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Abstract

The aim of this paper is to compute upper and lower bounds for convex value functions of derivative contracts. Laprise et al. (2006) compute bounds for American-style vanilla options by selected portfolios of call options. We provide an alternative interpretation of their numerical procedure as a stochastic dynamic program for which the Bellman value function is approximated by selected piecewise linear interpolations at each decision date. The stochastic dynamic program does not (directly) depend on portfolios of call options, but rather on a key ingredient: some transition parameters of the underlying asset. More in line with the literature on dynamic programming, our procedure is contract free and is well designed to accomodate all one-dimensional convex value functions of derivative contracts. In support of this, we revisit the numerical investigation of Laprise et al. (2006) and enlarge their findings to include options embedded in bonds under affine term-structure models of interest rates.

Key Words: American Options, Derivative Contracts, Convex Functions, Upper and Lower Bounds, Stochastic Dynamic Programming, Piecewise Linear Interpolations.

Résumé

L'objet de ce papier est de calculer des bornes supérieures et inférieures pour des fonctions valeur convexes de produits dérivés. Laprise et al. (2006) calculent de telles bornes pour des options américaines vanille à l'aide de portefeuilles choisis d'options d'achat dans des modèles multiplicatifs. On donne une interprétation alternative de leur procédure comme un programme dynamique pour lequel la fonction de Bellman est approchée par des interpolations linéaires par morceaux à chaque date de décision. Notre programme dynamique ne dépend pas directement de portefeuilles d'options d'achat mais, à la place, d'ingrédients clefs : des paramètres de transition du processus d'état. Notre approche s'insère bien dans le cadre théorique de la programmation dynamique en plus d'être bien adaptée pour accommoder tous les produits dérivés à fonctions valeurs unidimensionnelles et convexes. Notre investigation numérique comprend une reprise des résultats de Laprise et al. (2006) et une extension aux options implicites aux obligations dans les modèles affines de taux d'intérêt.

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1 Introduction

Models for financial derivatives are useful for practitioners since they provide fair values and sensitivities that may be used as guides for trading. Several derivative contracts cannot be valued in a closed form and have to be approximated in some way. Examples include options with early exercise opportunities. Several valuing procedures have been proposed in the literature. They assume a discretization of the state space and build on the property that the approximate value converges to the true value for finer and finer discretizations. In general, for a given discretization, the approximation error is unknown, which is a major disadvantage from a practical point of view. Enveloping the value function to be computed allows one to bound the approximation error from above.

Laprise et al. (2006) compute upper and lower bounds for American-style vanilla options using selected portfolios of call options in multiplicative models. Instead of building on portfolios of call options to value derivatives, we design an equivalent procedure, based on stochastic dynamic programming (SDP), for which the Bellman value function is approximated by selected piecewise linear interpolations at each decision date. Our construction has four main advantages with respect to (Laprise et al. 2006):

1. SDP belongs to a well-known family of numerical procedures, while Laprise et al. (2006) is an ad-hoc procedure. Thus, in using SDP, one benefits from the accumulated knowledge in the field of dynamic programming. See for example (Bertsekas 1995).
2. SDP separates the evaluation problem into two parts: the dynamics of the underlying asset, captured by some transition parameters, and the form of the derivative to be priced, captured by a convex function, while Laprise et al. (2006) mix between the two parts.
3. SDP accommodates all convex value functions of derivative contracts, while Laprise et al. (2006) consider only on vanilla options under the geometric Brownian motion.
4. SDP can be extended for pricing convex derivatives in high-dimensional state spaces, while the algorithm of Laprise et al. (2006), as it is designed, cannot.

An extensive literature on approximation methods for valuing derivatives is readily available. Commonly used methods are based on:

1. quasi-analytic approaches (Bunch and Johnson 1992, Geske and Johnson 1984, MacMillan 1986, Barone-Adesi and Whaley 1987, Carr, Jarrow, and Myneni 1992, Huang, Subrahmanyam, and Yu 1996, Carr 1998, Ju 1998, and Ju and Zhong 1999);
2. trees (Rendleman and Bartter 1979, Cox, Ross, and Rubinstein 1979, Breen 1991, and Komrad and Ritchken 1991);
3. finite-differences (Parkinson 1977, Brennan and Schwartz 1977 and 1978, Courtadon 1982, and Hull and White 1990);
4. finite-elements (Barone-Adesi, Bermudez, and Hatgioannides 2003, and de Frutos 2005 and 2006);
5. finite volumes(Zvan, Forsyth, and Vetzal 2001);
6. stochastic dynamic programming (Chen 1970 and Ben-Ameur, Breton, and François 2006);
7. Monte Carlo simulation (Boyle, Broadie, and Glasserman 1997 and Broadie and Glasserman 1997).

