}^n) = 0$, where the first inequality is a consequence of $\mathbb{E}(\|\xi_{i+j}'^n\|^q | \mathcal{F}_{i-1}^n) \leq \mathbb{E}(\|\xi_{i+j}^n\|^q | \mathcal{F}_{i-1}^n) \leq L_q L^q$ which can be proved using the Jensen inequality and the law of iterated expectation. Hence applying Lemma B.2 of [Aït-Sahalia and](#)

Jacod (2014) we have

$$\mathbb{E}(\|\sum_{j=1}^{2k_n-1} \xi_{i+j}''^n\|^q | \mathcal{F}_{i-1}^n) \leq K_q L^q L_q k_n^{q/2}.$$

To see the latter, we first prove that the required condition $\mathbb{E}(\|\xi_i^n\|^q | \mathcal{F}_{i-1}^n) \leq L_q L^q$ in the Lemma B.2 of Aït-Sahalia and Jacod (2014) can be replaced by $\mathbb{E}(\|\xi_{i+j}^n\|^q | \mathcal{F}_{i-1}^n) \leq L_q L^q$ for $1 \leq j \leq 2k_n - 1$ without altering the result.

Lemma 4. *We have:*

$$\begin{aligned} & \left| \mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \gamma_{i+2k_n}^{n,gh} \gamma_{i+2k_n}^{n,ab} | \mathcal{F}_i^n) - \frac{4}{k_n^2} (c_i^{n,ga} c_i^{n,hb} + c_i^{n,gb} c_i^{n,ha}) (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \right. \\ & - \frac{4\Delta_n}{3} (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \bar{c}_i^{n,gh,ab} - \frac{4\Delta_n}{3} (c_i^{n,ga} c_i^{n,hb} - c_i^{n,gb} c_i^{n,ha}) \bar{c}_i^{n,jk,lm} \\ & \left. - \frac{4(k_n \Delta_n)^2}{9} \bar{c}_i^{n,gh,ab} \bar{c}_i^{n,jk,lm} \right| \leq K \Delta_n (\Delta_n^{1/8} + \eta_{i,4k_n}^n). \end{aligned}$$

Throughout, we use the expression "successive conditioning" to refer to the following equalities,

$$\begin{aligned} x_1 y_1 - x_0 y_0 &= x_0 (y_1 - y_0) + y_0 (x_1 - x_0) + (x_1 - x_0)(y_1 - y_0) \\ x_1 y_1 z_1 - x_0 y_0 z_0 &= x_0 y_0 (z_1 - z_0) + x_0 z_0 (y_1 - y_0) + y_0 z_0 (x_1 - x_0) + x_0 (y_0 - y_1)(z_0 - z_1) \\ &\quad + y_0 (x_0 - x_1)(z_0 - z_1) + z_0 (x_0 - x_1)(y_0 - y_1) + (x_1 - x_0)(y_1 - y_0)(z_1 - z_0) \end{aligned}$$

which hold for any real numbers $x_0, y_0, z_0, x_1, y_1, z_1$.

Proof of Lemma 4

To prove Lemma 4, we first note that $\gamma_i^{n,jk} \gamma_i^{n,lm}$ is $\mathcal{F}_{i+2k_n}^n$ -measurable. Then, by the law of iterated expectation we have

$$\mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \gamma_{i+2k_n}^{n,gh} \gamma_{i+2k_n}^{n,ab} | \mathcal{F}_i^n) = \mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \mathbb{E}(\gamma_{i+2k_n}^{n,gh} \gamma_{i+2k_n}^{n,ab} | \mathcal{F}_{i+2k_n}^n) | \mathcal{F}_i^n).$$

From equation (3.27) in Jacod and Rosenbaum (2012), we have

$$\begin{aligned} & |\mathbb{E}(\gamma_{i+2k_n}^{n,gh} \gamma_{i+2k_n}^{n,ab} | \mathcal{F}_{i+2k_n}^n) - \frac{2}{k_n} (c_{i+2k_n}^{n,ga} c_{i+2k_n}^{n,hb} + c_{i+2k_n}^{n,gb} c_{i+2k_n}^{n,ha}) - \frac{2k_n \Delta_n}{3} \bar{c}_{i+2k_n}^{n,gh,ab}| \leq K \sqrt{\Delta_n} (\Delta_n^{1/8} + \eta_{i+2k_n,2k_n}^n), \\ & |\mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} | \mathcal{F}_i^n) - \frac{2}{k_n} (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) - \frac{2k_n \Delta_n}{3} \bar{c}_i^{n,jk,lm}| \leq K \sqrt{\Delta_n} (\Delta_n^{1/8} + \eta_{i,2k_n}^n). \end{aligned}$$

I also holds that

$$\begin{aligned} & |\mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \left[\mathbb{E}(\gamma_{i+2k_n}^{n,gh} \gamma_{i+2k_n}^{n,ab} | \mathcal{F}_{i+2k_n}^n) - \frac{2}{k_n} (c_{i+2k_n}^{n,ga} c_{i+2k_n}^{n,hb} + c_{i+2k_n}^{n,gb} c_{i+2k_n}^{n,ha}) - \frac{2k_n \Delta_n}{3} \bar{c}_{i+2k_n}^{n,gh,ab} \right] | \mathcal{F}_i^n)| \\ & \leq \sqrt{\Delta_n} \mathbb{E}(|\gamma_i^{n,jk}| |\gamma_i^{n,lm}| (\Delta_n^{1/8} + \eta_{i+2k_n,2k_n}^n) | \mathcal{F}_i^n) \leq K \sqrt{\Delta_n} \Delta_n^{1/8} \mathbb{E}(|\gamma_i^{n,jk}| |\gamma_i^{n,lm}| | \mathcal{F}_i^n) \\ & + K \sqrt{\Delta_n} \mathbb{E}(|\gamma_i^{n,jk}| |\gamma_i^{n,lm}| \eta_{i+2k_n,2k_n}^n | \mathcal{F}_i^n) \leq K \Delta_n (\Delta_n^{1/8} + \eta_{i,4k_n}^n), \end{aligned}$$

where the last inequality follows from Lemma 6. Using (39) successively with $Z = c$ and $Z = \bar{c}$ (recall that the latter holds under Assumption 2), together with the successive conditioning, we have

$$|\mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \left[\frac{2}{k_n} (c_{i+2k_n}^{n,ga} c_{i+2k_n}^{n,hb} + c_{i+2k_n}^{n,gb} c_{i+2k_n}^{n,ha}) + \frac{2k_n \Delta_n}{3} \bar{c}_{i+2k_n}^{n,gh,ab} - \frac{2}{k_n} (c_i^{n,ga} c_i^{n,hb} + c_i^{n,gb} c_i^{n,ha}) \right. \\ \left. - \frac{4\Delta_n}{3} (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \bar{c}_i^{n,gh,ab} - \frac{4\Delta_n}{3} (c_i^{n,ga} c_i^{n,hb} - c_i^{n,gb} c_i^{n,ha}) \bar{c}_i^{n,jk,lm} - \frac{4(k_n \Delta_n)^2}{9} \bar{c}_i^{n,gh,ab} \bar{c}_i^{n,jk,lm} \right] | \mathcal{F}_i^n)|$$

$$\begin{aligned}
& -\frac{2k_n\Delta_n}{3}\bar{c}_i^{n,gh,ab}\Big]\Big|\mathcal{F}_i^n\Big)| \leq K\Delta_n\Delta_n^{1/4}, \\
& |\mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\Big[\frac{2}{k_n}(c_i^{n,ga}c_i^{n,hb}+c_i^{n,gb}c_i^{n,ha})+\frac{2k_n\Delta_n}{3}\bar{c}_i^{n,gh,ab}\Big]-\Big[\frac{2}{k_n}(c_i^{n,jl}c_i^{n,km}+c_i^{n,jm}c_i^{n,kl})+\frac{2k_n\Delta_n}{3}\bar{c}_i^{n,jk,lm}\Big] \\
& \times \Big[\frac{2}{k_n}(c_i^{n,ga}c_i^{n,hb}+c_i^{n,gb}c_i^{n,ha})+\frac{2k_n\Delta_n}{3}\bar{c}_i^{n,gh,ab}\Big]\Big|\mathcal{F}_i^n\Big)| \leq K\Delta_n(\Delta_n^{1/8}+\eta_{i,2k_n}^n).
\end{aligned}$$

The last inequality yields the result.

Lemma 5. *Let ζ_i^n be a r -dimensional $\mathcal{F}_i^n$ measurable process satisfying $\|\mathbb{E}(\zeta_i^n|\mathcal{F}_{i-1}^n)\| \leq L'$ and $\mathbb{E}(\|\zeta_i^n\|^q|\mathcal{F}_{i-1}^n) \leq L_q$, let also φ_i^n be a real-valued $\mathcal{F}_i^n$ measurable process that fulfills $\mathbb{E}(\|\varphi_{i+j-1}^n\|^q|\mathcal{F}_{i-1}^n) \leq L^q$ for $q \geq 2$ and $1 \leq j \leq 2k_n - 1$, then we have*

$$\mathbb{E}\left(\left\|\sum_{j=1}^{2k_n-1}\varphi_{i+j-1}^n\zeta_{i+j}^n\right\|^q\Big|\mathcal{F}_{i-1}^n\right) \leq K_q L^q (L_q k_n^{q/2} + L'^q k_n^q).$$

We introduce some new notations. Following [Jacod and Rosenbaum \(2012\)](#), we define

$$\alpha_i^n = (\Delta_i^n Y')(\Delta_i^n Y')^\top - c_i^n \Delta_n, \quad \beta_i^n = \hat{c}_i'^n - c_i^n, \quad \text{and} \quad \gamma_i^n = \hat{c}_{i+k_n}'^n - \hat{c}_i'^n, \quad (37)$$

which satisfy

$$\beta_i^n = \frac{1}{k_n\Delta_n} \sum_{j=0}^{k_n-1} (\alpha_{i+j}^n + (c_{i+j}^n - c_i^n)\Delta_n) \quad \text{and} \quad \gamma_i^n = \beta_{i+k_n}^n - \beta_i^n + \Delta_n(c_{i+k_n}^n - c_i^n). \quad (38)$$

We recall some well-known results. For any continuous Itô process Z_t , we have

$$\mathbb{E}\left(\sup_{w \in [0,s]} \|Z_{t+w} - Z_t\|^q \Big| \mathcal{F}_t\right) \leq K_q s^{q/2}, \quad \text{and} \quad \mathbb{E}\left(\|Z_{t+s} - Z_t\| \Big| \mathcal{F}_t\right) \leq Ks. \quad (39)$$

Set $\mathcal{F}_i^n = \mathcal{F}_{(i-1)\Delta_n}$. By (4.10) in [Jacod and Rosenbaum \(2013\)](#) we have,

$$\mathbb{E}\left(\|\alpha_i^n\|^q \Big| \mathcal{F}_i^n\right) \leq K_q \Delta_n^q \text{ for all } q \geq 0 \text{ and } \mathbb{E}\left(\left\|\sum_{j=0}^{k_n-1} \alpha_{i+j}^n\right\|^q \Big| \mathcal{F}_i^n\right) \leq K_q \Delta_n^q k_n^{q/2} \text{ whenever } q \geq 2. \quad (40)$$

Combining (40), (38), (39) with $Z = c$ and the Hölder inequality yields for $q \geq 2$,

$$\mathbb{E}\left(\|\beta_i^n\|^q \Big| \mathcal{F}_i^n\right) \leq K_q \Delta_n^{q/4}, \quad \text{and} \quad \mathbb{E}\left(\|\gamma_i^n\|^q \Big| \mathcal{F}_i^n\right) \leq K_q \Delta_n^{q/4}. \quad (41)$$

For any càdlàg bounded process Z , we set

$$\begin{aligned}
\eta_{t,s}(Z) &= \sqrt{\mathbb{E}\left(\sup_{0 < u \leq s} \|Z_{t+u} - Z_t\|^2 \Big| \mathcal{F}_i^n\right)}, \\
\eta_{i,j}^n(Z) &= \sqrt{\mathbb{E}\left(\sup_{0 \leq u \leq j\Delta_n} \|Z_{(i-1)\Delta_n+u} - Z_{(i-1)\Delta_n}\|^2 \Big| \mathcal{F}_i^n\right)}.
\end{aligned}$$

Lemma 6. *For any càdlàg bounded process Z , for all $t, s > 0$, $j, k \geq 0$, set $\eta_{t,s} = \eta_{t,s}(Z)$, then we have:*

$$\begin{aligned}
\Delta_n \mathbb{E}\left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \eta_{i,k_n}\right) &\longrightarrow 0, \quad \Delta_n \mathbb{E}\left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \eta_{i,2k_n}\right) \longrightarrow 0, \\
\mathbb{E}\left(\eta_{i+j,k} \Big| \mathcal{F}_i^n\right) &\leq \eta_{i,j+k} \quad \text{and} \quad \Delta_n \mathbb{E}\left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \eta_{i,4k_n}\right) \longrightarrow 0.
\end{aligned}$$

The first three claims of Lemma 6 are proved in [Jacod and Rosenbaum \(2012\)](#). The last result can be proved

similarly to the first two.