The sandwich algorithms of Burkard, Hamacher, and Rote (1992) and Rote (1992) for convex functions based on secants and tangents cannot be applied directly in our context as they assume that the exact function values and their exact derivatives are known. However, we use the same concepts to envelop convex value functions of derivative contracts. Two approaches are used in the literature to derive upper and lower bounds for value functions of derivative contracts. In the first, closed-form envelopes are analytically derived (Johnson 1983, Lévy 1985, Broadie and Detemple 1996, and Davis, Schachermayer, and Tompkins 2001). In the second, upper and lower bounds are computed by means of numerical procedures (Broadie and Glasserman 1997, Broadie and Cao 2008, Chung and Chang 2007, Chung, Hung, and Wang 2010, Haugh and Kogan 2004, Magdon-Ismail 2003, and Laprise et al. 2006).

The rest of the paper is organized as follows. In Section 2, we present our stochastic dynamic program and show how to obtain upper and lower bounds for convex value functions of derivative contracts. In Section 3, we provide a numerical investigation. Section 4 is a conclusion.

2 SDP Formulation

2.1 Model and Notation

Let $\{X\}$ be the price process of an underlying asset, interpreted here as the state process. Examples include stocks and interest rates. An American-style option is defined by a known payoff function $\varphi(t,x) \geq 0$ under exercise, where $t \in \{t_0 = 0 < t_1 < \cdots < t_N = T\}$ lies in a finite set of decision dates and $x = X_t$ is the level of the state variable at time t. We consider herein convex value functions on x. For example, $\varphi(t,x) = \max(0, K - x)$, for an American put option, where K is the option strike price. The payoff function $\varphi(t,x)$ is called the exercise value and indicated in the following by $v^e(t,x)$. A European option is a particular American-style option with a unique decision date at the option maturity date $t_N = T$, that is, $v^e(t_n,x) = 0$, for $n < N$.

The option value function at t_n is defined by

$$v(t_n,x) = \max\left(v^e(t_n,x), v^h(t_n,x)\right), \quad \text{for all } x. \tag{1}$$

where $v^h(t_n,x)$ is the holding value, which depends on the future potentialities of the contract. No-arbitrage pricing gives:

$$v^h(t_n,x) = E\left[\rho_n v\left(t_{n+1}, X_{t_{n+1}}\right) \mid X_{t_n} = x\right], \quad \text{for all } x, \tag{2}$$

with the convention that $v^h(t_N,x) = 0$, for all x. Here, and in the sequel, ρ_n is the (possibly stochastic) risk-free discount factor for the time period $\Delta t_n = t_{n+1} - t_n$ and $E[\cdot \mid X_{t_n} = x]$ the conditional expectation symbol with respect to the risk-neutral probability measure. Eq. (1)–(2) say that the option holder must decide, in an optimal way, whether or not to exercise its right at each decision date and level of the state variable. The option value function at maturity is

$$v(t_N,x) = v^e(t_N,x), \quad \text{for all } x. \tag{3}$$

American-style options cannot be evaluated in closed form, except for very few particular cases. Their values have to be approximated in some way. We propose herein upper and lower bounds, and obtain

$$\widehat{v}(t_0,x_0) = \frac{\widehat{v}_{\sup}(t_0,x_0) + \widehat{v}_{\inf}(t_0,x_0)}{2}. \tag{4}$$

These bounds verify

$$\widehat{v}_{\inf}(t,x) \leq v(t,x) \leq \widehat{v}_{\sup}(t,x), \quad \text{for all } t \text{ and } x,$$

which results in an approximation error $e(t,x)$ that verifies

$$0 \leq e(t,x) \leq \widehat{v}_{\sup}(t,x) - \widehat{v}_{\inf}(t,x), \quad \text{for all } t \text{ and } x. \tag{5}$$

Our approach provides not only an approximation for the option value (eq. 4) but also an upper bound for the approximation error (eq. 5).

Let $\mathcal{G} = \{a_1, \ldots, a_p\}$ be a mesh of grid points that covers the state space. We select the gird points to be the quantiles of the underlying asset at $t_N = T$. For example, in the Black and Scholes' model, the gird points are constructed as follows:

$$a_0 = 0, \quad a_1 = x_0 e^{\left(r - \frac{\sigma^2}{2}\right)T - 7\sigma\sqrt{T}}, \quad a_2 = x_0 e^{\left(r - \frac{\sigma^2}{2}\right)T - 5\sigma\sqrt{T}},$$
$$a_{p-1} = x_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + 5\sigma\sqrt{T}}, \quad a_p = x_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + 7\sigma\sqrt{T}}, \quad a_{p+1} = \infty,$$

and

$$a_i = x_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} z_i}, \tag{6}$$

where z_i is the quantile of the standard normal distribution associated with fraction i/p, for $i = 3, \ldots, p-2$. This construction, although not unique, has the property to set more evaluation nodes in the most visited areas (eq. 6). The grid points $\mathcal{G}$ is kept fixed over time.

2.2 The Upper Envelope

Suppose that a convex upper envelope approximation $\widetilde{v}^h_{\sup}(t_{n+1}, .)$ of $v^h(t_{n+1}, .)$ is available at a certain future date t_{n+1}. One has

$$\widetilde{v}^h_{\sup}(t_{n+1}, x) \geq v^h(t_{n+1}, x), \quad \text{for all } x.$$

This assumption is not really a limitation since

$$\widetilde{v}^h_{\sup}(t_N, x) = v^h(t_N, x) = 0, \quad \text{for all } x.$$

That is to say that SDP starts at $t_N = T$, where

$$\widetilde{v}_{\sup}(t_N, x) = v(t_N, x) = v^e(t_N, x), \quad \text{for all } x. \tag{7}$$

Eq. (3) and (7) say that the SDP approximation $\widetilde{v}_{\sup}(t_N, .)$ at maturity is nothing else than the true value function of the derivative to be evaluated. Now, we show how SDP moves backward in time from time t_{n+1} to time t_n.

First, we construct $\widetilde{v}_{\sup}(t_{n+1}, x)$ as follows:

$$\widetilde{v}_{\sup}(t_{n+1}, x) = \max\left(v^e(t_{n+1}, x), \widetilde{v}^h_{\sup}(t_{n+1}, x)\right), \quad \text{for all } x. \tag{8}$$

The value function in eq. (8) is convex and over approximate $v(t_{n+1}, x)$, since the maximum of two convex functions is convex.

Then, we compute $\widetilde{v}_{\sup}(t_{n+1}, a_k)$, for all a_k in $\mathcal{G}$, so that we can store the required information in a computer. Next, we use a piecewise linear interpolation, and extend $\widetilde{v}_{\sup}(t_{n+1}, .)$ from $\mathcal{G}$ to the overall state space $\mathbb{R}_+$ as follows:

$$\widehat{v}_{\sup}(t_{n+1}, x) = \sum_{i=0}^{p} \left(\alpha_i^{n+1} + \beta_i^{n+1} X_{t_{n+1}}\right) \mathbb{I}\left(a_i \leq x < a_{i+1}\right), \quad \text{for all } x, \tag{9}$$

where $\mathbb{I}(\cdot)$ is the indicator function and

$$\begin{aligned}
\beta_i^{n+1} &= \frac{\tilde{v}(t_{n+1}, a_{i+1}) - \tilde{v}(t_{n+1}, a_i)}{a_{i+1} - a_i} \\
\alpha_i^{n+1} &= \frac{a_i \tilde{v}(t_{n+1}, a_{i+1}) - a_{i+1}\tilde{v}(t_{n+1}, a_i)}{a_{i+1} - a_i} \\
&\text{for } i = 1, \ldots, p-1\ ,
\end{aligned}$$

and

$$\alpha_0^{n+1} = \alpha_1^{n+1}, \quad \beta_0^{n+1} = \beta_1^{n+1}, \quad \alpha_p^{n+1} = \alpha_{p-1}^{n+1}, \quad \text{and } \beta_p^{n+1} = \beta_{p-1}^{n+1}.$$

In doing so, the value function $\widehat{v}_{\sup}(t_{n+1}, .)$, defined on the overall state space, is stored in the computer through the local coefficients α_i^{n+1} and β_i^{n+1}, for $i = 0, \ldots, p$. This equation is a direct application of the secant algorithm, which guarantees a convex upper envelope of $v(t_{n+1}, .)$ (Burkard, Hamacher, and Rote 1992 and Rote 1992). The secant envelope $\widehat{v}_{\sup}(t_{n+1}, .)$ to a convex upper envelope $\widetilde{v}_{\sup}(t_{n+1}, .)$ of a convex function $v(t_{n+1}, .)$ is a convex upper envelope:

$$\widehat{v}_{\sup}(t_{n+1}, x) \geq \widetilde{v}_{\sup}(t_{n+1}, x) \geq v(t_{n+1}, x), \quad \text{for all } x.$$

Finally, to move backward in time from t_{n+1} to t_n, we use eq. (2) and (9) and no-arbitrage pricing as follows:

$$\begin{aligned}\widetilde{v}^h_{\sup}(t_n, x) &= E\left[\rho_n \widehat{v}_{\sup}\left(t_{n+1}, X_{t_{n+1}}\right) \mid X_{t_n} = x\right] \\ &= \sum_{i=0}^{p} \left(\alpha_i^{n+1} E\left[\rho_n \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = x\right] + \right. \\ &\qquad \left. \beta_i^{n+1} E\left[\rho_n X_{t_{n+1}} \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = x\right]\right).\end{aligned} \tag{10}$$

The value function $\widetilde{v}^h_{\sup}(t_n, .)$ is a convex upper envelope of $v^h(t_n, .)$. Indeed, the weighted average of convex functions is a convex function, where the weighted average is the conditional expectation in eq. (10). Now, we compute $\widetilde{v}^h_{\sup}(t_n, .)$ on $\mathcal{G}$ so that it can be stored in the computer:

$$\begin{aligned}\widetilde{v}^h_{\sup}(t_n, a_k) &= \sum_{i=0}^{p} \left(\alpha_i^{n+1} E\left[\rho_n \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = a_m\right] + \right. \\ &\qquad \left. \beta_i^{n+1} E\left[\rho_n X_{t_{n+1}} \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = a_k\right]\right) \\ &= \sum_{i=0}^{p} \alpha_i^{n+1} A_{ki} + \beta_i^{n+1} B_{ki}, \quad \text{for } k = 1, \ldots, p,\end{aligned} \tag{11}$$

where the transition parameters A_{ki} and B_{ki} are

$$A_{ki} = E\left[\rho_n \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = a_k\right] \tag{12}$$

and

$$B_{ki} = E\left[\rho_n X_{t_{n+1}} \mathbb{I}\left(a_i \le X_{t_{n+1}} < a_{i+1}\right) \mid X_{t_n} = a_k\right]. \tag{13}$$