Lemma 7. *Let Z be a continuous Itô process with drift term b_t^Z and spot variance process c_t^Z , set $\eta_{t,s} = \eta_{t,s}(b^Z, c^Z)$, then the following bounds can be established:*

$$\begin{aligned}
|\mathbb{E}(Z_t | \mathcal{F}_0) - tb_0^Z| &\leq Kt\eta_{0,t} \\
|\mathbb{E}(Z_t^j Z_t^k - tc_0^{Z,jk} | \mathcal{F}_0)| &\leq Kt^{3/2}(\sqrt{\Delta_n} + \eta_{0,t}) \\
|\mathbb{E}((Z_t^j Z_t^k - tc_0^{Z,jk})(c_t^{Z,lm} - c_0^{Z,lm}) | \mathcal{F}_0)| &\leq Kt^2 \\
|\mathbb{E}(Z_t^j Z_t^k Z_t^l Z_t^m | \mathcal{F}_0) - \Delta_n^2(c_0^{Z,jk} c_0^{Z,lm} + c_0^{Z,jl} c_0^{Z,km} + c_0^{Z,jm} c_0^{Z,kl})| &\leq Kt^{5/2} \\
|\mathbb{E}(Z_t^j Z_t^k Z_t^l | \mathcal{F}_0)| &\leq Kt^2 \\
|\mathbb{E}(\prod_{l=1}^6 Z_t^{j_l} | \mathcal{F}_0) - \frac{\Delta_n^3}{6} \sum_{l < l'} \sum_{k < k'} \sum_{m < m'} c_0^{Z,j_l j_{l'}} c_0^{Z,j_k j_{k'}} c_0^{Z,j_m j_{m'}}| &\leq Kt^{7/2}
\end{aligned}$$

The first four claims of Lemma 7 are parts of Lemma 4.1 in [Jacod and Rosenbaum \(2012\)](#). The two remaining statements can be shown similarly.

Lemma 8. *The following results hold:*

$$|\mathbb{E}(\beta_i^{n,jk} \beta_i^{n,lm} \beta_i^{n,gh} | \mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}^n), \quad (42)$$

$$|\mathbb{E}(\beta_i^{n,jk} \beta_i^{n,lm} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}^n), \quad (43)$$

$$|\mathbb{E}(\beta_i^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm})(c_{i+k_n}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}^n), \quad (44)$$

$$|\mathbb{E}(\beta_i^{n,jk} \gamma_i^{n,lm} \gamma_i^{n,gh} | \mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,2k_n}^n), \quad (45)$$

$$|\mathbb{E}(\gamma_i^{n,jk} \gamma_i^{n,lm} \gamma_i^{n,gh} | \mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,2k_n}^n). \quad (46)$$

Proof of (42) in Lemma 8

We start by obtaining some useful bounds for some quantities of interest. First, using the second statement in Lemma 7 applied to $Z = Y'$, we have

$$|\mathbb{E}(\alpha_i^{n,jk} | \mathcal{F}_i^n)| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i,1}^n). \quad (47)$$

Second, by repeated application of the Cauchy-Schwartz inequality and making use of the third and last statements in Lemma 7 as well as (39) with $Z = c$, it can be shown that

$$\left| \mathbb{E}(\alpha_i^{n,jk} \alpha_i^{n,lm} | \mathcal{F}_i^n) - \Delta_n^2 (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \right| \leq K\Delta_n^{5/2}. \quad (48)$$

Next, by successive conditioning and using the bound in (39) for $Z = c$ as well as (47) and (48), we have for $0 \leq u \leq k_n - 1$,

$$\left| \mathbb{E}(\alpha_{i+u}^{n,jk} | \mathcal{F}_i^n) \right| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i,u}^n), \quad (49)$$

$$\left| \mathbb{E}(\alpha_{i+u}^{n,jk} \alpha_{i+u}^{n,lm} | \mathcal{F}_i^n) - \Delta_n^2 (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \right| \leq K\Delta_n^{5/2}, \quad (50)$$

To show (42), we first observe that $\beta_i^{n,jk} \beta_i^{n,lm} \beta_i^{n,gh}$ can be decomposed as

$$\begin{aligned}
\beta_i^{n,jk} \beta_i^{n,lm} \beta_i^{n,gh} &= \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,u}^{n,lm} \zeta_{i,u}^{n,gh} + \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-2} \sum_{v=u+1}^{k_n-1} \left[\zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm} \zeta_{i,v}^{n,gh} + \zeta_{i,u}^{n,gh} \zeta_{i,v}^{n,jk} \zeta_{i,v}^{n,lm} \right. \\
&\quad \left. + \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,gh} \zeta_{i,v}^{n,jk} \right] + \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-2} \sum_{v=u+1}^{k_n-1} \left[\zeta_{i,u}^{n,jk} \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,gh} + \zeta_{i,u}^{n,gh} \zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm} + \zeta_{i,u}^{n,lm} \zeta_{i,u}^{n,gh} \zeta_{i,v}^{n,jk} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-3} \sum_{v=u+1}^{k_n-2} \sum_{w=v+1}^{k_n-1} \left[\zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm} \zeta_{i,w}^{n,gh} + \zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,gh} \zeta_{i,w}^{n,lm} + \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,jk} \zeta_{i,w}^{n,gh} + \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,gh} \zeta_{i,w}^{n,jk} \right. \\
& \left. + \zeta_{i,u}^{n,gh} \zeta_{i,v}^{n,lm} \zeta_{i,w}^{n,jk} + \zeta_{i,u}^{n,gh} \zeta_{i,v}^{n,jk} \zeta_{i,w}^{n,lm} \right],
\end{aligned}$$

with $\zeta_{i,u}^n = \alpha_{i+u}^n + (c_{i+u}^n - c_i^n) \Delta_n$, which satisfies $\mathbb{E}(\|\zeta_{i,u}^n\|^q | \mathcal{F}_i^n) \leq K \Delta_n^q$ for $q \geq 2$.

Set

$$\begin{aligned}
\xi_i^n(1) &= \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,u}^{n,lm} \zeta_{i,u}^{n,gh}, \quad \xi_i^n(2) = \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-2} \sum_{v=u+1}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm} \zeta_{i,v}^{n,gh} \\
\xi_i^n(3) &= \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-2} \sum_{v=u+1}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,gh} \quad \text{and} \quad \xi_i^n(4) = \frac{1}{k_n^3 \Delta_n^3} \sum_{u=0}^{k_n-3} \sum_{v=u+1}^{k_n-2} \sum_{w=v+1}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm} \zeta_{i,w}^{n,gh}.
\end{aligned}$$

The following bounds can be established,

$$\begin{aligned}
|\mathbb{E}(\xi_i^n(1) | \mathcal{F}_i^n)| &\leq K \Delta_n, \quad |\mathbb{E}(\xi_i^n(2) | \mathcal{F}_i^n)| \leq K \Delta_n, \quad |\mathbb{E}(\xi_i^n(3) | \mathcal{F}_i^n)| \leq K \Delta_n \quad \text{and} \\
|\mathbb{E}(\xi_i^n(4) | \mathcal{F}_i^n)| &\leq K \Delta_n^{3/4} (\Delta_n^{1/4} + \eta_{i,k_n}).
\end{aligned}$$

Proof of $|\mathbb{E}(\xi_i^n(1) | \mathcal{F}_i^n)| \leq K \Delta_n$

The result readily follows from an application of the Cauchy Schwartz inequality together with the bound $\mathbb{E}(\|\zeta_{i+u}^n\|^q | \mathcal{F}_i^n) \leq K_q \Delta_n^q$ for $q \geq 2$.

Proof of $|\mathbb{E}(\xi_i^n(2) | \mathcal{F}_i^n)| \leq K \Delta_n$

Using the law of iterated expectation, we have, for $u < v$,

$$\mathbb{E}(\zeta_{i+u}^{n,jk} \zeta_{i+v}^{n,lm} \zeta_{i+v}^{n,gh} | \mathcal{F}_i^n) = \mathbb{E}(\zeta_{i+u}^{n,jk} \mathbb{E}(\zeta_{i+v}^{n,lm} \zeta_{i+v}^{n,gh} | \mathcal{F}_{i+u+1}^n) | \mathcal{F}_i^n). \quad (51)$$

By successive conditioning, (48) and the Cauchy-Schwartz inequality, we also have

$$|\mathbb{E}(\zeta_{i,v}^{n,lm} \zeta_{i,v}^{n,gh} | \mathcal{F}_{i+u+1}^n) - \Delta_n^2 (c_{i+u+1}^{n,lg} c_{i+u+1}^{n,mh} + c_{i+u+1}^{n,lh} c_{i+u+1}^{n,mg}) - \Delta_n^2 (c_{i+u+1}^{n,gh} - c_i^{n,gh})(c_{i+u+1}^{n,lm} - c_i^{n,lm})| \leq K \Delta_n^{5/2}.$$

Given that $\mathbb{E}(|\zeta_{i+u}^{n,jk}|^q | \mathcal{F}_i^n) \leq \Delta_n^q$, the approximation error involved in replacing $\mathbb{E}(\zeta_{i+v}^{n,lm} \zeta_{i+v}^{n,gh} | \mathcal{F}_{i+u+1}^n)$ by $\Delta_n^2 (c_{i+u+1}^{n,lg} c_{i+u+1}^{n,mh} + c_{i+u+1}^{n,lh} c_{i+u+1}^{n,mg}) + \Delta_n^2 (c_{i+u+1}^{n,gh} - c_i^{n,gh})(c_{i+u+1}^{n,lm} - c_i^{n,lm})$ in (51) is smaller than $\Delta_n^{7/2}$.

From (3.9) in Jacod and Rosenbaum (2012) we have

$$|\mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lm} - c_{i+u}^{n,lm}) | \mathcal{F}_i^n)| \leq K \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i,k_n}^n). \quad (52)$$

Since $(c_{i+u}^n - c_i^n)$ is $\mathcal{F}_{i+u}^n$ -measurable, we use the successive conditioning, the Cauchy-Schwartz inequality, (47), (48) and the fifth statement in Lemma 7 applied to $Z = c$ to obtain

$$\begin{aligned}
|\mathbb{E}(\alpha_{i+u}^{n,gh} (c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+u}^{n,jk} - c_i^{n,jk}) | \mathcal{F}_i^n)| &\leq K \Delta_n^{5/2} \\
|\mathbb{E}(\alpha_{i+u}^{n,jk} \alpha_{i+u}^{n,lm} (c_{i+u}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n)| &\leq K \Delta_n^{5/2} \\
|\mathbb{E}((c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n)| &\leq K \Delta_n
\end{aligned} \quad (53)$$

which can be proved using . The following inequalities can be established easily using (47), the successive conditioning together with (39) for $Z = c$,

$$\begin{aligned}
\left| \mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lg} c_{i+u+1}^{n,mh} + c_{i+u+1}^{n,lh} c_{i+u+1}^{n,mg}) | \mathcal{F}_i^n) \right| &\leq K \Delta_n^{3/2} \\
\left| \mathbb{E}((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u+1}^{n,lg} c_{i+u+1}^{n,mh} + c_{i+u+1}^{n,lh} c_{i+u+1}^{n,mg}) | \mathcal{F}_i^n) \right| &\leq K \Delta_n^{1/2}
\end{aligned}$$

$$\left| \mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,gh} - c_i^{n,gh})(c_{i+u+1}^{n,lm} - c_i^{n,lm}) | \mathcal{F}_i^n) \right| \leq K \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i,k_n}^n)$$

The last three inequalities together yield $|\mathbb{E}(\xi_i^n(2) | \mathcal{F}_i^n)| \leq K \Delta_n$.

Proof of $|\mathbb{E}(\xi_i^n(3) | \mathcal{F}_i^n)| \leq K \Delta_n$

First, note that, for $u < v$, we have

$$\mathbb{E}(\zeta_{i+u}^{n,jk} \zeta_{i+u}^{n,lm} \zeta_{i+v}^{n,gh} | \mathcal{F}_i^n) = \mathbb{E}(\zeta_{i+u}^{n,jk} \zeta_{i+u}^{n,lm} \mathbb{E}(\zeta_{i+v}^{n,gh} | \mathcal{F}_{i+u+1}^n) | \mathcal{F}_i^n). \quad (54)$$

By successive conditioning and (47), we have

$$|\mathbb{E}(\alpha_{i+w}^{n,gh} | \mathcal{F}_{i+v+1}^n)| \leq K \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i+v+1,w-v}). \quad (55)$$

Using the first statement of Lemma applied to $Z = c$, it can be shown that

$$|\mathbb{E}((c_{i+w}^{n,gh} - c_{i+v+1}^{n,gh}) | \mathcal{F}_i^n) - \Delta_n(w - v - 1) \tilde{b}_{i+v+1}^{n,gh}| \leq K(w - v - 1) \Delta_n \eta_{i+v+1,w-v} \leq K \Delta_n^{1/2} \eta_{i+v+1,w-v}.$$

The last two inequalities together imply

$$\left| \mathbb{E}(\zeta_{i+w}^{n,gh} | \mathcal{F}_{i+v+1}^n) - (c_{i+v+1}^{n,gh} - c_i^{n,gh}) \Delta_n - \Delta_n^2(w - v - 1) \tilde{b}_{i+v+1}^{n,gh} \right| \leq K \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i+v+1,w-v}). \quad (56)$$

Since $\mathbb{E}(|\zeta_{i,u}^{n,jk}|^q | \mathcal{F}_i^n) \leq \Delta_n^q$, the error induced by replacing $\mathbb{E}(\zeta_{i+v}^{n,gh} | \mathcal{F}_{i+u+1}^n)$ by $(c_{i+v+1}^{n,gh} - c_i^{n,gh}) \Delta_n + \Delta_n^2(w - v - 1) \tilde{b}_{i+v+1}^{n,gh}$ in (54) is smaller than $\Delta_n^{7/2}$.

Using Cauchy Schwartz inequality, successive conditioning, (53), (39) for $Z = c$ and the boundedness of $\tilde{b}_t$ and c_t we obtain

$$\begin{aligned} & \left| \mathbb{E}(\alpha_{i+u}^{n,jk} \alpha_{i+u}^{n,lm} (c_{i+u+1}^{n,jk} - c_i^{n,gh}) | \mathcal{F}_{i+u}^n) \right| \leq K \Delta_n^{5/2} \\ & \left| \mathbb{E}(\alpha_{i+u}^{n,jk} \alpha_{i+u}^{n,lm} \tilde{b}_{i+u+1}^{n,gh} | \mathcal{F}_{i+u}^n) \right| \leq K \Delta_n^2 \\ & \left| \mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+u+1}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n) \right| \leq K \Delta_n^{1/4} \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i,k_n}^n) \\ & \left| \mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u}^{n,lm} - c_i^{n,lm}) \tilde{b}_{i+u+1}^{n,gh} | \mathcal{F}_i^n) \right| \leq \Delta_n^{5/4} \\ & \left| \mathbb{E}((c_{i+u}^{n,jk} - c_i^{n,gh})(c_{i+u}^{n,lm} - c_i^{n,lm}) \tilde{b}_{i+u+1}^{n,gh} | \mathcal{F}_i^n) \right| \leq K \Delta_n^{1/2} \\ & \left| \mathbb{E}((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+u+1}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n) \right| \leq K \Delta_n. \end{aligned}$$

The above inequalities together yield $|\mathbb{E}(\xi_i^n(3) | \mathcal{F}_i^n)| \leq K \Delta_n$.

Proof of $|\mathbb{E}(\xi_i^n(4) | \mathcal{F}_i^n)| \leq K \Delta_n^{3/4} (\Delta_n^{1/4} + \eta_{i,k_n}^n)$

We first observe that $\xi_i^n(4)$ can be rewritten as

$$\xi_i^n(4) = \frac{1}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \zeta_{i+u}^{n,jk} \zeta_{i+v}^{n,lm} \zeta_{i+w}^{n,gh},$$

where

$$\begin{aligned} \zeta_{i+u}^{n,jk} \zeta_{i+v}^{n,lm} \zeta_{i+w}^{n,gh} = & \left[\alpha_{i+u}^{n,jk} \alpha_{i+v}^{n,lm} \alpha_{i+w}^{n,gh} + \alpha_{i+u}^{n,jk} \Delta_n \alpha_{i+v}^{n,lm} (c_{i+w}^{n,gh} - c_i^{n,gh}) + \alpha_{i+u}^{n,jk} \Delta_n (c_{i+v}^{n,lm} - c_i^{n,lm}) \alpha_{i+w}^{n,gh} \right. \\ & + \Delta_n^2 \alpha_{i+u}^{n,jk} (c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+w}^{n,gh} - c_i^{n,gh}) + \Delta_n (c_{i+u}^{n,jk} - c_i^{n,jk}) \alpha_{i+v}^{n,lm} \alpha_{i+w}^{n,gh} + \Delta_n^2 (c_{i+u}^{n,jk} - c_i^{n,jk}) \alpha_{i+v}^{n,lm} (c_{i+w}^{n,gh} - c_i^{n,gh}) \\ & \left. + \Delta_n^2 (c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm}) \alpha_{i+w}^{n,gh} + \Delta_n^3 (c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+w}^{n,gh} - c_i^{n,gh}) \right]. \end{aligned}$$

Based on the above decomposition, we set

$$\xi_i^n(4) = \sum_{j=1}^8 \chi(j),$$

with $\chi(j)$ defined below. Our target is show that $|\mathbb{E}(\chi(j)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}^n)$, $j = 1, \dots, 8$. To start, set

$$\chi(1) = \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \alpha_{i+v}^{n,lm} \alpha_{i+w}^{n,gh}.$$

Upon changing the order of the summation, we have

$$\chi(1) = \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \alpha_{i+w}^{n,gh}.$$

Define also

$$\chi'(1) = \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \mathbb{E}(\alpha_{i+w}^{n,gh} | \mathcal{F}_{i+v+1}^n).$$

Note that $\mathbb{E}(\chi(1)|\mathcal{F}_i^n) = \mathbb{E}(\chi'(1)|\mathcal{F}_i^n)$.

It is easy to see that, by Lemma 5, we have for $q \geq 2$,

$$\mathbb{E}\left(\left\|\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk}\right\|^q \middle| \mathcal{F}_i^n\right) \leq K_q \Delta_n^{3q/4}.$$

The Cauchy-Schwartz inequality yields,

$$\begin{aligned} \mathbb{E}\left(\left|\sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk}\right) \alpha_{i+v}^{n,lm} \mathbb{E}(\alpha_{i+w}^{n,gh} | \mathcal{F}_{i+v+1}^n)\right|\right| \mathcal{F}_i^n\right) &\leq K k_n^2 \left[\mathbb{E}\left(\left|\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk}\right|^4 \middle| \mathcal{F}_i^n\right)\right]^{1/4} \left[\mathbb{E}\left(\left|\alpha_{i+v}^{n,lm}\right|^4 \middle| \mathcal{F}_i^n\right)\right]^{1/4} \\ &\times \left[\mathbb{E}\left(\left|\mathbb{E}(\alpha_{i+w}^{n,gh} | \mathcal{F}_{i+v+1}^n)\right|^2 \middle| \mathcal{F}_i^n\right)\right]^{1/2} \leq K \Delta_n k_n^2 \Delta_n^{3/4} \Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_{i,k_n}^n), \end{aligned}$$

where the last iteration is obtained using (55) as well as the inequality $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ which holds for a and b positive real numbers and the third statement in Lemma 6.

It follows from this result that

$$|\mathbb{E}(\chi(1)|\mathcal{F}_i^n)| \leq K \Delta_n^{3/4} (\sqrt{\Delta_n} + \eta_{i,k_n}^n).$$

Next set,

$$\begin{aligned} \chi(2) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \Delta_n (c_{i+u}^{n,jk} - c_i^{n,jk}) \right) \alpha_{i+v}^{n,lm} \alpha_{i+w}^{n,gh}, \\ \chi(3) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+v}^{n,jk} \right) \Delta_n (c_{i+u}^{n,lm} - c_i^{n,lm}) \alpha_{i+w}^{n,gh}, \\ \chi(4) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \Delta_n (c_{i+u}^{n,jk} - c_i^{n,jk}) \right) \Delta_n (c_{i+u}^{n,lm} - c_i^{n,lm}) \alpha_{i+w}^{n,gh}. \end{aligned}$$

Given that, for $q \geq 2$, we have,

$$\mathbb{E}\left(\left\|\sum_{u=0}^{v-1} \Delta_n (c_{i+u}^{n,jk} - c_i^{n,jk})\right\|^q \middle| \mathcal{F}_i^n\right) \leq K_q \Delta_n^{3q/4} \quad \text{and} \quad \mathbb{E}(\|c_{i+u}^{n,jk} - c_i^{n,jk}\|^q | \mathcal{F}_i^n) \leq K_q \Delta_n^{q/4},$$

one can follow essentially the same steps as for $\chi(1)$ to show that

$$|\mathbb{E}(\chi(2)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n} + \eta_{i,k_n}^n) \quad \text{and} \quad |\mathbb{E}(\chi(j)|\mathcal{F}_i^n)| \leq K\Delta_n(\sqrt{\Delta_n} + \eta_{i,k_n}^n) \quad \text{for } j = 3, 4.$$

Define

$$\begin{aligned} \chi(5) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \Delta_n(c_{i+w}^{n,gh} - c_i^{n,gh}) \\ \chi'(5) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \Delta_n \mathbb{E}((c_{i+w}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_{i+v+1}^n) \\ \chi(6) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \Delta_n(c_{i+u}^{n,jk} - c_i^{n,jk}) \right) \alpha_{i+v}^{n,lm} \Delta_n(c_{i+w}^{n,gh} - c_i^{n,gh}) \\ \chi(7) &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \Delta_n(c_{i+w}^{n,gh} - c_i^{n,gh}), \end{aligned}$$

where we have $\mathbb{E}(\chi(5)|\mathcal{F}_i^n) = \mathbb{E}(\chi'(5)|\mathcal{F}_i^n)$. Recalling (56), we further decompose $\chi'(5)$ as,

$$\chi'(5) = \sum_{j=1}^5 \chi(5)[j],$$

with

$$\begin{aligned} \chi'(5)[1] &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \left(\mathbb{E}(c_{i+w}^{n,gh} - c_i^{n,gh} | \mathcal{F}_{i+v+1}^n) - (c_{i+v+1}^{n,gh} - c_i^{n,gh}) \Delta_n - \tilde{b}_{i+v+1}^{n,gh} \Delta_n^2(w-v-1) \right) \\ \chi'(5)[2] &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \Delta_n(c_{i+v}^{n,gh} - c_i^{n,gh}) \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm} \\ \chi'(5)[3] &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v+1}^{n,gh} - c_{i+v}^{n,gh}) \alpha_{i+v}^{n,lm} \\ \chi'(5)[4] &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n^2(w-v-1) (\tilde{b}_{i+v+1}^{n,gh} - \tilde{b}_{i+v}^{n,gh}) \alpha_{i+v}^{n,lm} \\ \chi'(5)[5] &= \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \Delta_n^2(w-v-1) \tilde{b}_{i+v}^{n,gh} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \alpha_{i+v}^{n,lm}. \end{aligned}$$

Using (56), (55), (52) and following the same strategy proof as for $\chi(1)$, it can be shown that

$$|\mathbb{E}(\chi'(5)[j]|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n} + \eta_{i,k_n}^n), \quad \text{for } j = 1, \dots, 5.$$

which in turn implies

$$|\mathbb{E}(\chi(5)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n} + \eta_{i,k_n}^n), \quad \text{for } j = 1, \dots, 5.$$

The term $\chi(6)$ can be handled similarly to $\chi(5)$, hence we conclude that

$$|\mathbb{E}(\chi(6)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n} + \eta_{i,k_n}^n).$$

Next, we set

$$\chi(7) = \frac{1}{(k_n\Delta_n)^3} \sum_{w=2}^{k_n-1} \left(\sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \Delta_n(c_{i+w}^{n,gh} - c_i^{n,gh}) \right).$$

To handle this term, we define,

$$\begin{aligned}
\chi(7)[1] &= \frac{1}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \left(\sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \Delta_n(c_{i+v+1}^{n,gh} - c_{i+v}^{n,gh}) \right) \\
\chi(7)[2] &= \frac{1}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \left(\sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \Delta_n(c_{i+v}^{n,gh} - c_i^{n,gh}) \right) \\
\chi(7)[3] &= \frac{1}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \left(\sum_{v=0}^{w-1} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \Delta_n^2(w-v-1) (\tilde{b}_{i+v+1}^{n,gh} - \tilde{b}_{i+v}^{n,gh}) \right) \\
\chi(7)[4] &= \frac{1}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \left(\sum_{v=0}^{w-1} \Delta_n^2(w-v-1) \tilde{b}_{i+v}^{n,gh} \left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} \right) \Delta_n(c_{i+v}^{n,lm} - c_i^{n,lm}) \right),
\end{aligned}$$

so that

$$\chi(7) = \sum_{j=1}^4 \chi(7)[j].$$

Using arguments similar to that used to handle $\chi(1)$, it can be shown that,

$$|\mathbb{E}(\chi(7)[j] | \mathcal{F}_i^n)| \leq K \Delta_n^{1/4} (\Delta_n^{1/4} + \eta_{i,k_n}), \quad \text{for } j = 1, \dots, 3,$$