The transition tables A and B in eq. (12)–(13) can be seen as the minimum information required on $\{X\}$ to play the role of a state process. These tables are available in closed form under geometric Brownian motions (Ben-Ameur, Breton, and François 2006) and mean-reverting Gaussian and chi-squared processes (Ben-Ameur et al. 2007). For time-homogeneous diffusions, they do not depend on time as long as $\Delta t_n = t_{n+1} - t_n$ is kept fixed. For simplicity, we assume this assumption herein, but clearly, our SDP still works even though Δt_n does change over time.

A key feature of SDP is that $\widehat{v}(t_n, x)$ is known, for all n and x. The Greeks follow by finite-difference approximations, as shown in eq. (14)–(15). The SDP approximation of the Delta coefficient, Δ, is based on a first-order central finite-difference approximation:

$$\widehat{\Delta} = \frac{\widehat{v}(t_0, x_0 + h) - \widehat{v}(t_0, x_0 - h)}{2h}, \quad \text{for a selected small } h, \tag{14}$$

where Δ is the sensitivity coefficient of $v(t_0, x)$ with respect to x, evaluated at $x_0 = X_{t_0}$. The SDP approximation of the Theta coefficient, Θ, is

$$\widehat{\Theta} = \frac{\widehat{v}(t_1, x_0) - \widehat{v}(t_0, x_0)}{h}, \tag{15}$$

where t_1 is an evaluation date in the right neighborhood of t_0, which does not need to be a decision date. Gamma, Γ, is computed as well by a second-order central finite-difference approximation. Vega, $\mathcal{V}$, and Rho, ρ, can be computed by first-order central finite-difference approximations, but they require SDP to run twice.

2.3 The Lower Envelope

We construct the lower envelope in the same spirit. The tangent envelope $\widehat{v}^h_{\inf}(t_{n+1}, .)$ is a piecewise linear interpolation that is constructed from the tangents to $\widetilde{v}^h_{\inf}(t_{n+1}, .)$ at the evaluation nodes in $\mathcal{G}$. The local successive tangents do intersect at a_i^*, where $i = 1, \ldots, p$, as indicated in (Burkard, Hamacher, and Rote 1992 and Rote 1992). We consider the new grid points $\mathcal{G}^* = \left\{a_1, a_1^*, a_2, \ldots, a_{p-1}, a_{p-1}^*, a_p\right\} = \mathcal{G} \cup \left\{a_1^*, \ldots, a_{p-1}^*\right\}$.

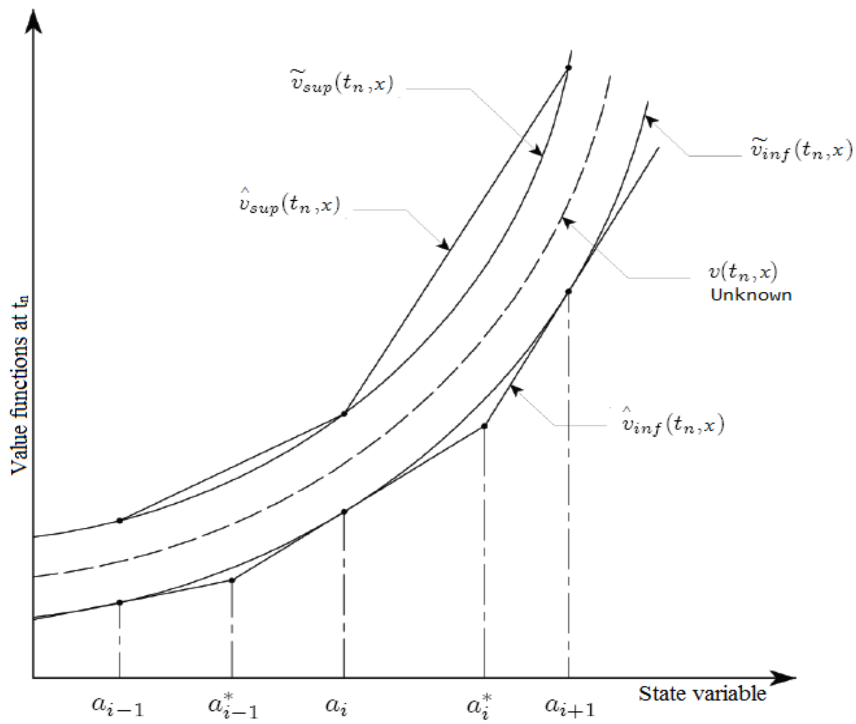

Figure 1: Upper and Lower Envelopes

For the tangent algorithm to run, we need the first derivatives of $\widetilde{v}^h(t_n, x)$ with respect to x, for all $a_k \in \mathcal{G}$. We compute these delta-based coefficients from their associated first derivatives in all cases, except for the CIR model, where we use first-order central finite-difference approximations. Their associated transition parameters are not known in closed form.