To handle the remaining term $\chi(7)[4]$, we set,

$$\begin{aligned}
\chi(7)[4][1] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lm} - c_{i+u}^{n,lm}) (c_{i+u+1}^{n,gh} - c_{i+u}^{n,gh}) \\
\chi(7)[4][2] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,gh} - c_i^{n,gh}) \alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lm} - c_{i+u}^{n,lm}) \\
\chi'(7)[4][2] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,gh} - c_i^{n,gh}) \mathbb{E}(\alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lm} - c_{i+u}^{n,lm}) | \mathcal{F}_{i+u}^n) \\
\chi(7)[4][3] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,lm} - c_i^{n,lm}) \alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,gh} - c_{i+u}^{n,gh}) \\
\chi(7)[4][4] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,lm} - c_i^{n,lm}) (c_{i+u}^{n,gh} - c_i^{n,gh}) \alpha_{i+u}^{n,jk} \\
\chi(7)[4][5] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,lm} - c_i^{n,lm}) \alpha_{i+u}^{n,jk} (c_{i+v}^{n,gh} - c_{i+u+1}^{n,gh}) \\
\chi'(7)[2][5] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,lm} - c_i^{n,lm}) \alpha_{i+u}^{n,jk} \mathbb{E}((c_{i+v}^{n,gh} - c_{i+u+1}^{n,gh}) | \mathcal{F}_{i+u}^n) \\
\chi(7)[4][6] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,lm} - c_{i+u}^{n,lm}) (c_{i+v}^{n,gh} - c_{i+u+1}^{n,gh}) \\
\chi(7)[4][7] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,gh} - c_i^{n,gh}) \alpha_{i+u}^{n,jk} (c_{i+v}^{n,lm} - c_{i+u+1}^{n,lm}) \\
\chi(7)[4][8] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} (c_{i+u+1}^{n,gh} - c_{i+u}^{n,gh}) (c_{i+v}^{n,lm} - c_{i+u+1}^{n,lm}) \\
\chi(7)[4][9] &= \frac{\Delta_n^2}{(k_n \Delta_n)^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n,jk} (c_{i+v}^{n,lm} - c_{i+u+1}^{n,lm}) (c_{i+v}^{n,gh} - c_{i+u+1}^{n,gh}),
\end{aligned}$$

which satisfy,

$$\chi(7)[4] = \sum_{j=1}^9 \chi(7)[4][j].$$

Using arguments similar to that used to handle $\chi(1)$, it can be shown that,

$$|\mathbb{E}(\chi(7)[4][j]|\mathcal{F}_i^n)| \leq K\Delta_n^{1/4}(\Delta_n^{1/4} + \eta_{i,k_n}), \quad \text{for } j = 1, \dots, 8,$$

which yields

$$|\mathbb{E}(\chi(7)|\mathcal{F}_i^n)| \leq K\Delta_n^{1/4}(\Delta_n^{1/4} + \eta_{i,k_n}),$$

Now set,

$$\chi(8) = \frac{1}{k_n^3} \sum_{w=2}^{k_n-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} (c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+w}^{n,gh} - c_i^{n,gh}).$$

This term can be further decomposed in 6 (non overlapping) components. Then using the following bounds,

$$\begin{aligned} |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+w}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n \\ |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+v}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}) \\ |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+v}^{n,lm} - c_i^{n,lm})(c_{i+u}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n \\ |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+w}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n \\ |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+v}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n \\ |\mathbb{E}\left((c_{i+u}^{n,jk} - c_i^{n,jk})(c_{i+u}^{n,lm} - c_i^{n,lm})(c_{i+u}^{n,gh} - c_i^{n,gh})|\mathcal{F}_i^n\right)| &\leq K\Delta_n, \end{aligned}$$

which follow from successive conditioning and existing bounds, we deduce that

$$|\mathbb{E}(\chi(8)|\mathcal{F}_i^n)| \leq K\Delta_n,$$

This completes the proof.

Proof of (43) and (44) in Lemma 8

Observing that

$$\begin{aligned} \beta_i^{n,jk}(c_{i+k_n}^{n,lm} - c_i^{n,lm})(c_{i+k_n}^{n,gh} - c_i^{n,gh}) &= \frac{1}{k_n\Delta_n} \sum_{u=0}^{k_n-1} \zeta_{i,u}^{n,jk}(c_{i+k_n}^{n,lm} - c_i^{n,lm})(c_{i+k_n}^{n,gh} - c_i^{n,gh}), \\ \beta_i^{n,jk} \beta_i^{n,lm}(c_{i+k_n}^{n,gh} - c_i^{n,gh}) &= \frac{1}{k_n^2\Delta_n^2} \sum_{u=0}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,u}^{n,lm}(c_{i+k_n}^{n,gh} - c_i^{n,gh}) + \frac{1}{k_n^2\Delta_n^2} \sum_{u=0}^{k_n-2} \sum_{v=0}^{k_n-1} \zeta_{i,u}^{n,jk} \zeta_{i,v}^{n,lm}(c_{i+k_n}^{n,gh} - c_i^{n,gh}) \\ &\quad + \frac{1}{k_n^2\Delta_n^2} \sum_{u=0}^{k_n-2} \sum_{v=0}^{k_n-1} \zeta_{i,u}^{n,lm} \zeta_{i,v}^{n,jk}(c_{i+k_n}^{n,gh} - c_i^{n,gh}), \end{aligned}$$

(43) and (44) can be proved using the same strategy proof as for (42).

Proof of (45) and (46) in Lemma 8

Note that we have,

$$\begin{aligned} \gamma_i^{n,jk} \gamma_i^{n,lm} \beta_i^{n,gh} &= \beta_i^{n,gh} \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} + \beta_i^{n,gh} \beta_i^{n,jk} \beta_i^{n,lm} - \beta_i^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} - \beta_i^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} \\ &+ \beta_i^{n,gh} \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) - \beta_i^{n,gh} \beta_i^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) + \beta_i^{n,gh} \beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) - \beta_i^{n,gh} \beta_i^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) \\ &+ \beta_i^{n,gh} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,lm} - c_i^{n,lm}), \end{aligned}$$

and

$$\begin{aligned} \gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_i^{n,lm} &= \beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} + \beta_{i+k_n}^{n,gh} \beta_i^{n,jk} \beta_i^{n,lm} - \beta_{i+k_n}^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} - \beta_{i+k_n}^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} \\ &+ \beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) - \beta_{i+k_n}^{n,gh} \beta_i^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) + \beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) - \beta_{i+k_n}^{n,gh} \beta_i^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) \\ &+ \beta_{i+k_n}^{n,gh} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,lm} - c_i^{n,lm}) - \beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} - \beta_{i+k_n}^{n,gh} \beta_i^{n,jk} \beta_i^{n,lm} + \beta_{i+k_n}^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} + \beta_{i+k_n}^{n,gh} \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} \\ &- \beta_i^{n,gh} \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) + \beta_i^{n,gh} \beta_i^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) - \beta_i^{n,gh} \beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) + \beta_i^{n,gh} \beta_i^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) \\ &- \beta_i^{n,gh} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,lm} - c_i^{n,lm}) + \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) + \beta_{i+k_n}^{n,jk} \beta_i^{n,lm} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) \\ &- \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) - \beta_i^{n,lm} \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) + \beta_{i+k_n}^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}) \\ &- \beta_i^{n,jk} (c_{i+k_n}^{n,lm} - c_i^{n,lm}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}) + \beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}) - \beta_i^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}) \\ &+ (c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,lm} - c_i^{n,lm}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}). \end{aligned}$$

From (38), it is easy to see that observe that β_i^n is $\mathcal{F}_{i+k_n}^n$ -measurable and satisfies $\|\mathbb{E}(\beta_i^n | \mathcal{F}_i^n)\| \leq K \Delta_n^{1/2}$.

Using the law of iterated expectation and existing bounds, it can be shown that

$$\begin{aligned} |\mathbb{E}(\beta_i^{n,lm} \beta_{i+k_n}^{n,jk} | \mathcal{F}_i^n)| &\leq K \Delta_n^{3/4}. \\ |\mathbb{E}(\beta_i^{n,lm} \beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,jk} | \mathcal{F}_i^n)| &\leq K \Delta_n \\ |\mathbb{E}(\beta_i^{n,lm} (c_{i+k_n}^{n,gh} - c_i^{n,gh}) \beta_{i+k_n}^{n,jk} | \mathcal{F}_i^n)| &\leq K \Delta_n \\ |\mathbb{E}(\beta_{i+k_n}^{n,lm} (c_{i+k_n}^{n,jk} - c_i^{n,jk}) | \mathcal{F}_i^n)| &\leq K \Delta_n^{3/4} \\ |\mathbb{E}((c_{i+k_n}^{n,jk} - c_i^{n,jk}) (c_{i+k_n}^{n,lm} - c_i^{n,lm}) (c_{i+k_n}^{n,gh} - c_i^{n,gh}) | \mathcal{F}_i^n)| &\leq K \Delta_n. \end{aligned} \tag{57}$$

By Lemma 3.3 in Jacod and Rosenbaum (2012), we have

$$|\mathbb{E}(\beta_{i+k_n}^{n,gh} \beta_{i+k_n}^{n,ab} | \mathcal{F}_{i+k_n}^n) - \frac{1}{k_n} (c_{i+k_n}^{n,ga} c_{i+k_n}^{n,hb} + c_{i+k_n}^{n,gb} c_{i+k_n}^{n,ha}) - \frac{k_n \Delta_n}{3} \bar{c}_{i+k_n}^{n,gh,ab}| \leq K \sqrt{\Delta_n} (\Delta_n^{1/8} + \eta_{i+k_n, k_n}^n).$$

Hence, for $\varphi_i^{n,gh} \in \{\beta_i^{n,gh}, c_{i+k_n}^{n,gh} - c_i^{n,gh}\}$ which satisfies $\mathbb{E}(|\varphi_i^{n,gh}|^q | \mathcal{F}_i^n) \leq K \Delta_n^{q/4}$ and $\mathbb{E}(\varphi_i^{n,gh} | \mathcal{F}_i^n) \leq K \Delta_n^{1/2}$, it can be proved that

$$|\mathbb{E}(\varphi_i^{n,gh} \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} | \mathcal{F}_i^n) - \mathbb{E}\left(\varphi_i^{n,gh} \left[\frac{1}{k_n} (c_{i+k_n}^{n,jl} c_{i+k_n}^{n,km} + c_{i+k_n}^{n,jm} c_{i+k_n}^{n,kl}) - \frac{k_n \Delta_n}{3} \bar{c}_{i+k_n}^{n,jk,lm} \right] | \mathcal{F}_i^n\right)| \leq K \Delta_n^{3/4} (\Delta_n^{1/4} + \eta_{i, 2k_n}^n).$$

Next by successive conditioning and making use of existing bounds one obtains,

$$\begin{aligned} |\mathbb{E}(\varphi_i^{n,gh} \bar{c}_{i+k_n}^{n,jk,lm})| &\leq K \Delta_n^{1/4} (\Delta_n^{1/4} + \eta_{i, k_n}^n) \\ |\mathbb{E}(\varphi_i^{n,gh} c_{i+k_n}^{n,jl} c_{i+k_n}^{n,km})| &\leq K \Delta_n^{1/2}, \end{aligned}$$

which implies

$$|\mathbb{E}(\varphi_i^{n,gh} \beta_{i+k_n}^{n,jk} \beta_{i+k_n}^{n,lm} | \mathcal{F}_i^n)| \leq K \Delta_n^{3/4} (\Delta_n^{1/4} + \eta_{i, 2k_n}^n). \tag{58}$$

It is easy to see that (42), (57) and (58) and the inequality $\eta_{i, k_n}^n \leq \eta_{i, 2k_n}^n$ together yields (45) and (46).

Step 3: Asymptotic Distribution of the approximate estimator

To start, we decompose the approximate estimator as

$$[\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A)} = [\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A1)} - [\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A2)},$$

with

$$[\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A1)} = \frac{3}{2k_n} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial_{gh} H \partial_{ab} G)(c_{i-1}^n) (\hat{c}_{i+k_n}'^{n,gh} - \hat{c}_i'^{n,gh}) (\hat{c}_{i+k_n}'^{n,ab} - \hat{c}_i'^{n,ab}),$$

and

$$[\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A2)} = \frac{3}{k_n^2} \sum_{g,h,a,b=1}^d \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial_{gh} H \partial_{ab} G)(\hat{c}_i'^n) (\hat{c}_i'^{n,ga} \hat{c}_i'^{n,hb} + \hat{c}_i'^{n,gb} \hat{c}_i'^{n,ha}).$$

In this section, we set for convenience, $c_{i-1}^n = c_{(i-1)\Delta_n}$ and $\mathcal{F}_i = \mathcal{F}_{(i-1)\Delta_n}$. Given the polynomial growth assumption satisfied by H and G and the fact that $k_n = \theta(\Delta_n)^{-1/2}$, by Theorem 2.2 in [Jacod and Rosenbaum \(2012\)](#) we have

$$\frac{1}{\sqrt{\Delta_n}} \left([\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A2)} - \frac{3}{\theta^2} \sum_{g,h,a,b=1}^d \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) dt \right) = O_p(1),$$

which yields

$$\frac{1}{\Delta_n^{1/4}} \left([\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A2)} - \frac{3}{\theta^2} \sum_{g,h,a,b=1}^d \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) dt \right) \xrightarrow{\mathbb{P}} 0.$$

To study the asymptotic behavior of $[\widehat{H(\bar{c})}, \widehat{G(\bar{c})}]_T^{(A1)}$, we follow [Aït-Sahalia and Jacod \(2014\)](#) and define the following multidimensional quantities

$$\begin{aligned} \zeta(1)_i^n &= \frac{1}{\Delta_n} \Delta_i^n Y' (\Delta_i^n Y')^\top - c_{i-1}^n, & \zeta(2)_i^n &= \Delta_i^n c, \\ \zeta'(u)_i^n &= \mathbb{E}(\zeta(u)_i^n | \mathcal{F}_{i-1}^n), & \zeta''(u)_i^n &= \zeta(u)_i^n - \zeta'(u)_i^n, \end{aligned}$$

with

$$\zeta^r(u)_i^n = \left(\zeta^r(u)_i^{n,gh} \right)_{1 \leq g,h \leq d}.$$

We also define for $m \in \{0, \dots, 2k_n - 1\}$ and $j, l \in \mathbb{Z}$,

$$\varepsilon(1)_m^n = \begin{cases} -1 & \text{if } 0 \leq m < k_n \\ +1 & \text{if } k_n \leq m < 2k_n, \end{cases}$$

$$\varepsilon(2)_m^n = \sum_{q=m+1}^{2k_n-1} \varepsilon(1)_q^n = (m+1) \wedge (2k_n - m - 1),$$

$$z_{u,v}^n = \begin{cases} 1/\Delta_n & \text{if } u = v = 1 \\ 1 & \text{otherwise,} \end{cases}$$

$$\gamma(u, v; m)_{j,l}^n = \frac{3}{2k_n^3} \sum_{q=0 \vee (j-m)}^{(l-m-1) \vee (2k_n-m-1)} \varepsilon(u)_q^n \varepsilon(u)_{q+m}^n, \quad \Gamma(u, v)_m^n = \gamma(u, v; m)_{0,2k_n}^n,$$

$$M(u, v; u', v')_n = z_{u,v}^n z_{u',v'}^n \sum_{m=1}^{2k_n-1} \Gamma(u, v)_m^n \Gamma(u', v')_m^n.$$

The following decompositions hold,

$$\begin{aligned} \hat{c}_i'^n &= c_{i-1}^n + \frac{1}{k_n} \sum_{j=0}^{k_n-1} \sum_{u=1}^2 \bar{\varepsilon}(u)_j^n \zeta(u)_{i+j}^n, \quad \hat{c}_{i+k_n}'^n - \hat{c}_i'^n = \frac{1}{k_n} \sum_{j=0}^{2k_n-1} \sum_{u=1}^2 \varepsilon(u)_j^n \zeta(u)_{i+j}^n, \\ \gamma_i^{n,gh} \gamma_i^{n,ab} &= \frac{1}{k_n^2} \sum_{u=1}^2 \sum_{v=1}^2 \left(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+j}^{n,ab} + \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+q}^{n,ab} \right. \\ &\quad \left. + \sum_{j=1}^{2k_n-1} \sum_{q=0}^{j-1} \varepsilon(u)_j^n \varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+q}^{n,ab} \right). \end{aligned}$$

Changing the order of the summation in the last term, we obtain

$$\begin{aligned} \gamma_i^{n,gh} \gamma_i^{n,ab} &= \frac{1}{k_n^2} \sum_{u=1}^2 \sum_{v=1}^2 \left(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+j}^{n,ab} + \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+q}^{n,ab} \right. \\ &\quad \left. + \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} \varepsilon(v)_j^n \varepsilon(u)_q^n \zeta(v)_{i+j}^{n,ab} \zeta(u)_{i+q}^{n,gh} \right). \end{aligned}$$

Therefore, we can further rewrite $[H(\widehat{c}), \widehat{G}(c)]_T^{(A1)}$ as

$$\begin{aligned} [H(\widehat{c}), \widehat{G}(c)]_T^{(A1)} &= [H(\widehat{c}), \widehat{G}(c)]_T^{(A11)} + [H(\widehat{c}), \widehat{G}(c)]_T^{(A12)} + [H(\widehat{c}), \widehat{G}(c)]_T^{(A13)}, \text{ with} \\ [H(\widehat{c}), \widehat{G}(c)]_T^{(A1w)} &= \sum_{g,h,a,b=1}^d \sum_{u,v=1}^2 \widehat{A1w}(H, gh, u; G, ab, v)_T^n, \quad w = 1, 2, 3, \end{aligned}$$

and,

$$\begin{aligned} \widehat{A11}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-1} (\partial_{gh} H \partial_{ab} G)(c_{i-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+j}^{n,ab}, \\ \widehat{A12}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} (\partial_{gh} H \partial_{ab} G)(c_{i-1}^n) \varepsilon(u)_j^n \varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh} \zeta(v)_{i+q}^{n,ab}, \\ \widehat{A13}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} (\partial_{gh} H \partial_{ab} G)(c_{i-1}^n) \varepsilon(v)_j^n \varepsilon(u)_q^n \zeta(v)_{i+j}^{n,ab} \zeta(u)_{i+q}^{n,gh}, \end{aligned}$$

where we clearly have $\widehat{A13}(H, gh, u; G, ab, v)_T^n = \widehat{A12}(G, ab, v; H, gh, u)_T^n$. Changing the order of the summations we have,

$$\begin{aligned} \widehat{A11}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]} \sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}, \\ \widehat{A12}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=2}^{[T/\Delta_n]} \sum_{m=1}^{(i-1) \wedge (2k_n-1)} \sum_{j=0 \vee (i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1) \wedge (i-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \times \end{aligned}$$

$$\varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n.$$

Set

$$\widetilde{A11}(H, gh, u; G, ab, v)_T^n = \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \sum_{j=0}^{2k_n-1} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab},$$

$$\widetilde{A12}(H, gh, u; G, ab, v)_T^n = \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \sum_{m=1}^{(i-1) \wedge (2k_n-1)} \sum_{j=0}^{(2k_n-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n,$$

and

$$\begin{aligned} \overline{A11}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \left(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_j^n \right) (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}, \\ &= \Gamma(u, v)_0^n \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab} \\ \overline{A12}(H, gh, u; G, ab, v)_T^n &= \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \sum_{m=1}^{(i-1) \wedge (2k_n-1)} \sum_{j=0}^{(2k_n-m-1)} \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n, \\ &= \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \rho_{gh}(u, v)_i^n \zeta_{ab}(v)_i^n, \end{aligned}$$

with

$$\rho_{gh}(u, v)_i^n = \sum_{m=1}^{2k_n-1} \Gamma(u, v)_m^n \zeta_{gh}(u)_{i-m}^n.$$

The following results hold:

$$\frac{1}{\Delta_n^{1/4}} \left(\widetilde{A1w}(H, gh, u; G, ab, v)_T^n - \overline{A1w}(H, gh, u; G, ab, v)_T^n \right) \xrightarrow{\mathbb{P}} 0 \quad \text{for all } (H, gh, u, G, ab, v) \text{ and } w = 1, 2. \quad (59)$$

$$\frac{1}{\Delta_n^{1/4}} \left(\widetilde{A1w}(H, gh, u; G, ab, v)_T^n - \overline{A1w}(H, gh, u; G, ab, v)_T^n \right) \xrightarrow{\mathbb{P}} 0 \quad \text{for all } (H, gh, u, G, ab, v) \text{ and } w = 1, 2. \quad (60)$$

Proof of (59) for $w = 1$

The proof is similar to Step5 on page 548 of [Aït-Sahalia and Jacod \(2014\)](#). Our proof deviates from the latter reference by the fact that, in all the sums, the terms $\zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}$ are scaled by random variables rather than constant real numbers. First observe that we can write,

$$\begin{aligned} \widetilde{A11} - \overline{A11} &= \widetilde{\widetilde{A11}}(1) + \widetilde{\widetilde{A11}}(2) + \widetilde{\widetilde{A11}}(3) \quad \text{with} \\ \widetilde{\widetilde{A11}}(1) &= \sum_{i=1}^{(2k_n-1) \wedge [T/\Delta_n]} \left(\frac{3}{2k_n^3} \sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab} \\ \widetilde{\widetilde{A11}}(2) &= \sum_{i=[T/\Delta_n]-2k_n+2}^{[T/\Delta_n]} \frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right. \\ &\quad \left. - \sum_{j=0}^{(2k_n-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}. \end{aligned}$$

$$\begin{aligned} \widetilde{\widetilde{A11}}(3) = & \sum_{i=2k_n}^{[T/\Delta_n]-2k_n+1} \frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right. \\ & \left. - \sum_{j=0}^{(2k_n-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}. \end{aligned}$$

It is easy to see that $\widetilde{\widetilde{A12}}(3) = 0$. Using (39) with $Z = c$ and (40), it can be shown that

$$\mathbb{E}(\|\zeta(1)_i^n\|^q | \mathcal{F}_{i-1}^n) \leq K_q, \quad \mathbb{E}(\|\zeta(2)_i^n\|^q | \mathcal{F}_{i-1}^n) \leq K_q \Delta_n^{q/2}. \quad (61)$$

The polynomial growth assumption on H and G and the boundedness of c_t imply that $|(\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n)| \leq K$. Hence, the random quantities $\left(\frac{3}{2k_n^3} \sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right)$ and $\frac{3}{2k_n^3} \sum_{j=0}^{(2k_n-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n$ are $\mathcal{F}_{i-1}^n$ -measurable and are bounded by $\tilde{\gamma}_{u,v}^n$ defined as

$$\tilde{\gamma}_{u,v}^n = \begin{cases} K & \text{if } (u, v) = (2, 2) \\ K/k_n & \text{if } (u, v) = (1, 2), (2, 1) \\ K/k_n^2 & \text{if } (u, v) = (1, 1). \end{cases}$$

Similarly, the quantity,

$$\frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-[T/\Delta_n])}^{(2k_n-1) \wedge (i-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n - \sum_{j=0}^{(2k_n-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) \varepsilon(u)_j^n \varepsilon(v)_j^n \right),$$

is $\mathcal{F}_{i-1}^n$ -measurable and bounded by $2\tilde{\gamma}_{u,v}^n$. Note also that, by (61) and the Cauchy Schwartz inequality, we have,

$$\mathbb{E}(\|\zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}\| | \mathcal{F}_{i-1}^n) \leq \mathbb{E}(\|\zeta(u)_i^n\|^2 | \mathcal{F}_{i-1}^n)^{1/2} \mathbb{E}(\|\zeta(v)_i^n\|^2 | \mathcal{F}_{i-1}^n)^{1/2} \leq \begin{cases} K \Delta_n & \text{if } (u, v) = (2, 2) \\ K \Delta_n^{1/2} & \text{if } (u, v) = (1, 2), (2, 1) \\ K & \text{if } (u, v) = (1, 1). \end{cases}$$

Making use of the above bounds and the fact that $k_n = \theta \Delta_n^{-1/2}$, we have $\mathbb{E}(|\widetilde{\widetilde{A11}}(1)|) \leq K \Delta_n^{1/2}$ and $\mathbb{E}(|\widetilde{\widetilde{A11}}(2)|) \leq K \Delta_n^{1/2}$ for all (u, v) . These two results together imply $\widetilde{\widetilde{A11}}(1) = o(\Delta_n^{-1/4})$ and $\widetilde{\widetilde{A11}}(2) = o(\Delta_n^{-1/4})$ which yields the result.