The sum in eq. (9) and the one in eq. (11) are now computed on $\mathcal{G}^*$, instead of $\mathcal{G}$. We compute consequently the new transition tables A^* and B^*, and run backward the recursion. Unfortunately, the lower envelope requires to recompute the transition tables A^* and B^* at each time step, since $\mathcal{G}^*$, in fact the set $\left\{a_1^*, \ldots, a_{p-1}^*\right\}$, does change over time.

All in all, we set $\widetilde{v}_{\inf}(t_N, .) = v(t_N, .) = \widetilde{v}_{\sup}(t_N, .)$ at the very beginning. Then, we construct the secant and the tangent envelopes at t_N on the overall state space $\mathbb{R}_+$ by piecewise linear interpolations, which are convex and verify

$$\widehat{v}_{\inf}(t_N, .) \leq v(t_N, .) \leq \widehat{v}_{\sup}(t_N, .).$$

Next, we build $\widetilde{v}_{\inf}(t_{N-1}, .)$ on $\widehat{v}_{\inf}(t_N, .)$, and $\widetilde{v}_{\sup}(t_{N-1}, .)$ on $\widehat{v}_{\sup}(t_N, .)$, which are convex and verify

$$\widetilde{v}_{\inf}(t_{N-1}, .) \leq v(t_{N-1}, .) \leq \widetilde{v}_{\sup}(t_{N-1}, .).$$

The secant and the tangent envelopes at t_{N-1} come, and so on.

Finally, we move backward in time from t_N to t_{N-1}, down to t_0, where we obtain SDP approximations for the option value and its derivatives. Figure 1 explains how SDP approximations behave at time t_n.

3 Numerical Investigation

We run our stochastic dynamic program (SDP) with vanilla options under geometric Brownian motions. First, we consider European vanilla options. SDP results are compared to the closed-form solution of Black and Scholes (1973). SDP does extremely well. Then, we consider American-style vanilla options. SDP competes well against various alternatives in the literature. Finally, SDP is compared with Laprise et al. (2006).

Thereafter, we consider a bond analysis under affine term-structure models of interest rates. We start with straight bonds for which closed-form solutions are available under mean-reverting Gaussian processes (Vasicek 1977) and mean-reverting square-root processes (Cox, Ingersoll, and Ross 1985). SDP approximate values converge to the "true" values. Then, we consider the put option embedded in bonds. Likewise, SDP does extremely well. To conclude, SDP shows consistency, robustness, and efficiency. This section shows that our presentation outperforms the one of Laprise et al. (2006). SDP directly applies for all European- as well as American-style derivatives with convex value functions. Examples include vanilla and embedded put options.

We write the code lines in C and compile it with GCC. We use some C functions from the GSL library. We run the software under Windows XP and a 1.7 Ghz laptop computer. CPU times (in seconds) are informative but cannot be directly compared to those of Laprise et al. (2006), which we report as well.

3.1 Vanilla Options under Black-Scholes

Table 1.1 deals with European vanilla put options in the Black and Scholes (BS) setting. The parameters are $\sigma = 20\%$ per year, $\delta = 0$ per year, $r = 5\%$ per year, $K = \$100$, $T = 3$ years, and $N = 6$, where σ is the volatility of the stock return, δ is its (continuous) dividend rate, and r is the (constant) interest rate compounded continuously. Whereas one step is sufficient, we run SDP backward in time through the six virtual decision dates.

From Table 1.1, SDP_{sup} and SDP_{inf} converge extremely well, both to the true values. The relative error of the dynamic program never exceeds .0001. CPU times (in seconds) indicate that the complexity of SDP_{sup} is in p^2. Higher CPU times for SDP_{inf} are observed, as a result of the additional effort of computing A^* and B^* at each step of the recursion.

Table 1.2 allows for early exercising. Clearly, obtained SDP_{sup} and SDP_{inf} approximate values converge to the same values, as the grid size increases. Our results are compared to those of Laprise et al. (2006), indicated here by L_{inf} and L_{sup}.

Table 2 also deals with American-style vanilla put options. The parameters are $\sigma = 20\%$ per year, $\delta = 8\%$ per year, $r = 8\%$ per year, $K = \$100$, $T = 3$ years, $N = 10,000$, and $p = 1600$.