Proof of (59) for $w = 2$

We proceed similarly to Step 6 on page (548) of Ait-Sahalia and Jacod (2014). First, observe that we have

$$\begin{aligned} \widetilde{\widetilde{A12}} - \widetilde{\widetilde{A12}} &= \widetilde{\widetilde{A12}}(1) + \widetilde{\widetilde{A12}}(2) \quad \text{with} \\ \widetilde{\widetilde{A12}}(1) &= \sum_{i=2}^{(2k_n-1) \wedge [T/\Delta_n]} \left(\sum_{m=1}^{(i-1)} \frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1) \wedge (i-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right) \right. \\ &\quad \left. \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n \right) \\ \widetilde{\widetilde{A12}}(2) &= \sum_{i=[T/\Delta_n]-2k_n+2}^{[T/\Delta_n]} \left(\sum_{m=1}^{(i-1) \wedge (2k_n-1)} \left(\frac{3}{2k_n^3} \sum_{j=0 \vee (i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1) \wedge (i-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right) \right) \end{aligned}$$

$$- \sum_{j=0}^{(2k_n-m-1)} \left((\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right) \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n.$$

It is easy to see that the quantity,

$$\kappa_i^{m,n} = \frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1) \wedge (i-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right)$$

is $\mathcal{F}_{i-m-1}^n$ measurable and bounded by $\tilde{\gamma}_{u,v}^n$. Let,

$$\kappa_i^n = \sum_{m=1}^{(i-1)} \frac{3}{2k_n^3} \left(\sum_{j=0 \vee (i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1) \wedge (i-m-1)} (\partial_{gh} H \partial_{ab} G)(c_{i-1-j-m}^n) \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right) \zeta_{gh}(u)_{i-m}^n.$$

It follows that, κ_i^n is $\mathcal{F}_{i-1}^n$ -measurable. We have,

$$\mathbb{E}(|\kappa_i^{m,n}|^z | \mathcal{F}_0) \leq (\tilde{\gamma}_{u,v}^n)^z$$

$$|\mathbb{E}(\zeta(u)_{i-m}^n | \mathcal{F}_{i-m-1})| \leq \begin{cases} K\sqrt{\Delta_n} & \text{if } u = 1 \\ K\Delta_n & \text{if } u = 2 \end{cases}, \quad \mathbb{E}(\|\zeta(u)_{i-m}^n\|^z | \mathcal{F}_{i-m-1}) \leq \begin{cases} K_z & \text{if } u = 1 \\ K_z \Delta_n^{z/2} & \text{if } u = 2 \end{cases}$$

Using Lemma 5, we deduce that for $z \geq 2$

$$\mathbb{E}(|\kappa_i^n|^z) \leq \begin{cases} K_z (\tilde{\gamma}_{u,v}^n)^z k_n^{z/2} & \text{if } u = 1 \\ K_z (\tilde{\gamma}_{u,v}^n)^z / k_n^{z/2} & \text{if } u = 2 \end{cases} \leq \begin{cases} K_z / k_n^{-3z/2} & \text{if } v = 1 \\ K_z k_n^{-z/2} & \text{if } v = 2 \end{cases}$$

Using the above result, and similarly to step 6 on page 548 of [Aït-Sahalia and Jacod \(2014\)](#), we obtain that, $\frac{1}{\Delta_n^{1/4}} \widetilde{A12}(1) \xrightarrow{\mathbb{P}} 0$. A similar argument yields $\frac{1}{\Delta_n^{1/4}} \widetilde{A12}(2) \xrightarrow{\mathbb{P}} 0$ which completes the proof of (59) for $w = 2$.

Proof of (60) for $w = 1$

Define

$$\Theta(u, v)_0^{(c), i, n} = \frac{3}{2k_n^3} \sum_{j=0}^{2k_n-1} \left((\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \right) \varepsilon(u)_j^n \varepsilon(v)_j^n$$

By Taylor expansion, the polynomial growth assumption on H and G and using (39) with $Z = c$ we have

$$\left| \mathbb{E} \left((\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) | \mathcal{F}_{i-2k_n}^n \right) \right| \leq K(k_n \Delta_n) \leq K\sqrt{\Delta_n} \quad \text{for } j = 0, \dots, 2k_n - 1$$

$$\mathbb{E}(|(\partial_{gh} H \partial_{ab} G)(c_{i-j-1}^n) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n)|^q | \mathcal{F}_{i-2k_n}^n) \leq K(k_n \Delta_n)^{q/2} \leq K\Delta_n^{q/4} \quad \text{for } q \geq 2$$

Next, observe that $\Theta(u, v)_0^{(c), i, n}$ is $\mathcal{F}_{i-1}^n$ -measurable and satisfies $|\Theta(u, v)_0^{(c), i, n}| \leq \tilde{\gamma}_{u,v}^n$, $|\mathbb{E}(\Theta(u, v)_0^{(c), i, n} | \mathcal{F}_{i-2k_n}^n)| \leq K\Delta_n^{1/2} \tilde{\gamma}_{u,v}^n$ and $\mathbb{E}(|\Theta(u, v)_0^{(c), i, n}|^q | \mathcal{F}_{i-2k_n}^n) \leq K_q \Delta_n^{q/4} (\tilde{\gamma}_{u,v}^n)^q$ where the latter follows from the Hölder inequality. We aim to prove that,

$$\hat{E} = \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} \zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} \right],$$

goes to zero in probability for any H, G, g, h, a, b ; $u, v = 1, 2$.¹⁵

To show this result, we first introduce the following quantities:

$$\begin{aligned}\widehat{E}(1) &= \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} \mathbb{E}(\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n) \right] \\ \widehat{E}(2) &= \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} (\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} - \mathbb{E}(\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n)) \right]\end{aligned}$$

with $\widehat{E} = \widehat{E}(1) + \widehat{E}(2)$. By Cauchy Schwartz inequality, we have,

$$\mathbb{E}(|\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab}|^q) \leq (\widehat{\gamma}_{u, v}^n)^{q/2}, \text{ where } \widehat{\gamma}_{u, v}^n = \begin{cases} K & \text{if } (u, v) = (1, 1) \\ K\Delta_n & \text{if } (u, v) = (1, 2), (2, 1) \\ K\Delta_n^2 & \text{if } (u, v) = (2, 2) \end{cases}$$

Since $\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab}$ is $\mathcal{F}_i^n$ -measurable, we use the martingale property of $\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} - \mathbb{E}(\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n)$ to deduce

$$\mathbb{E}(|\widehat{E}(2)|^2) \leq K\Delta_n^{-3/2} (\Delta_n^{1/4} \widehat{\gamma}_{u, v}^n)^2 \widehat{\gamma}_{u, v}^n \leq K\Delta_n \quad \text{in all cases.}$$

The latter inequality implies $\widehat{E}(2) \xrightarrow{\mathbb{P}} 0$ for all (u, v) . We are left to show that $\widehat{E}(1) \xrightarrow{\mathbb{P}} 0$.

We recall some estimates under Assumption 2, see (B.83) in [Ait-Sahalia and Jacod \(2014\)](#)

$$|\mathbb{E}(\zeta(1)_i^{n, gh} \zeta(2)_i^{n, ab} | \mathcal{F}_{i-1}^n)| \leq K\Delta_n, \quad (62)$$

$$|\mathbb{E}(\zeta(1)_i^{n, gh} \zeta(1)_i^{n, ab} | \mathcal{F}_{i-1}^n) - (c_{i-1}^{n, ga} c_{i-1}^{n, hb} + c_{i-1}^{n, gb} c_{i-1}^{n, ha})| \leq K\Delta_n^{1/2}, \quad (63)$$

$$|\mathbb{E}(\zeta(2)_i^{n, gh} \zeta(2)_i^{n, ab} | \mathcal{F}_{i-1}^n - \bar{c}_{i-1}^{n, gh, ab} \Delta_n)| \leq K\Delta_n^{3/2} (\sqrt{\Delta_n} + \eta_i^n). \quad (64)$$

Case $(u, v) \in \{(1, 2), (2, 1)\}$. By (62) we have

$$\mathbb{E}(|\widehat{E}(1)|) \leq K \frac{T}{\Delta_n} \frac{1}{\Delta_n^{1/4}} (\Delta_n^{1/4} \widehat{\gamma}_{u, v}^n \Delta_n) \leq K\Delta_n^{1/2} \quad \text{so } \widehat{E}(1) \xrightarrow{\mathbb{P}} 0.$$

Case $(u, v) \in \{(1, 1), (2, 2)\}$. Set

$$\begin{aligned}\widehat{E}'(1) &= \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} V_{i-2k_n}^n \right] \\ \widehat{E}''(1) &= \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} (V_{i-1}^n - V_{i-2k_n}^n) \right] \\ \widehat{E}'''(1) &= \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=2k_n}^{[T/\Delta_n]} \Theta(u, v)_0^{(c), i, n} (\mathbb{E}(\zeta(u)_i^{n, gh} \zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n) - V_{i-1}^n) \right]\end{aligned}$$

where

$$V_{i-1}^n = \begin{cases} c_{i-1}^{n, ga} c_{i-1}^{n, hb} + c_{i-1}^{n, gb} c_{i-1}^{n, ha} & \text{if } (u, v) = (2, 2) \\ \bar{c}_{i-1}^{n, gh, ab} \Delta_n & \text{if } (u, v) = (1, 1) \\ 0 & \text{otherwise} \end{cases}$$

¹⁵It turns out that given our restrictions on the functions H and G and the different estimates derived for multidimensional processes, the most relevant arguments on which depend the convergence are u and v .

Note that we have $\widehat{E}(1) = \widehat{E}'(1) + \widehat{E}''(1) + \widehat{E}'''(1)$. Using (63) and (64), it can be shown that

$$\mathbb{E}(|\widehat{E}'''(1)|) \leq \begin{cases} K \frac{1}{\Delta_n^{5/4}} (\Delta_n^{1/4} \widetilde{\gamma}_{u,v}^n) \Delta_n^{1/2} & \text{if } (u, v) = (1, 1) \\ K \frac{1}{\Delta_n^{5/4}} (\Delta_n^{1/4} \widetilde{\gamma}_{u,v}^n) \Delta_n^{3/2} & \text{if } (u, v) = (2, 2) \end{cases} \leq K \Delta_n^{1/2} \quad \text{in all cases.}$$

We make use of lemma B.8 in Aït-Sahalia and Jacod (2014) to prove that $\widehat{E}'(1) \xrightarrow{\mathbb{P}} 0$. To this end, we write

$$\widehat{E}'(1) = \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1} \Theta(u, v)_0^{(c), i-1+2k_n, n} V_{(i-1)\Delta_n} \right].$$

Noting that the summand in the last sum is $\mathcal{F}_{i+2k_n-2}^n$ -measurable, then applying lemma B.8 in Aït-Sahalia and Jacod (2014) requires showing the following two results

$$\begin{aligned} \frac{1}{\Delta_n^{1/4}} \left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1} |\mathbb{E}(\Theta(u, v)_0^{(c), i-1+2k_n, n} V_{(i-1)\Delta_n} | \mathcal{F}_{i-1}^n)| \right] &\xrightarrow{\mathbb{P}} 0 \quad \text{and} \\ \frac{2k_n-2}{\Delta_n^{1/2}} \left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1} \mathbb{E}(|\Theta(u, v)_0^{(c), i-1+2k_n, n} V_{(i-1)\Delta_n}|^2) \right] &\Rightarrow 0. \end{aligned}$$

The first point readily follows from the inequality

$$|\mathbb{E}(\Theta(u, v)_0^{(c), i-1+2k_n, n} V_{(i-1)\Delta_n} | \mathcal{F}_{i-1}^n)| \leq \begin{cases} K \Delta_n^{1/2} \widetilde{\gamma}_{u,v}^n & \text{if } (u, v) = (1, 1) \\ K \Delta_n^{1/2} \widetilde{\gamma}_{u,v}^n \Delta_n & \text{if } (u, v) = (2, 2) \end{cases} \leq K \Delta_n^{3/2} \quad \text{in all cases}$$

while the second is a direct consequence of

$$\mathbb{E}(|\Theta(u, v)_0^{(c), i-1+2k_n, n} V_{(i-1)\Delta_n}|^2) \leq \begin{cases} K \Delta_n^{1/2} (\widetilde{\gamma}_{u,v}^n)^2 & \text{if } (u, v) = (1, 1) \\ K \Delta_n^{1/2} (\widetilde{\gamma}_{u,v}^n)^2 \Delta_n^2 & \text{if } (u, v) = (2, 2) \end{cases} \leq K \Delta_n^{5/2} \quad \text{in all cases.}$$

Finally to prove that $\widehat{E}''(1) \xrightarrow{\mathbb{P}} 0$, we exploit the fact that

$$\begin{aligned} \mathbb{E}(|\Theta(u, v)_0^{(c), i, n} (V_{(i-1)\Delta_n} - V_{(i-2k_n)\Delta_n})|) &\leq \mathbb{E}(|\Theta(u, v)_0^{(c), i, n}|^2)^{1/2} \mathbb{E}(|V_{(i-1)\Delta_n} - V_{(i-2k_n)\Delta_n}|^2)^{1/2} \\ &\leq \begin{cases} K \Delta_n^{1/2} \widetilde{\gamma}_{u,v}^n & \text{if } (u, v) = (1, 1) \\ K \Delta_n^{1/4} \widetilde{\gamma}_{u,v}^n \Delta_n \Delta_n^{1/4} & \text{if } (u, v) = (2, 2) \end{cases} \end{aligned}$$

which follows from an application of Cauchy-Schwartz inequality combined with existing estimates. Indeed using successive conditioning one can prove that for $(u, v) = (1, 1)$ and $(2, 2)$ $\mathbb{E}(|V_{(i-1)\Delta_n} - V_{(i-2k_n)\Delta_n}|^2) \leq \Delta_n^{1/2}$ under Assumption 2.