SDP is compared to main known results in the literature. In the following, GJ stands for the four-point extrapolation of Geske and Johnson (1984), BJ for the modified two-point Geske-Johnson method of Bunch and Johnson (1992), HSY for the four-point extrapolation recursive method of Huang, Subrahmanyam, and Yu (1996), BD for the quasi-closed-form solution of Broadie and Detemple (1996), BT for a binomial tree with $N = 150$, B for the accelerated binomial tree of Breen (1991) with $N = 150$ time steps, BW for the quadratic analytical approximation of Barone-Adesi and Whaley (1987), and JZ for the modified quadratic approximation of Ju and Zhong (1999). The true value of the option to be priced, indicated by TRUE, is based on a binomial tree with $10,000$ time steps.

Here, the accuracy has decreased a bit if we compare our results with those reported in Table 1.1 and Table 1.2. We reach the limit in terms of our computer capability. However, SDP_{sup} and SDP_{inf} still compare well to the TRUE values and to alternative approximations. On the one hand, SDP values envelop well the true value, and, on the other hand, SDP values are always in between and never at the extremes of their competitors.

3.2 Bonds under Affine Term-Structure Models of Interest Rates

3.2.1 The Vasicek Model

Table 3.1 and Table 3.2 report upper and lower bounds for the straight bond analyzed in (Ben-Ameur et al. 2007) under the Vasicek model, where the risk-free interest rate moves according to a mean-reverting

Table 1.1: Bounds for European Put Options under Black and Scholes

		x_0					
	p	60	90	100	110	140	CPU
SDP_{inf}	200	27.96551	10.23792	6.99239	4.70677	1.36516	0.77
	400	27.96694	10.23985	6.99409	4.70865	1.36570	2.38
	800	27.96718	10.24034	6.99483	4.70955	1.36639	9.41
	1600	27.96725	10.24045	6.99513	4.70962	1.36651	36.24
	3200	27.96726	10.24047	6.99515	4.70964	1.36653	134.16
BS		27.96726	10.24048	6.99516	4.70965	1.36653	
SDP_{sup}	3200	27.96727	10.24049	6.99517	4.70966	1.36654	11.53
	1600	27.96729	10.24054	6.99522	4.70971	1.36657	2.87
	800	27.96741	10.24074	6.99536	4.70986	1.36668	0.73
	400	27.96797	10.24185	6.99655	4.71092	1.36724	0.29
	200	27.97080	10.24600	7.00142	4.71563	1.36969	0.09

Table 1.2: Bounds for American-style Put Options under Black and Scholes

		x_0					
	p	60	90	100	110	140	CPU
SDP_{inf}	200	37.55345	12.89931	8.44096	5.49378	1.49818	0.70
	400	37.55382	12.90464	8.44470	5.49704	1.49917	2.29
	800	37.55383	12.90555	8.44569	5.49794	1.49984	8.78
	1600	37.55384	12.90561	8.44595	5.49801	1.49997	36.17
	3200	37.55384	12.90566	8.44598	5.49803	1.49999	139.29
L_{inf}	200	37.55400	12.90500	8.44500	5.49700	1.49700	3.50
L_{sup}	200	37.55400	12.90600	8.44700	5.50000	1.50600	2.30
SDP_{sup}	3200	37.55384	12.90569	8.44600	5.49806	1.50000	10.36
	1600	37.55384	12.90573	8.44606	5.49811	1.50003	2.62
	800	37.55385	12.90591	8.44622	5.49828	1.50015	0.90
	400	37.55389	12.90709	8.44748	5.49950	1.50079	0.32
	200	37.55402	12.91138	8.45290	5.50443	1.50351	0.09

Table 2: SDP vs alternatives for American-style Put Options

x_0	80	90	100	110	120
SDP_{inf}	22.175	16.167	11.660	8.323	5.889
TRUE	22.205	16.207	11.704	8.367	5.930
SDP_{sup}	22.265	16.286	11.790	8.453	6.011
GJ	22.208	16.164	11.705	8.389	5.944
BJ	22.711	16.531	11.811	8.407	5.931
HSY	22.245	16.134	11.718	8.436	5.988
BD	22.199	16.199	11.699	8.363	5.926
BT	22.199	16.205	11.690	8.378	5.933
B	22.384	16.201	11.662	8.356	5.932
BW	22.395	16.498	12.030	8.687	6.222
JZ	22.148	16.170	11.700	8.390	5.968

Gaussian process. The bond analyzed has a principal amount of \$1, a maturity of 20.172 years, a yearly coupon distributed at the rate of 4.25% per year, and 21 remaining coupon dates. The next coupon date is at $t_1 = 0.172$ years. SDP is clearly convergent to Vasicek's formula. The relative error never exceeds .00003 with $p = 800$.

Now, we consider a (hypothetical) embedded put option. Unfortunately, the embedded call option cannot be analyzed in that way. The value function of the bond with its call feature is not convex. The put option gives the investor the right to return the bond to the issuer at certain specified coupon dates for specified put prices (in dollars): $p(12) = .78914$, $p(13) = .80749$, $p(14) = .83040$, $p(15) = .85824$, $p(16) = .88039$, $p(17) = .90311$, $p(18) = .92641$, $p(19) = .95032$, $p(20) = .97484$, and, by convention, $p(21) = 1$. Here, the protection period is $[t_0, t_{12}) = [0, 11.72)$ (in years). Thus, the first early-exercise opportunity is at the twelfth coupon date, that is, $t_{12} = 11,72$ years.