Proof of (60) for $w = 2$

The target is to show that

$$\begin{aligned} \widehat{E}(2) &= \frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \left(\sum_{m=1}^{2k_n-1} \left(\frac{3}{2k_n^3} \sum_{j=0}^{2k_n-m-1} [(\partial_{gh} H \partial_{ab} G)(c_{i-j-m-1}^n) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n)] \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \right) \times \right. \\ &\quad \left. \zeta(u)_{i-m}^{n, gh} \right) \zeta(v)_i^{n, ab} \xrightarrow{\mathbb{P}} 0. \end{aligned}$$

In order to achieve this goal we introduce some new notation, for any $0 \leq m \leq 2k_n - 1$

$$\begin{aligned}\Theta(u, v)_m^{(c), i, n} &= \frac{3}{2k_n^3} \sum_{j=0}^{2k_n-m-1} [(\partial_{gh} H \partial_{ab} G)(c_{i-j-m-1}^n) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n)] \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \\ \rho(u, v)^{(c), i, n, gh} &= \sum_{m=1}^{2k_n-1} \Theta(u, v)_m^{(c), i, n} \zeta(u)_{i-m}^{n, gh}.\end{aligned}$$

It is easy to see that $\Theta(u, v)_m^{(c), i, n}$ is $\mathcal{F}_{i-m-1}^n$ measurable and satisfies by Hölder inequality

$$|\Theta(u, v)_m^{(c), i, n}| \leq \tilde{\gamma}_{u, v}^n \quad \text{and} \quad \mathbb{E}\left(|\Theta(u, v)_m^{(c), i, n}|^q | \mathcal{F}_{i-2k_n}^n\right) \leq K_q \Delta_n^{q/4} (\tilde{\gamma}_{u, v}^n)^q.$$

Using Lemma 5, we deduce that for $q \geq 2$

$$\mathbb{E}(|\rho(u, v)^{(c), i, n, gh}|^q) \leq \begin{cases} K_q (\Delta_n^{1/4} \tilde{\gamma}_{u, v}^n)^q k_n^{q/2} & \text{if } u = 1 \\ K_q (\Delta_n^{1/4} \tilde{\gamma}_{u, v}^n)^q / k_n^{q/2} & \text{if } u = 2 \end{cases} \leq \begin{cases} K_q / k_n^{2q} & \text{if } v = 1 \\ K_q k_n^q & \text{if } v = 2 \end{cases} \quad (65)$$

Set

$$\begin{aligned}\hat{E}'(2) &= \frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u, v)^{(c), i, n, gh} \mathbb{E}(\zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n), \\ \hat{E}''(2) &= \frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u, v)^{(c), i, n, gh} (\zeta(v)_i^{n, ab} - \mathbb{E}(\zeta(v)_i^{n, ab} | \mathcal{F}_{i-1}^n)).\end{aligned}$$

The martingale increments property implies $\mathbb{E}(|\hat{E}''(2)|^2) \leq K \Delta_n^{1/2}$ in all the cases implying $\hat{E}''(2) \xrightarrow{\mathbb{P}} 0$. Next using the bounds on $\rho(u, v)^{(c), i, n, gh}$ and similarly to step 7 on page 549 of [Aït-Sahalia and Jacod \(2014\)](#), we obtain that $\hat{E}'(2) \xrightarrow{\mathbb{P}} 0$.

Return to the proof of Theorem 1

So far, we have proved that,

$$\begin{aligned}\frac{1}{\Delta_n^{1/4}} \left(\widehat{[H(\bar{c}), G(\bar{c})]}_T^{(A1)} - \sum_{g, h, a, b=1}^d \sum_{u, v=1}^2 \overline{A11}(H, gh, u; G, ab, v)_T^n + \overline{A12}(H, gh, u; G, ab, v)_T^n \right. \\ \left. + \overline{A12}(G, ab, v; H, gh, u)_T^n \right) \xrightarrow{\mathbb{P}} 0.\end{aligned}$$

We next show that,

$$\frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \rho_{gh}(u, v)_i^n \zeta'_{ab}(v)_i^n \xrightarrow{\mathbb{P}} 0, \quad \forall (u, v) \quad (66)$$

$$\frac{1}{\Delta_n^{1/4}} \left(\overline{A11}(H, gh, u; G, ab, v) - \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) \bar{c}_t^{gh, ab} dt \right) \xrightarrow{\mathbb{P}} 0 \quad \text{when } (u, v) = (2, 2) \quad (67)$$

$$\frac{1}{\Delta_n^{1/4}} \left(\overline{A11}(H, gh, u; G, ab, v) - \frac{3}{\theta^2} \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) dt \right) \xrightarrow{\mathbb{P}} 0 \quad \text{when } (u, v) = (1, 1) \quad (68)$$

$$\frac{1}{\Delta_n^{1/4}} \overline{A11}(H, gh, u; G, ab, v) \xrightarrow{\mathbb{P}} 0 \quad \text{when } (u, v) = (1, 2), (2, 1) \quad (69)$$

which in turn will imply,

$$\frac{1}{\Delta_n^{1/4}} \left(\widehat{[H(c), G(c)]_T}^{(A)} - [H(c), G(c)]_T - \frac{3}{2k_n^3} \sum_{g,h,a,b}^d \sum_{u,v=1}^2 \sum_{i=2k_n}^{[T/\Delta_n]} \left[(\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \rho_{gh}(u, v) \zeta_{ab}''(v)_i^n \right. \right. \quad (70)$$

$$\left. + (\partial_{ab} H \partial_{gh} G)(c_{i-2k_n}^n) \rho_{ab}(v, u) \zeta_{gh}''(v)_i^n \right] \Big) \xrightarrow{\mathbb{P}} 0. \quad (71)$$

(66) can be proved easily following steps similar to step 7 on page 549 of [Aït-Sahalia and Jacod \(2014\)](#) and using the bounds of $\rho(u, v)_i^{n,gh}$ in (65). To show (67), (68) and (69), we set

$$\overline{\overline{A11}}(H, gh, u; G, ab, v) = \Gamma(u, v)_0^n \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) \zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab}.$$

Then it holds that,

$$\frac{1}{\Delta_n^{1/4}} \left(\overline{\overline{A11}}(H, gh, u; G, ab, v) - \overline{A11}(H, gh, u; G, ab, v) \right) \xrightarrow{\mathbb{P}} 0.$$

This result can be proved following similar steps as for (59) in case $w = 1$ by replacing $\Theta(u, v)_0^{(c),i,n}$ by $\Gamma(u, v)_0^n ((\partial_{gh} H \partial_{ab} G)(c_{i-1}) - (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}))$ which has the same bounds as the former. Next we decompose $\overline{\overline{A11}}$ as

$$\begin{aligned} \overline{\overline{A11}}(H, gh, u; G, ab, v) &= \Gamma(u, v)_0^n \left[\sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) V_{i-1}^n \right. \\ &\quad + \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) \left(\mathbb{E}(\zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab} | \mathcal{F}_{i-1}^n) - V_{i-1}^n \right) \\ &\quad \left. + \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) \left(\zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab} - \mathbb{E}(\zeta(u)_i^{n,gh} \zeta(v)_i^{n,ab} | \mathcal{F}_{i-1}^n) \right) \right] \end{aligned}$$

Following the proof of (60) for $w = 1$ at this time replacing $\Theta(u, v)_0^{(c),i,n}$ by $\Gamma(u, v)_0^n (\partial_{gh} H \partial_{ab} G)(c_{i-1})$ which satisfies only the condition $|\Gamma(u, v)_0^n (\partial_{gh} H \partial_{ab} G)(c_{i-1})| \leq \tilde{\gamma}_{u,v}^n$ we can see that the last two terms in the above decomposition of vanish to zero at a rate slower than $\Delta_n^{1/4}$ thus we have

$$\frac{1}{\Delta_n^{1/4}} \left(\overline{\overline{A11}}(H, gh, u; G, ab, v) - \Gamma(u, v)_0^n \left(\sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) V_{i-1}^n \right) \right) \Rightarrow 0$$

As a consequence for $(u, v) = (1, 2), (2, 1)$

$$\frac{1}{\Delta_n^{1/4}} \overline{\overline{A11}}(H, gh, u; G, ab, v) \Rightarrow 0$$

The results follows from the following observation,

$$\frac{1}{\Delta_n^{1/4}} \left(\Gamma(u, v)_0^n \left(\sum_{g,h,a,b=1}^d \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) V_{i-1}^n(u, v) \right) - \frac{3}{\theta^2} \int_0^T (\partial_{gh} H \partial_{ab} G)(c_t) (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) dt \right) \Rightarrow 0,$$

$$\text{for } (u, v) = (2, 2)$$

$$\frac{1}{\Delta_n^{1/4}} \left(\sum_{g,h,a,b=1}^d \Gamma(u, v)_0^n \left(\sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh} H \partial_{ab} G)(c_{i-1}) V_{i-1}^n(u, v) \right) - [H(c), G(c)]_T \right) \Rightarrow 0, \quad \text{for } (u, v) = (1, 1)$$

Set

$$\begin{aligned}\xi(H, gh, u; G, ab, v)_i^n &= \frac{1}{\Delta_n^{1/4}} (\partial_{gh} H \partial_{ab} G)(c_{i-2k_n}^n) \rho_{gh}(u, v)_i^n \zeta_{ab}''(v)_i^n \\ Z(H, gh, u; G, ab, v)_t^n &= \Delta_n^{1/4} \sum_{i=2k_n}^{[t/\Delta_n]} \xi(H, gh, u; G, ab, v)_i^n.\end{aligned}$$

(70) implies

$$\frac{1}{\Delta_n^{1/4}} \left([H(\widehat{c}), \widehat{G}(c)]_T^{(A)} - [H(c), G(c)]_T \right) \stackrel{\mathcal{L}}{=} \sum_{g,h,a,b=1}^d \sum_{u,v=1}^2 \frac{1}{\Delta_n^{1/4}} \left(Z(H, gh, u; G, ab, v)_T^n + Z(H, ab, v; G, gh, u)_T^n \right). \quad (72)$$

Next, observe that to derive the asymptotic distribution of $\left([H_1(\widehat{c}), \widehat{G}_1(c)]_T^{(A)}, \dots, [H_\kappa(\widehat{c}), \widehat{G}_\kappa(c)]_T^{(A)} \right)$, it suffices to study the joint asymptotic behavior of the family of processes $\frac{1}{\Delta_n^{1/4}} Z(H, gh, u; G, ab, v)_t^n$.

It is easy to see that the $\xi(H, gh, u; G, ab, v)_i^n$ are martingale increments, relative to the discrete filtration $(\mathcal{F}_i^n)$. Therefore, by Theorem 2.2.15 of [Jacod and Protter \(2012\)](#), to obtain the joint asymptotic distribution of $\frac{1}{\Delta_n^{1/4}} Z(H, gh, u; G, ab, v)_t^n$, it is enough to prove the following three properties, for all $t > 0$, all $(H, gh, u; G, ab, v)$, $(H', g'h', u'; G', a'b', v')$ and all martingales N which are either bounded and orthogonal to W , or equal to one component W^j ,

$$\begin{aligned}A\left((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\right)_t^n &:= \sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(\xi(H, gh, u; G, ab, v)_i^n \xi(H', g'h', u'; G', a'b', v')_i^n | \mathcal{F}_{i-1}^n) \\ &\stackrel{\mathbb{P}}{\Rightarrow} A\left((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\right)_t \\ \sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(|\xi(H, gh, u; G, ab, v)_i^n|^4 | \mathcal{F}_{i-1}^n) &\stackrel{\mathbb{P}}{\Rightarrow} 0 \\ B(N; H, gh, u; G, ab, v)_t^n &:= \sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(\xi(H, gh, u; G, ab, v)_i^n \Delta_i^n N | \mathcal{F}_{i-1}^n) \stackrel{\mathbb{P}}{\Rightarrow} 0.\end{aligned}$$

Using the polynomial growth assumption on H_r and G_r , the second and the third results can be proved by a natural extension to the multivariate case of (B.105) and (B.106) in [Ait-Sahalia and Jacod \(2014\)](#).