Table 3.1: SDP Upper Bounds for Straight Bonds under Vasicek

(r_0, p)	200	400	800	1600	3200	Formula
0.01	.927729	.927503	.927442	.927427	.927424	.927422
0.02	.909268	.909025	.908973	.908958	.908955	.908953
0.03	.891190	.890954	.890895	.890881	.890878	.890877
0.04	.873488	.873258	.873201	.873189	.873185	.873184
0.05	.856154	.855936	.855885	.855871	.855868	.855867
0.06	.839179	.838990	.838935	.838921	.838918	.838917
0.07	.822599	.822395	.822343	.822331	.822328	.822327
0.08	.806365	.806154	.806105	.806092	.806089	.806088
0.09	.790468	.790262	.790211	.790198	.790195	.790194
0.10	.774900	.774698	.774652	.774640	.774637	.774636
CPU	0.08	0.29	1.04	3.88	14.82	

Table 3.2: SDP Lower Bounds for Straight Bonds under Vasicek

(r_0, p)	200	400	800	1600	3200	Formula
0.01	.927275	.927382	.927413	.927420	.927422	.927422
0.02	.908799	.908918	.908942	.908951	.908953	.908953
0.03	.890707	.890837	.890868	.890874	.890876	.890877
0.04	.873030	.873148	.873175	.873181	.873183	.873184
0.05	.855729	.855833	.855857	.855864	.855866	.855867
0.06	.838786	.838878	.838908	.838915	.838916	.838917
0.07	.822196	.822294	.822319	.822324	.822326	.822327
0.08	.805949	.806056	.806080	.806086	.806087	.806088
0.09	.790043	.790156	.790184	.790191	.790193	.790194
0.10	.774505	.774606	.774628	.774634	.774635	.774636
CPU	2.26	9.01	35.18	137.80	548.95	

Table 3.3: SDP Upper Bounds for Putable Bonds under Vasicek

(r_0, p)	200	400	800	1600	3200
0.01	1.012603	1.012511	1.012486	1.012479	1.012477
0.02	.992265	.992150	.992134	.992128	.992126
0.03	.972350	.972240	.972215	.972209	.972208
0.04	.952851	.952744	.952720	.952716	.952714
0.05	.933758	.933661	.933643	.933637	.933636
0.06	.915064	.914995	.914972	.914966	.914964
0.07	.896805	.896717	.896696	.896692	.896690
0.08	.878929	.878831	.878812	.878807	.878805
0.09	.861426	.861329	.861308	.861303	.861301
0.10	.844287	.844192	.844175	.844170	.844169
CPU	0.06	0.25	0.98	3.75	14.97

Table 3.4: SDP Lower Bounds for Putable Bonds under Vasicek

(r_0, p)	200	400	800	1600	3200
0.01	1.012351	1.012451	1.012469	1.012476	1.012477
0.02	.991991	.992104	.992115	.992123	.992125
0.03	.972056	.972181	.972199	.972206	.972207
0.04	.952578	.952691	.952706	.952712	.952713
0.05	.933517	.933615	.933627	.933634	.933635
0.06	.914853	.914937	.914955	.914962	.914964
0.07	.896578	.896670	.896682	.896688	.896690
0.08	.878683	.878785	.878797	.878803	.878805
0.09	.861165	.861275	.861292	.861299	.861300
0.10	.844054	.844150	.844161	.844167	.844168
CPU	1.23	4.70	18.59	74.26	262.95

SDP_{sup} and SDP_{inf} envelop the true value of the bond extremely well. The SDP approximation error can be computed in this case; it is almost equal to zero for the highest grid size $p = 3200$.

The put option embedded in bonds cannot be valued in closed form. However, SDP_{sup} and SDP_{inf} envelop the "true" value extremely well (see Table 3.3 and Table 3.4). Since SDP values do converge, we conclude that the SDP approximation error converges to zero as the grid size p increases.

3.2.2 The CIR Model

We report the same bond analysis under CIR specifications, where the risk-free short-term interest rate moves according to a mean-reverting square-root process. To construct the lower envelope, we need the first derivatives of $\widetilde{v}(t_n, x)$ with respect to x, for all a_k on $\mathcal{G}$. We use central finite-difference approximations since the transition parameters are known in quasi-closed form.

Table 4.1 and Table 4.2 compare SDP approximations to CIR true values. SDP behaves extremely well. The upper and lower envelopes surround well the true values with a relative error that never exceeds .0004 with $p = 800$. Again, SDP shows robustness, consistency, and convergence.

Now, we envelop the value of the putable bond presented previously. Recall that the true value is unknown in this case. Again, as seen earlier, the SDP upper and lower bounds do converge, and the computing time is much higher for the lower envelope.