Define

$$V_{ab}^{a'b'}(v, v')_t = \begin{cases} (c_t^{aa'} c_t^{bb'} + c_t^{ab'} c_t^{ba'}) & \text{if } (v, v') = (1, 1) \\ \bar{c}_t^{ab, a'b'} & \text{if } (v, v') = (2, 2) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{V}_{gh}^{g'h'}(u, u')_t = \begin{cases} (c_t^{gg'} c_t^{hh'} + c_t^{gh'} c_t^{hg'}) & \text{if } (u, u') = (1, 1) \\ \bar{c}_t^{gh, g'h'} & \text{if } (u, u') = (2, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Once again using the polynomial growth assumption on H_r and G_r and following steps similar to the proof of (B.104) in [Ait-Sahalia and Jacod \(2014\)](#), one can show that

$$\begin{aligned}A\left((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\right)_t &= \\ M(u, v; u', v') \int_0^t (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H \partial_{a'b'} G)(c_s) V_{ab}^{a'b'}(v, v')_s \bar{V}_{gh}^{g'h'}(u, u')_s ds,\end{aligned}$$

with

$$M(u, v; u', v') = \begin{cases} 3/\theta^3 & \text{if } (u, v; u', v') = (1, 1; 1, 1) \\ 3/4\theta & \text{if } (u, v; u', v') = (1, 2; 1, 2), (2, 1; 2, 1) \\ 151\theta/280 & \text{if } (u, v; u', v') = (2, 2; 2, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$A\left((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\right)_T = \begin{cases} \frac{3}{\beta^3} \int_0^T (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H' \partial_{a'b'} G')(c_t) (c_t^{gg'} c_t^{hh'} + c_t^{gh'} c_t^{hg'}) (c_t^{aa'} c_t^{bb'} + c_t^{ab'} c_t^{ba'}) dt & \text{if } (u, v; u', v') = (1, 1; 1, 1) \\ \frac{3}{4\beta} \int_0^T (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H' \partial_{a'b'} G')(c_t) (c_t^{gg'} c_t^{hh'} + c_t^{gh'} c_t^{hg'}) \bar{c}_t^{ab, a'b'} dt & \text{if } (u, v; u', v') = (1, 2; 1, 2) \\ \frac{3}{4\beta} \int_0^T (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H' \partial_{a'b'} G')(c_t) (c_t^{aa'} c_t^{bb'} + c_t^{ab'} c_t^{ba'}) \bar{t}_s^{gh, g'h'} dt & \text{if } (u, v; u', v') = (2, 1; 2, 1) \\ \frac{151\beta}{280} \int_0^T (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H' \partial_{a'b'} G')(c_t) \bar{c}_s^{ab, a'b'} \bar{c}_t^{gh, g'h'} dt & \text{if } (u, v; u', v') = (2, 2; 2, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Using (72), we deduce that the asymptotic covariance between $[H_r(\widehat{c}), \widehat{G}_r(c)]_T^{(A)}$ and $[H_s(\widehat{c}), \widehat{G}_s(c)]_T^{(A)}$ is given by

$$\begin{aligned} & \sum_{g,h,a,b=1}^d \sum_{g',h',a',b'=1}^d \sum_{u,v,u',v'=1}^2 \left(A\left((H_r, gh, u; G_r, ab, v), (H_s, g'h', u'; G_s, a'b', v')\right)_T \right. \\ & + A\left((H_r, gh, u; G_r, ab, v), (H_s, a'b', v'; G_s, g'h', u')\right)_T + A\left((H_r, ab, v; G_r, gh, u), (H_s, g'h', u'; G_s, a'b', v')\right)_T \\ & \left. + A\left((H_r, ab, v; G_r, gh, u), (H_s, a'b', v'; G_s, g'h', u')\right)_T \right). \end{aligned}$$

After some simple calculations, the above expression can be rewritten as

$$\begin{aligned} & \sum_{g,h,a,b=1}^d \sum_{j,k,l,m=1}^d \left(\frac{6}{\theta^3} \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_t)) \left[(c_t^{gj} c_t^{hk} + c_t^{gk} c_t^{hj}) (c_t^{al} c_t^{bm} + c_t^{am} c_t^{bl}) \right. \right. \\ & + (c_t^{aj} c_t^{bk} + c_t^{ak} c_t^{bj}) (c_t^{gl} c_t^{hm} + c_t^{gm} c_t^{hl}) \left. \right] dt \\ & + \frac{151\theta}{140} \int_0^t (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_t)) \left[\bar{c}^{gh,jk} \bar{c}^{ab,lm} + \bar{c}^{ab,jk} \bar{c}^{gh,lm} \right] dt \\ & + \frac{3}{2\theta} \int_0^t (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(c_t)) \left[(c_t^{gj} c_t^{hk} + c_t^{gk} c_t^{hj}) \bar{c}_t^{ab,lm} + (c_t^{al} c_t^{bm} + c_t^{am} c_t^{bl}) \bar{c}_t^{gh,jk} \right. \\ & \left. \left. + (c_t^{gl} c_t^{hm} + c_t^{gm} c_t^{hl}) \bar{c}_t^{ab,jk} + (c_t^{aj} c_t^{bk} + c_t^{ak} c_t^{bj}) \bar{c}_t^{gh,lm} \right] dt \right), \end{aligned}$$

which completes the proof.

B.2 Proof of Theorem 2

Using the polynomial growth assumption on H_r, G_r, H_s and G_s and Theorem 2.2 in [Jacod and Rosenbaum \(2012\)](#), one can show that

$$\frac{6}{\theta^3} \widehat{\Omega}_T^{r,s,(1)} \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(1)}.$$

Next, making use of equation (3.27) in [Jacod and Rosenbaum \(2012\)](#), it can be shown that

$$\frac{3}{2\theta}[\widehat{\Omega}_T^{r,s,(3)} - \frac{6}{\theta}\widehat{\Omega}_T^{r,s,(1)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(3)}.$$

Finally, to show that

$$\frac{151\theta}{140} \frac{9}{4\theta^2}[\widehat{\Omega}_T^{r,s,(2)} + \frac{4}{\theta^2}\widehat{\Omega}_T^{r,s,(1)} - \frac{4}{3}\widehat{\Omega}_T^{r,s,(3)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(2)},$$

we first observe that as in Step 1, the approximation error induced by replacing $\widehat{c}_i^n$ by $\widehat{c}_i'^n$ is negligible. For $1 \leq g, h, a, b, j, k, l, m \leq d$ and $1 \leq r, s \leq d$, we define

$$\begin{aligned} \widehat{W}_T^n &= \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(\widehat{c}_i^n) \gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_{i+2k_n}^{n,ab} \gamma_{i+2k_n}^{n,lm} \\ \widehat{w}(1)_i^n &= (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_i^n) \mathbb{E}(\gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_{i+2k_n}^{n,ab} \gamma_{i+2k_n}^{n,lm} | \mathcal{F}_i^n) \\ \widehat{w}(2)_i^n &= (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_i^n) (\gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_{i+2k_n}^{n,ab} \gamma_{i+2k_n}^{n,lm} - \mathbb{E}(\gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_{i+2k_n}^{n,ab} \gamma_{i+2k_n}^{n,lm} | \mathcal{F}_i^n)) \\ \widehat{w}(3)_i^n &= \left((\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(\widehat{c}_i^n) - (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_i^n) \right) \gamma_i^{n,gh} \gamma_i^{n,jk} \gamma_{i+2k_n}^{n,ab} \gamma_{i+2k_n}^{n,lm} \\ \widehat{W}(u)_t^n &= \sum_{i=1}^{[T/\Delta_n]-4k_n+1} \widehat{w}_i(u), \quad u = 1, 2, 3. \end{aligned}$$

Note that we have $\widehat{W}_t^n = \widehat{W}(1)_t^n + \widehat{W}(2)_t^n + \widehat{W}(3)_t^n$. By Taylor expansion and using repeatedly the boundedness of c_t we have

$$|\widehat{w}(3)_i^n| \leq (1 + \|\beta_i^n\|^{4(p-1)}) \|\beta_i^n\| \|\gamma_i^n\|^2 \|\gamma_{i+2k_n}^n\|^2,$$

which implies $\mathbb{E}(|\widehat{w}(3)_i^n|) \leq K\Delta_n^{5/4}$ and $\widehat{W}(3)_t^n \xrightarrow{\mathbb{P}} 0$. Using Cauchy-Schwartz inequality and the bound $\mathbb{E}(\|\gamma_i^n\|^q | \mathcal{F}_i^n) \leq K\Delta_n^{q/4}$, we have $\mathbb{E}(|\widehat{w}(2)_i^n|^2) \leq K\Delta_n^2$. Observing furthermore that $\widehat{w}(2)_i^n$ is $\mathcal{F}_{i+4k_n}$ -measurable, we use Lemma B.8 in [Aït-Sahalia and Jacod \(2014\)](#) to show that $\widehat{W}(2)_t^n \xrightarrow{\mathbb{P}} 0$. Also, define

$$\begin{aligned} w_i^n &= (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_i^n) \left[\frac{4}{k_n^2 \Delta_n} (c_i^{n,ga} c_i^{n,hb} + c_i^{n,gb} c_i^{n,ha}) (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \right. \\ &\quad \left. + \frac{4}{3} (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \bar{c}_i^{n,gh,ab} + \frac{4}{3} (c_i^{n,ga} c_i^{n,hb} + c_i^{n,gb} c_i^{n,ha}) \bar{c}_i^{n,jk,lm} + \frac{4(k_n^2 \Delta_n)}{9} \bar{c}_i^{n,gh,ab} \bar{c}_i^{n,jk,lm} \right], \\ W_T^n &= \Delta_n \sum_{i=1}^{[T/\Delta_n]-4k_n+1} w_i^n. \end{aligned}$$

The cadlag property of c and $\bar{c}$ and $k_n \sqrt{\Delta_n} \rightarrow \theta$ and the Riemann integral argument imply $W_T^n \xrightarrow{\mathbb{P}} W_T$ defined as

$$\begin{aligned} W_T &= \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_t) \left[\frac{4}{\theta^2} (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) (c_t^{jl} c_t^{km} + c_t^{jm} c_t^{kl}) + \frac{4}{3} (c_t^{jl} c_t^{km} + c_t^{jm} c_t^{kl}) \bar{c}_t^{gh,ab} \right. \\ &\quad \left. + \frac{4}{3} (c_t^{ga} c_t^{hb} + c_t^{gb} c_t^{ha}) \bar{c}_t^{jk,lm} + \frac{4\theta^2}{9} \bar{c}_t^{gh,ab} \bar{c}_t^{jk,lm} \right] dt. \end{aligned}$$

In addition, by Lemma 4 we have

$$\mathbb{E}(|\widehat{W}(1)_T^n - W_T^n|) \leq \Delta_n \mathbb{E} \left(\sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\Delta_n^{1/8} + \eta_{i,4k_n}) \right).$$

Hence, by the third point of Lemma 6 we have $\widehat{W}_T^n \xrightarrow{\mathbb{P}} W_t$ from which it can be deduced that

$$\begin{aligned}
& \frac{9}{4\theta^2} \left[\widehat{W}(1)_T^n + \frac{4}{k_n^2} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(\widehat{c}_i^n) [C_i^n(jk, lm) C_i^n(gh, ab)] \right. \\
& - \frac{2}{k_n} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(\widehat{c}_i^n) C_i^n(gh, ab) \gamma_i^{n,jk} \gamma_i^{n,lm} \\
& - \left. \frac{2}{k_n} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(\widehat{c}_i^n) C_i^n(jk, lm) \gamma_i^{n,gh} \gamma_i^{n,ab} \right] \\
& \xrightarrow{\mathbb{P}} \int_0^T (\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s)(c_t) \bar{c}_t^{gh,ab} \bar{c}_t^{jk,lm} dt.
\end{aligned}$$

The result follows from the above convergence, a symmetry argument and straightforward calculations.

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