4 Conclusion

We propose an alternative stochastic dynamic program to Laprise et al. (2006) for enveloping convex derivatives. Our construction has four main advantages:

1. SDP belongs to a well-known family of numerical procedures, while Laprise et al. (2006) is an ad hoc procedure. Thus, in using SDP, one benefits from the accumulated knowledge in the field of dynamic programming. See for example (Bertsekas 1995).
2. SDP separates the evaluation problem into two parts: the dynamics of the underlying asset, captured by some transition parameters, and the form of the derivative to be priced, captured by a convex function, while Laprise et al. (2006) mix between the two parts.
3. SDP accommodates all convex value functions of derivative contracts, while Laprise et al. (2006) consider only on vanilla options under the geometric Brownian motion. Examples include putable bonds under affine term-structure models of interest rates.
4. SDP can be extended for pricing convex derivatives in high-dimensional state spaces, while the algorithm of Laprise et al. (2006), as it is designed, cannot.

Future work includes an extension to two-dimensional state spaces, which may be achieved by a stochastic dynamic program in which piecewise linear approximations are exchanged for piecewise bilinear approximations. The consideration of upper and lower envelopes via portfolios of vanilla (or basket) call options become irrelevant in high-dimensional state spaces. Examples include Asian options under geometric Brownian motions, as in (Ben-Ameur, Breton, and L'Écuyer 2002), and locally convex vanilla options under GARCH processes, as in (Ben-Ameur, Breton, and Martinez 2009). There, efficient implementations for upper envelopes are already available, but remain a challenge for lower envelopes.

Table 4.1: SDP Upper Bounds for Straight Bonds under CIR

(r_0, p)	200	400	800	1600	3200	Formula
0.01	0.956450	0.955565	0.955330	0.955268	0.955252	0.955247
0.02	0.932726	0.931849	0.931616	0.931556	0.931540	0.931535
0.03	0.909478	0.908723	0.908522	0.908470	0.908456	0.908452
0.04	0.887133	0.886280	0.886059	0.886001	0.885986	0.885981
0.05	0.865223	0.864399	0.864181	0.864125	0.864110	0.864105
0.06	0.843775	0.843063	0.842875	0.842826	0.842813	0.842809
0.07	0.823151	0.822354	0.822149	0.822095	0.822081	0.822076
0.08	0.802929	0.802166	0.801963	0.801911	0.801897	0.801893
0.09	0.783146	0.782479	0.782304	0.782258	0.782247	0.782243
0.10	0.764110	0.763369	0.763179	0.763129	0.763116	0.763112
CPU	0.26	1.03	4.23	16.06	65.65	

Table 4.2: SDP Lower Bounds for Straight Bonds under CIR

(r_0, p)	200	400	800	1600	3200	Formula
0.01	0.955133	0.954729	0.955043	0.955180	0.955227	0.955247
0.02	0.930605	0.930963	0.931320	0.931466	0.931514	0.931535
0.03	0.907388	0.907860	0.908236	0.908383	0.908432	0.908452
0.04	0.884658	0.885344	0.885757	0.885912	0.885960	0.885981
0.05	0.862650	0.863451	0.863882	0.864036	0.864085	0.864105
0.06	0.841319	0.842166	0.842592	0.842742	0.842790	0.842809
0.07	0.820432	0.821416	0.821857	0.822010	0.822057	0.822076
0.08	0.800210	0.801230	0.801677	0.801827	0.801874	0.801893
0.09	0.780590	0.781603	0.782034	0.782180	0.782225	0.782243
0.10	0.761366	0.762466	0.762902	0.763050	0.763094	0.763112
CPU	0.99	4.65	25.24	104.64	450.77	

Table 4.3: SDP Upper Bounds for Putable Bonds under CIR

(r_0, p)	200	400	800	1600	3200
0.01	1.022094	1.021725	1.021624	1.021599	1.021592
0.02	.996544	.996191	.996091	.996067	.996060
0.03	.971521	.971294	.971227	.971211	.971207
0.04	.947478	.947137	.947044	.947021	.947015
0.05	.923907	.923585	.923493	.923471	.923465
0.06	.900835	.900622	.900559	.900544	.900540
0.07	.878653	.878336	.878251	.878230	.878225
0.08	.856905	.856612	.856526	.856506	.856501
0.09	.835630	.835429	.835370	.835356	.835352
0.10	.815161	.814867	.814790	.814771	.814766
CPU	0.05	0.27	1.09	4.64	19.43

Table 4.4: SDP Lower Bounds for Puttable Bonds under CIR

(r_0, p)	200	400	800	1600	3200
0.01	1.021547	1.021283	1.021488	1.021560	1.021582
0.02	.995253	.995759	.995959	.996029	.996050
0.03	.970507	.970936	.971114	.971178	.971198
0.04	.946250	.946732	.946918	.946985	.947005
0.05	.922728	.923187	.923372	.923436	.923456
0.06	.899896	.900291	.900454	.900514	.900532
0.07	.877517	.877965	.878136	.878197	.878216
0.08	.855827	.856244	.856415	.856474	.856492
0.09	.834760	.835123	.835274	.835328	.835345
0.10	.814111	.814528	.814683	.814741	.814757
CPU	0.42	2.01	8.04	38.35	168.88

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